The Solow growth model

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Background

- Most European nations were industrialized in the dawn of the XX century, and the main driver of growth was the accumulation of capital (both physical and financial)
- After the IIWW: definition of the ideia of the GDP and first Statistics Agencies to measure it (see a nice history of the concept Coyle (2014)
- Some ideia of the stylized facts (covering a short time span) appeared: v.g. Kaldor's stylized facts
- The Solow (1956) paper tried to explain those facts
- When the "Keynesian" model was the state of the art
- Most of the economic growth theory and empirics has this models as a reference point.
- He was awarded the Nobel Prize in 1987

Kaldor's stylized facts (1963)

Fact K1 per capita GDP (y) grows along time, and its rate of growth shows no decreasing tendency (debatable: for mature countries);

Fact K2 K grows along time;

Fact K3 r (r.o.r of capital) is roughly constant (debatable: it shows a slightly downward tendency for most developing countries);

Fact K4 K/Y is roughly constant;

Fact K5 the shares of capital and labor in the aggregate income are approximately constant (debatable: this is not the case after the early 1980's) ;

Fact K6 the growth of Y (p.c.) varies substantially between countries.

Solow (1956) model

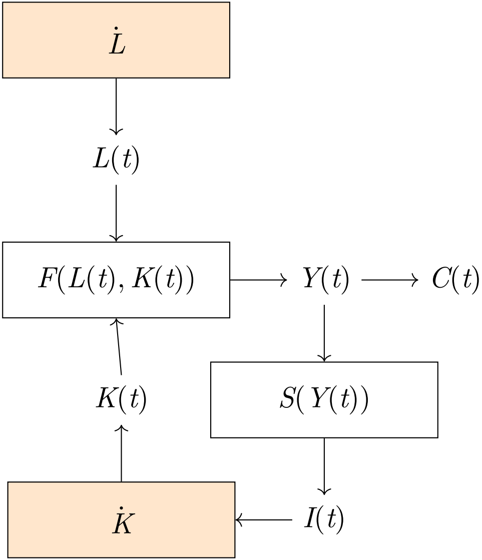
Structure of the economy

- Environment:
  - closed economy producing a single composite good
  - there is only one reproducible factor: capital
  - there are no idle factors (no unemployment)
- Population:
  - exogenous
- Growth engine: capital accumulation

Solow (1956) model

Assumptions

- Production:
  - production uses two factors: labor and physical capital
  - production technology: neoclassical (increasing, concave, Inada, CRS)
- Households: add-hoc behaviour
  - inelastically supply labor
  - ad-hoc savings proportional to income
  - static expectations (no anticipations)
- There is macroeconomic consistency (market clearing), but not necessarily microeconomic consistency (decisions on labor supply, consumption and finance are disconnected)


$\dot{L}$
$L(t)$
$F(L(t), K(t))$
$Y(t)$
$C(t)$
$K(t)$
$S(Y(t))$
$\dot{K}$
$I(t)$

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The model: production technology

- **Neo-classical production function**

$$Y(t) = F(A, K(t), L(t)) = AK(t)^{\alpha}L(t)^{1-\alpha},\ 0 < \alpha < 1$$

  where: A productivity, K stock of capital, L labor input
- properties
  - constant returns to scale
  - increasing in both factors: $\nabla F(K, L) = (F_K, F_L)^{\top} > \mathbf{0}$
  - concave in (K, L)
  - Inada

$$\lim_{K \to 0} F_K(K, L) = \lim_{L \to 0} F_K(K, L) = +\infty$$

$$\lim_{K \to \infty} F_K(K, L) = \lim_{L \to \infty} F_K(K, L) = 0$$

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The model: factor demand and distribution

- Inverse factor demand functions
  - the demand K is such that the rate of return of capital equals the marginal productivity of capital

$$r(t) = F_K(K, L) = \alpha \frac{Y(t)}{K(t)}$$

  - the demand L is such that the wage rate equals the marginal productivity of labor

$$w(t) = F_L(K, L) = (1 - \alpha) \frac{Y(t)}{L(t)}$$

- from CRS and Euler's theorem the distribution of income is

$$Y(t) = r(t)K(t) + w(t)L(t)$$

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The model: factor dynamics

- **Population growth**

$$\dot{N}(t) = nN(t)$$

n is the exogenous rate of growth

- **No unemployment** (or demand and supply of labor)

