The Malthusian growth model

Paulo Brito
pbrito@iseg.ulisboa.pt

3.3.2021
Malthusian theory

- Popular definition of ”Malthusian economics”: population growth exponentially and food grows linearly
- This would lead either to catastrophe or to the existence of natural (not nice) stabilization mechanisms, in the absence of ”moral restraint”
- The idea there is an endogenous mechanism relating population and wages is consistent with events in pre-industrial W. Europe
Wages and population in historical data

Figure 5: Population and Real Wages in England, 1250-1750 CE
(Source: Clark, 2005)
Malthusian theory

- We will see that the existence of marginal decreasing returns to labor is a necessary (although not sufficient) condition.
- The idea that the existence of a fixed resources and decreasing returns to production implies that growth processes eventually stop is present in most Classical economists (Quesnay, Smith, Ricardo, Marx) and, possibly, in modern ecologists.
- But it was Thomas Malthus who stated it more clearly in An Essay on the Principle of Population (1798) and systematically gathered data to sustain it.
- We next provide a modern view of the theory
Malthusian model

The general idea:

- It presents the joint dynamics of production and population growth
- In pre-industrial societies: there are two main factors of production labor and land
- Labor is the reproducible factor of production (no capital accumulation, no R&D)
- The basic dynamic mechanism is: increase in income leads to increase in population and in labor supply; this increases aggregate income, but income per capita does not increase at the same pace, leading eventually to a steady state (positive extensive effect but negative intensive effect).
- **Decreasing marginal returns for the reproducible factor** is the main driving force behind the non-existence of growth in the long run.
- The conditions for the existence of long run growth are very specific (learning-by-doing)
Assumptions

▶ Production:
  ▶ production uses two factors: labor and land
  ▶ the production function has constant returns to scale
  ▶ the only reproducible factor is labor, and it faces decreasing marginal returns

▶ Population:
  ▶ fertility is endogenous and mortality is exogenous

▶ Farmers:
  ▶ households are land-owners
  ▶ they choose among consumption and child-rearing
  ▶ there are no savings
$v(y(t)) \rightarrow c(t)$

$v(y) = \max \{ u(c, b) : c + pb \leq y \}$ and $y = \frac{Y}{L}$

where
The model
Production

- **Production function**

  \[ Y(t) = (AX)^\alpha L(t)^{1-\alpha}, \ 0 < \alpha < 1 \]

  where: \( A \) productivity, \( X \) stock of land, \( L \) labor input

- displays constant returns to scale

  \[ (\lambda AX)^\alpha (\lambda L)^{1-\alpha} = \lambda Y \]

- implication: the Euler theorem holds

  \[ Y = \frac{\partial Y}{\partial L} L + \frac{\partial Y}{\partial X} X \]
The model
Production technology

- positive marginal returns for labor and land

\[
\frac{\partial Y}{\partial L} = (1 - \alpha) \frac{Y}{L} > 0, \quad \frac{\partial Y}{\partial X} = \alpha \frac{Y}{X} > 0
\]

- Inada production function: \( \lim_{L \to 0} \frac{\partial Y}{\partial L} = \infty \) and \( \lim_{L \to \infty} \frac{\partial Y}{\partial L} = 0 \)

- No bias in technical change (why?)

\[
MRS_{L,X} = \frac{(1 - \alpha) X}{\alpha L}
\]
The model

Inada property regarding the marginal productivity of labor
The model

Production technology

- Decreasing marginal returns

\[
\frac{\partial^2 Y}{\partial L^2} = -\alpha(1 - \alpha) \frac{Y}{L^2} < 0, \quad \frac{\partial^2 Y}{\partial X^2} = -\alpha(1 - \alpha) \frac{Y}{X^2} < 0
\]

- The two factors are Edgeworth complements

\[
\frac{\partial^2 Y}{\partial X \partial L} = \alpha(1 - \alpha) \frac{Y}{LX} > 0
\]

- The technology is linear

\[
\frac{\partial^2 Y}{\partial L^2} \frac{\partial^2 Y}{\partial X^2} - \left(\frac{\partial^2 Y}{\partial X \partial L}\right)^2 = 0
\]

- The AU elasticities are

\[
\varepsilon_{LL} = \alpha, \quad \varepsilon_{XX} = 1 - \alpha, \quad \varepsilon_{LX} = -\alpha
\]

