

Closed book exam. No auxiliary material ( on paper, electronic or any other form) is allowed.

1. [6 points] Please answer **two** of the following three questions.
  - (a) Describe the most common structure of a Malthusian growth model. Which kind of growth facts can these models account for ? Why can't they account for the stylized facts associated to "modern economic growth"?
  - (b) The Ramsey model is the workhorse for modern macroeconomics and growth theory. Describe its main assumptions and give an intuition for the behaviour of its solution.
  - (c) The Uzawa-Lucas model displays transition dynamics while the AK model only displays long-run growth. After presenting a short description of the two models, provide a justification for that difference between the two models.
2. [7 points] Consider a version of the Solow model, in which there are two types of labor: skilled  $L_s$ , and unskilled labor  $L_u$ . The proportion of population with each skill is constant, such that  $\ell = L_u/L$  and  $1 - \ell = L_s/L$ , where  $0 < \ell < 1$ . The total population,  $L$ , grows at a constant rate  $n > 0$ . The technology of production involves a complementarity between capital and unskilled labor and a substitution between them and skilled labor. It is represented by the production function

$$Y(t) = (K(t) + L_u(t))^\alpha (AL_s(t))^{1-\alpha}$$

where  $0 < \alpha < 1$  and  $A > 1$  measures the specific productivity of skilled labor. The savings function is  $S(t) = sY(t)$ , with  $0 < s < 1$ , and there is no depreciation of capital.

- (a) Derive the accumulation equation for the detrended capital stock  $k(t) \equiv K(t)/L(t)$ .
  - (b) Prove there is a unique long run level for  $k$ . Is uniqueness related to the Inada properties , for  $k$  , of the production function ?
  - (c) Describe the properties of the model regarding the existence of a balanced growth path, of transition dynamics and of endogenous growth.
  - (d) Assume there is a permanent increase in the proportion of unskilled labour  $\ell$ . Determine the effects over long run growth, the level effects, and the transitional dynamics. (Hint: assume that  $\ell < \alpha$  and  $s\alpha^\alpha(A(1-\alpha))^{1-\alpha} > n$ ).
3. [7 points] Consider an economy in which physical and human capital are perfect substitutes in production and investment. We denote aggregate physical and human capital by  $K^a$  and  $H^a$ , respectively, and aggregate total capital by  $W^a = K^a + H^a$ . The production function is  $Y^a = AW^a$ , where  $Y^a$  is aggregate output, and the accumulation equation is  $\dot{W}^a = Y^a - C^a$ . We assume that total population follows the equation  $N(t) = e^{nt}$  with  $n > 0$ . Consider a centralized economy model in where the central planner has the utility functional

$$\max_{(C^a(t))_{t \geq 0}} \int_0^\infty \frac{C^a(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt,$$

where  $\theta > 1$  and  $\rho > 0$ , given  $W^a(0) = W_0 > 0$  and  $\lim_{t \rightarrow \infty} e^{-\rho t} W^a(t) \geq 0$ .

- (a) Determine the optimality conditions as an initial-terminal value problem for per capita consumption and total wealth.
- (b) Discuss the verification of the necessary conditions for the existence of a balanced growth path.
- (c) Specify the model in detrended variables, and determine the long-run (endogenous) growth rate.
- (d) Solve the planner's problem. Determine the solution for the optimal per capita output.
- (e) Discuss the growth properties of the model, and, in particular, the implications of changes in parameter  $A$ .