# The $AK$ model

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# Exogenous and endogenous growth

- Almost all the models we presented previously
  - did **not** display long run growth: i.e. $\lim_{t\to\infty} g(t) > 0$
  - or, if they did, it was **exogenous**, i.e., introduced by some external exponential mechanism (growth in productivity or growth in population)
  - this is a description not an explanation
- From now on we present **endogenous growth models**
  - we already saw that in order to have long-run growth the main accumulating factor should be exponential
  - the question is: how to generate this in an aggregate model ?

# The $AK$ model

- It is the simplest **endogenous growth model**
- The economy has the following features:
  1. population is constant and normalised to one $N = 1$
  2. there is one reproducible input: physical capital
     (we can see all types of capital as being perfect substitutes or having the relative prices constant, which requires having the same reproduction mechanism)
  3. the economy produces one good with a CRS technology
     (using only capital)
  4. the good is used in consumption and investment (it is a closed economy)
  5. the consumer solves an intertemporal optimization problem

# Version of the $AK$ model

- Decentralized version: there is no state and the allocation of capital through time is determined by market equilibrium
- Centralized version: there is a central planner ("benevolent dictator") that determines the optimal allocation of capital by maximizing the intertemporal social welfare
- As there are no externalities or other distortions, the two versions are equivalent: in this case we say that the **equilibrium allocations are Pareto optimal**
- When there are externalities (see the Romer model) the two economies lead to different allocations: then **equilibrium allocations are not Pareto optimal**

# Assumptions

- Technology: linear production function

$$Y = AK,\ A > 0$$

  $Y$ **per capita GDP,** $K$ **capital intensity**, $A =$ average $=$ marginal productivity

- Preferences: intertemporal utility functional

$$\int_0^\infty \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt,\ \theta > 0,\ \text{and } \rho > 0$$

  $C$ **per capita consumption**
  As in the Ramsey model: allows for an efficient choice of consumption and rules out over-accumulation of capital

# Equilibrium/Constraint

- As an equilibrium condition for the market for the good (decentralized economy): supply = demand

$$Y = C + I$$

$I = \dot{K} + \delta\, K$ gross investment = net investment plus capital depreciation ($\delta > 0$ depreciation rate)

- Or can be seen as a constraint (centralized problem): origin = use of resources

$$Y = C + \dot{K} + \delta K$$

- In both cases we have the differential equation

$$\dot{K} = (A - \delta)\, K - C$$

# The model: centralized version

- The central planner determines the optimal paths $(C(t), K(t))_{t\in[0,\infty)}$ by solving the problem

$$\max_{[C(t)]_{t\geq 0}} \int_0^\infty \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{K} = AK - C - \delta K,$$

$$K(0) = k_0, \text{ given, } \text{ t} = 0$$

$$\lim_{t\to+\infty} e^{-At} K(t) \geq 0.$$

- Assumption: $A > \rho + \delta$

# Solving the model

Solution by the Pontriyagin's maximum principle

- The current-value Hamiltonian

$$H(C, K, Q) = \frac{C^{1-\theta} - 1}{1-\theta} + Q\Big(\big(A - \delta\big)\, K - C\Big)$$

- The first order optimality conditions

$$\frac{\partial H}{\partial C} = 0 \iff C^{-\theta} = Q$$

$$\dot{Q} = \rho\, Q - \frac{\partial H}{\partial K} \iff \dot{Q} = \big(\rho + \delta - A\big)\, Q$$

$$\lim_{t\to\infty} K(t)\, Q(t)\, e^{-\rho t} = 0$$

- the admissibility conditions

$$\dot{K} = (A-\delta)K - C,$$

$$K(0) = k_0, \;\; \text{given}, \;\; \text{t} = 0$$

# The MHDS

- The MHDS

$$\dot{C} = \frac{C}{\theta}(A - \rho - \delta) \tag{1}$$
$$\dot{K} = AK - C - \delta K, \tag{2}$$
$$\tag{3}$$

- initial and the transversality conditions

$$\lim_{t\to\infty} C(t)^{-\theta} K(t) e^{-\rho t} = 0 \tag{4}$$
$$K(0) = K_0, \text{ given} \tag{5}$$