- We already known that the elasticity of substitution is equal to one:

\[
ES_{LX} = 1
\]
The model
Production efficiency

- production efficiency:

$$\max_{L,X} \{ Y(L, X) - wL - RX \}$$

where $w$ is the wage rate and $R$ are land rent

- and competitive markets lead to

$$w(L, X) = \frac{\partial Y}{\partial L} = (1 - \alpha) \frac{Y}{L} > 0$$
$$R(L, X) = \frac{\partial Y}{\partial X} = \alpha \frac{Y}{X} > 0$$

- Factors are Hicksian substitutables

$$w_L < 0, w_X > 0$$ wages decrease with labor and increase with land;
$$R_X < 0, R_L > 0$$ rents decrease with land and increase with labor.
Farmers’ problem
Endogenous rate of population growth

- There are \( L \) farmers; who receive (per capita) income from farming and decide which part to consume and which part to allocate to raising offspring, by deciding the number of offspring (Beckerian model)

- **Household’s (farmer’s) problem**

\[
\max_{c(t), b(t)} \{ c(t)^{1-\psi} b(t)^\psi : \ c(t) + pb(t) = y(t) \}
\]

\( 0 < \psi < 1 \) (relative) love for children, \( 1/\psi = ”moral\ restraint” \)
\( p > 0 \) relative cost of raising children

- **solution**

\[
c(t) = (1 - \psi)y(t) \text{ (consumption increases with income)}
\]
\[
b(t) = \frac{\psi}{p} y(t) \text{ (number of children increases with income)}
\]
Population dynamics

- Population growth

\[ \dot{L} \equiv \frac{dL(t)}{dt} = (b(t) - m)L(t) \]

- where the fertility rate is endogenous: \( b(t) = \frac{\psi}{p} y(t) \)
- the mortality rate is exogenous: \( m \) is given
- the initial level of population is assumed to be given by number \( L_0 \)

\[ L(t)|_{t=0} = L(0) = L_0 \]
The Malthusian model

Endogenous rate of population growth

Then

\[
\dot{L} = \left( \frac{\psi}{p} y(t) - m \right) L(t), \text{ for } t \in [0, \infty) \\
L(0) = L_0 \text{ given}
\]

where the per capita GDP is

\[
y(t) \equiv \frac{Y(t)}{L(t)} = \left( \frac{AX}{L(t)} \right)^\alpha
\]
Detour
Per-capita rate of growth arithmetics

- taking log-derivatives w.r.t time we have

\[ \frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} \]

- that we denote by

\[ g(t) = g_Y(t) - n(t) \]

- as the per capita GDP is

\[ y(t) \equiv \frac{Y(t)}{L(t)} = \left( \frac{AX}{L(t)} \right)^\alpha \]

- Then: the rate of growth is exactly negatively correlated to the rate of growth of population

\[ g(t) = \frac{\dot{y}}{y} = -\alpha \frac{\dot{L}}{L} \]
Take home

Remember:

- We are interested in what this model can tell us about economic growth
- For us growth is related to the dynamics of GDP per capita

\[ y(t) = \frac{Y(t)}{L(t)} \]

- we want to know the implications for:
  - the rate of growth of GDP \( g(t) = \frac{\dot{y}(t)}{y(t)} \)
  - the steady state level of GDP \( \bar{y} \)
  - and the dynamics: i.e. separating \( g(t) \) into transition and long-run components
Solving the Malthusian model

There are two approaches to solving the model

- Approach 0: get a geometric representation of the solution of the model (phase diagram)
- Approach 1: solve the differential equation for $L$ and substitute in $y$ to get the dynamics of growth
- Approach 2: obtain a differential equation for $y$ and solve it
The model

The phase diagram: geometric intuition
Solving the Malthusian model

Approach 1: solving for $L$

If we substitute $y$ in the dynamic equation for $L$ we have the initial value problem

$$\begin{cases} \dot{L} = \left( \frac{\psi}{p} \left( \frac{AX}{L(t)} \right)^\alpha - m \right) L(t), & t \geq 0 \\ L(0) = L_0 \text{ given} & t = 0 \end{cases}$$

we can solve it

then differentiate it and substitute in

$$g(t) = -\alpha \frac{\dot{L}}{L}$$

to get model’s explanation for the per-capita growth of the economy
Solving the Malthusian model

Approach 2: solving for $y$ directly

- from

$$\frac{iy}{y} = -\alpha \frac{\dot{L}}{L}$$

- we get the dynamic equation for the GDP per capita

$$i(y) = -\alpha \left( \frac{\psi}{p} y(t) - m \right) y(t)$$

(1)

together with the initial value

$$y(0) = y_0 = (AX)\alpha L_0^{1-\alpha}$$
Solving the Malthusian model

Explicit solution for $y$

- Equation (1) has two steady states $y^* = \{0, \bar{y}\}$ where

$$\bar{y} = \frac{mp}{\psi}$$

- we can re-write the growth equation as

$$\dot{y} = \alpha \frac{\psi}{p} (\bar{y} - y(t)) y(t)$$
Explicit solution for $y$

- This is a Bernoulli differential equation with has an explicit solution

$$y(t) = \left[ \frac{1}{\bar{y}} + \left( \frac{1}{y(0)} - \frac{1}{\bar{y}} \right) e^{-\alpha mt} \right]^{-1}, \text{ for } 0 \leq t < \infty$$

- satisfies $\lim_{t \to \infty} y(t) = \bar{y}$
Explicit solution for $g$

$g(t) = \frac{dy(t)}{dt} = \alpha m \left[ 1 - \left( 1 + \left( \frac{y}{y(0)} - 1 \right) e^{-\alpha mt} \right)^{-1} \right]$, for $0 \leq t < \infty$
Malthusian model

Properties

1. there is no long run growth, because \( \lim_{t \to \infty} g(t) = 0 \)
2. the long run level of GDP per capita is

\[
\bar{y} = \frac{mp}{\psi}
\]

increases with the mortality rate, the cost or rearing children and the "moral restraint" (no productivity effects)

3. there is only transitional dynamics (i.e., adjustments towards the steady state):
   - if the initial GDP is small, \( y(0) < \bar{y} \), then there is an increase in time of the GDP \( g(t) > 0 \)
   - if the initial GDP \( y(0) \) is large, \( y(0) > \bar{y} \), then there is an decrease in time of the GDP \( g(t) < 0 \)
Malthusian model
Mechanics of the model

- if $y(0)$ is large so is the wage rate $w(0) = (1 - \alpha) y(0)$
- this implies that the initial fertility rate is higher, $b(0) = \frac{\psi}{p} y(0)$
- population increases, which increases output,
- but decreases the rate of growth of GDP

$$g(t) = -\alpha n(t)$$

because there are decreasing marginal returns due to the fact that $X$ is fixed.
Malthusian model
Trajectories: $y$, $L$ and $w$

Figure: Parameter values: $\alpha = 2/3$, $m = 0.03$, $\psi = 0.01$, $p = 10$, $A = 1$, $X = 100$, and $y(0) < \bar{y}$
Malthusian model
Exponential increase in land productivity

Can increases in land-productivity generate long-run growth?

- now $Y(t) = (A(t)X)^\alpha L(t)^{1-\alpha}$
- where $\dot{A} = g_A A$, $g_A > 0$
- Taking logarithmic derivatives of the production function, this implies

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{A}}{A} + (1 - \alpha) \frac{\dot{L}}{L} - \frac{\dot{L}}{L} = \alpha \left( m + g_A - \frac{\psi}{p} y(t) \right)$$

- there is no increase in the long run growth rate (why ?)

$$\lim_{t \to \infty} g(t) = 0$$

- there is an increase in GDP level

$$\bar{y} = \frac{(g_A + m)p}{\psi}$$
Malthusian model and land productivity

Phase diagram for an increase in $A$
Malthusian model
Exponential increase in labor productivity

Can an increase in the productivity of labor generate long-run growth?

- now \( Y(t) = X^\alpha(h(t) L(t))^{1-\alpha} \)
- where \( \dot{h} = g_h h, \ g_h > 0 \)
- Taking logarithmic derivatives of the production function, this implies

\[
\frac{\dot{y}}{y} = (1 - \alpha) \left( \frac{\dot{h}}{h} + \frac{\dot{L}}{L} \right) - \frac{\dot{L}}{L} = \alpha \left[ \frac{1 - \alpha}{\alpha} g_h + m - \frac{\psi}{p} y(t) \right] y(t)
\]

- there is no increase in the long run growth rate (why?)

\[
\lim_{t \to \infty} g(t) = 0
\]

- there is an increase in GDP level

\[
\bar{y} = \frac{((1 - \alpha)g_h + \alpha m)p}{\alpha \psi}
\]
Malthusian model
Learning by doing

Can learning by doing generate long-run growth?