# The MHDS: solution

- Balanced growth path:

$$K(t) = \bar{K}(t) = k_0 \, e^{\gamma t}, \; t \in [0, \infty)$$
$$C(t) = \bar{C}(t) = \beta \, k_0 \, e^{\gamma t}, \; t \in [0, \infty)$$

where

$$\beta = \frac{(A - \delta)(\theta - 1) + \rho}{\rho}$$

# Growth in the AK model: conclusions

- We determine the growth facts on $Y(t) = AK(t)$ from the solution of the MHDS
- which implies that the **efficient output solution** is

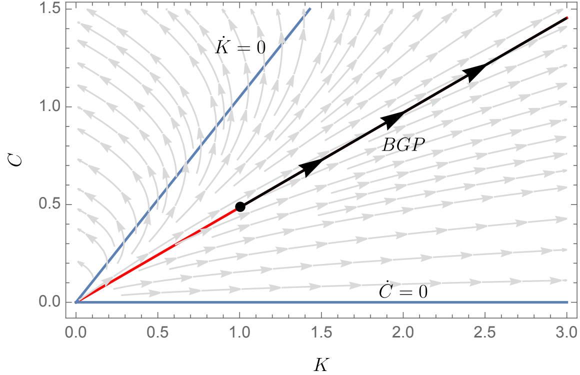
$$Y(t) = \bar{Y}(t) = Ak_0 e^{\gamma t},\ t \in [0, \infty)$$

- Conclusion (growth facts):
  1. the **endogenous long run rate of growth** is

$$\boxed{\bar{\gamma} = \frac{A - \delta - \rho}{\theta} > 0}$$

     (can be consistent with facts)
  2. the **long run level** is $\bar{y} = A\, k_0$ (factual)
  3. there is **no transitional dynamics** $\lambda = 0$ (counterfactual)

# Phase diagram



# Solution method

for endogenous growth models

How do we obtain the previous results ?

1. Write variables as: $X(t) = x(t)e^{\gamma_x t}$
   (level = detrended × trend)
2. Rewrite the MHDS for the detrended variables by introducing assumptions on the rates of growth (call it **detrended MHDS**) such that it is an autonomous ODE
3. Determine the long run growth rate ($\bar{\gamma}$) from the steady state of the detrended MHDS
4. Introduce the long run growth rate in the detrended MHDS and solve for the detrended variables, $k$ and $y = Ak$
5. Obtain the final solution for $K$ and, therefore, for $Y = AK$

# Step 1 : detrending variables

- Separate transition, $(k, c)$, and long-run trend $(e^{\gamma_k t}, e^{\gamma_c t})$ by writing
$$K(t) = k(t)e^{\gamma_k t}, \quad C(t) = c(t)e^{\gamma_c t},$$
- Then (because $c(t) = C(t)e^{-\gamma_c t}$ and $k(t) = K(t)e^{-\gamma_k t}$)

$$\frac{\dot{c}}{c} = \frac{\dot{C}}{C} - \gamma_c$$
$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \gamma_k$$

## Step 2 : building the detrended MHDS

- Substituting $\dot{C}/C$ and $\dot{K}/K$ we get

$$\frac{\dot{c}}{c} = \frac{A - \rho - \delta}{\theta} - \gamma_c$$
$$\frac{\dot{k}}{k} = A - \delta - \frac{c}{k} e^{(\gamma_c - \gamma_k)t} \gamma_k$$

- A necessary condition for the MHDS to be autonomous (time-independent) is $e^{(\gamma_c - \gamma_k)t} = 1$, which implies

$$\boxed{\gamma = \gamma_k = \gamma_c}$$

  The long run growth rates for the three variables is the same

- We call

$$\bar{K}(t) = \bar{k}\, e^{\gamma t}, \; \bar{C}(t) = \bar{c}\, e^{\gamma t}$$

  a **balanced growth path (BGP)**

# Step 2 : building the detrended MHDS, cont.

- Therefore, the **detrended MHDS** is

$$\dot{c} = c\left(\frac{A-\rho-\delta}{\theta} - \gamma\right)$$
$$\dot{k} = (A-\delta-\gamma)k - c$$