- now \( Y(t) = (A(t)X)\alpha L(t)^{1-\alpha} \)
- learning-by-doing: past production generates knowledge which increases land productivity
- Formally: \( A(t) = \beta \int_{-\infty}^{t} e^{-\mu(t-s)} A(s) y(s) ds \)
  where \( \beta \) reproduction of knowledge, \( \mu \) rate of oblivion
- taking derivatives at regards \( t \) (Leibniz formula)

\[
\dot{A} = (\beta y(t) - \mu) A(t)
\]

- the dynamic equation for per-capita GDP becomes

\[
\frac{\dot{y}}{y} = \left( \beta - \alpha \frac{\psi}{p} \right) y(t) + \alpha m - \mu
\]
Malthusian model
Learning by doing: continuation

▶ If we assume $\beta = \alpha \frac{\psi}{p} (\varepsilon_{LL} \times \frac{b}{y})$ then

$$\dot{y} = (\alpha m - \mu) y$$

▶ there is long run growth if $\alpha m > \mu$ because

$$g(t) = \alpha m - \mu > 0, \text{ for all } t > 0$$

▶ the GDP level is exogenous

$$y(t) = y_0 e^{(\alpha m - \mu)t}$$
Conclusions

- the existence of decreasing marginal returns to the reproducible factor of production (labor, $L$) implies that the Malthusian model does not feature long-run growth: there is only transitional dynamics (if initial population is too high, wages will be too low, which generates a fall in fertility and therefore a decrease in population until population is constant)

- **exogenous permanent increases** in productivity will only increase the long-run GDP **level** but will not generate long-run growth

- however, **endogenous** increases in productivity (v.g, generated by learning-by-doing) **may** generate long run growth (but in this case there is not transition dynamics). Learning-by-doing generates a reproduction mechanism.
Questions

▶ Are those conclusions robust to changes in the preferences between consumption and fertility?
▶ Are those conclusions robust to changes in the technology? In particular are they robust to the existence of biased technical change?
▶ Solving the problem set may provide answers to those questions.
References

- Original work: Malthus (1798)
- Textbook: (Galor, 2011, ch 2, 3)


Appendix
Solving a linear ODE’s

- The linear ODE
  \[ \dot{x} = \lambda (x(t) - \bar{x}) \]
  has an exact solution
  \[ x(t) = \bar{x} + (x(0) - \bar{x}) e^{\lambda t} \]
  where \( k \) is an arbitrary constant

- The initial value problem
  \[ \begin{cases} 
  \dot{x} = \lambda (x(t) - \bar{x}) \\
  x(0) = x_0 \text{ given} 
  \end{cases} \]
  has the exact solution
  \[ x(t) = \bar{x} + (x_0 - \bar{x}) e^{\lambda t} \]
Appendix

The linear and Bernoulli ODE’s

▶ The Bernoulli equation is

\[ \dot{x} = \alpha x(t) - \beta x^n \]

▶ If we set \( z(t) = x(t)^{1-n} \) and differentiate

\[ \dot{z} = (1 - \eta) x(t)^{-n} \dot{x} = \]

\[ = (1 - \eta) x(t)^{-n} (\alpha x(t) - \beta x^n) = \]

\[ = \lambda (z(t) - \bar{z}) \]

▶ is a linear ODE with solution with \( \lambda = (1 - \eta)\alpha \) and \( \bar{z} = \frac{\beta}{\alpha} \)

\[ z(t) = \bar{z} + (z(0) - \bar{z}) e^{\alpha(1-\eta)t} \]

▶ transforming back by making \( x(t) = z(t)^{\frac{1}{1-n}} \)

\[ x(t) = \left( \frac{\beta}{\alpha} + \left( x(0)^{1-\eta} - \frac{\beta}{\alpha} \right) e^{\alpha(1-\eta)t} \right)^{\frac{1}{1-n}} \]