# Step 3 : the long-run growth rates

- Setting $\dot{c} = 0$ we get the long run growth rate

$$\bar{\gamma} = \frac{A - \delta - \rho}{\theta} > 0$$

- Setting $\dot{k} = 0$ we get the long run ratio

$$\frac{\bar{c}}{\bar{k}} = \beta,$$

where

$$\beta \equiv A - \delta - \bar{\gamma} = \frac{1}{\theta}\left((A - \delta)(\theta - 1) + \rho\right) > 0$$

# Step 4: solving the detrended MHDS

- if we substitute the rate of growth $\gamma = \bar{\gamma}$ in the detrended MHDS we have

$$\dot{c} = 0 \tag{6}$$
$$\dot{k} = \beta k - c \tag{7}$$
$$0 = \lim_{t \to +\infty} e^{-\beta t} k(t) c(t)^{-\theta} \tag{8}$$

because

$$\lim_{t \to +\infty} e^{-(\rho + \bar{\gamma}(\theta - 1))t} k(t) c(t)^{-\theta} = \lim_{t \to +\infty} e^{-\beta t} k(t) c(t)^{-\theta}$$

# Step 4: solving the detrended MHDS (cont.)

- the solution of equation (6) is an unknown constant

$$c(t) = c(0)$$

  where $c(0)$ is an arbitrary constant

- substituting $c$ and solving equation (7) we find

$$k(t) = \left(k_0 - \frac{c(0)}{\beta}\right) e^{\beta t} + \frac{c(0)}{\beta}.$$

- to determine $c(0)$ we substitute in the TVC

$$\begin{aligned}
\lim_{t \to +\infty} e^{-\beta t} \frac{k(t)}{c(t)^{\theta}} &= \lim_{t \to +\infty} e^{-\beta t} \left[\left(k_0 - \frac{c(0)}{\beta}\right) e^{\beta t} + \frac{c(0)}{\beta}\right] \frac{1}{c(0)^{\theta}} \\
&= \lim_{t \to +\infty} \left[k_0 - \frac{c(0)}{\beta}\right] \frac{1}{c(0)^{\theta}} \\
&= 0 \Leftrightarrow c(0) = \beta k_0
\end{aligned}$$

# Step 4: solving the detrended MHDS (cont.)

- Therefore the detrended consumption is

$$c(t) = \bar{c} = \beta\, k_0, \text{ for all } t \in [0, \infty)$$

- and the detrended capital stock is

$$k(t) = \bar{k} = \frac{c(0)}{\beta} = k_0 \text{ for all } t \in [0, \infty)$$

- This means that there is no transitional dynamics

# Step 5: the solution to the AK model

- The **balanced growth path** BGP is

$$\bar{K}(t) = \bar{k}e^{\gamma t}, \quad \bar{C}(t) = \bar{c}e^{\gamma t}.$$

- where $\gamma = \bar{\gamma}$ is determined from the steady state of the detrended MHDS
- the endogenous rate of growth is

$$\bar{\gamma} = \frac{A - \delta - \rho}{\theta} > 0$$

- we get additionally the ratio of the levels along the BGP

$$\bar{c} = \beta\bar{k}, \ \bar{k} = k_0$$

- Observe that there is an indeterminacy here: we have two equations ($\dot{c} = 0$ and $\dot{k} = 0$) and three variables ($\gamma, c, k$)). However, the value for $k$ is given at the initial level
- **this is a typical property of the endogenous growth models**.

# Discussion

- **Conclusion**: this model provides a **theory for the balanced growth path**.
- Differently from the Ramsey model:
  - it displays long run growth
  - but does not display transition (i.e., business cycle) dynamics
- applying to different countries, it provides a **theory for the long run trend in the growth rates**, provided that growth is only explained by capital accumulation:
  - growth depends **positively** on total factor **productivity** $A$ and on the elasticity of **intertemporal substitution** in consumption $(1/\theta)$
  - growth depends **negatively** on the rate of **time preference** $\rho$ and on capital depreciation $\delta$ (wear and tear of infrastructures)

# References

- (Acemoglu, 2009, ch.11.1)
- Sometimes, researchers call this the Rebelo (1991) model

Daron Acemoglu. *Introduction to Modern Economic Growth*. Princeton University Press, 2009.

Sérgio Rebelo. Long run policy analysis and long run growth. *Journal of Political Economy*, 99(3):500–21, 1991.