$$L(t) = N(t)$$

- **Capital accumulation**

$$\dot{K} = I(t) - \delta K(t)$$

net investment =gorss investment - capital depreciation

$\delta > 0$ rate of depreciation of capital

Solow model: labour market

Consumption and investment

- **"Keynesian" consumption function**

$$C(t) = (1 - s)\,Y(t)$$

  $0 < s < 1$ is the marginal propensity to consume

- **savings decisions**

$$S(t) = s\,Y(t)$$

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Macroeconomic equilibrium

- **Equilibrium in the product market**

$$Y(t) = C(t) + I(t)$$

  aggregate supply = aggregate demand

- By Walras's law we could "close the model" by the **equilibrium in the capital market**

$$S(t) = I(t)$$

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GDP per capita

- The per capita GDP is

$$y(t) \equiv \frac{Y(t)}{N(t)}$$

- taking log-derivatives w.r.t time we have

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{N}}{N} \Leftrightarrow g(t) = g_Y(t) - n(t)$$

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The model: the rate of growth

- The per capita GDP is

$$y(t) \equiv \frac{Y(t)}{N(t)} = A\left(\frac{K(t)}{N(t)}\right)^{\alpha} = Ak(t)^{\alpha}$$

  defining the capital intensity by

$$k \equiv \frac{K}{L} = \frac{K}{N}$$

- Then

$$\boxed{g(t) = \frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} = \alpha g_k(t)}$$

- the rate of growth is a linear function of the rate of growth of the capital intensity
- but the ratio between the two is less than one

$$\frac{g(y)}{g_k(t)} = \alpha \in (0, 1)$$

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The capital accumulation equation

- the dynamic equations of the model are

$$\begin{cases} \dot{K} & = sAK^{\alpha}N^{1-\alpha} - \delta K \\ \dot{N} & = nN \end{cases}$$

- using the definition of capital intensity, k, we obtain

$$\boxed{\begin{cases} \dot{k} = sAk^{\alpha} - (n+\delta)k & t \geq 0 \\ k(0) = k_0, & t = 0 \end{cases}}$$

- Then the dynamics for per capita GDP is given by

$$\boxed{\begin{cases} \dot{y} = \alpha \left(sA^{\frac{1}{\alpha}} y^{1-\frac{1}{\alpha}}(t) - (n+\delta) \right) y(t) & t > 0 \\ y(0) = y_0 = Ak_0^{\alpha}, & t = 0 \end{cases}}$$

- We can solve the model for k or for y

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Solving the model

There are **several approaches** to solve the model, i.e, finding the relationship of k (or y) with time

- We can solve it by linearization in the neighborhood of the steady state(s)
- Sometimes, we can solve it explicitly (which is the case in this model)
- We can solve it numerically
- It is always a good idea to have a geometrical illustration of the model (if it has a low dimension)

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Solving for k by linearization

- Write the Solow accumulation equation as

$$\dot{k} = G(k) = s\,A\,k^{\alpha} - (n+\delta)k$$

- We start by determining the **steady state(s)**:
$k^* = \{k \geq 0 : G(k) = 0\} = \{0, \bar{k}\}$ where

$$\bar{k} = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- We consider the positive steady state $\bar{k}$, and take the variations $\Delta k(t) = k(t) - \bar{k}$
- We performing a first-order Taylor approximation in the neighborhood of $\bar{k}$

$$\frac{d\Delta k(t)}{dt} = \frac{dG}{dk}(\bar{k})\,\Delta k(t)$$

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Solving for k by linearization

- The approximated (linearized) capital accumulation equation is

$$\dot{k} = \lambda\,(k(t) - \bar{k})$$

where the coefficient is

$$\lambda = \frac{dG}{dk}(\bar{k}) = \alpha\, s\, A\, \bar{k}^{\alpha-1} - (n+\delta) = -(1-\alpha)\,(n+\delta) < 0$$

- Given $k(0) = k_0$ known **the approximate solution** is

$$\boxed{k(t) = \bar{k} + \left(k_0 - \bar{k}\right) e^{\lambda\, t},\ \text{for } t \in [0, \infty)}$$

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Explicit solution for k

- The explicit (exact) solution is proof

$$\boxed{k(t) = \left[\bar{k}^{1-\alpha} + \left(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\right) e^{\lambda t}\right]^{\frac{1}{1-\alpha}}, \; t \in [0, \infty)}$$

where

$$\lambda \equiv -(1-\alpha)(n+\delta) < 0$$

- The growth rate of the capital intensity is

$$g_k(t) = -(n+\delta)\left(\frac{\left(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\right) e^{\lambda t}}{\bar{k}^{1-\alpha} + \left(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\right) e^{\lambda t}}\right)$$

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Properties of the solution

- The solution is continuous in k_0

$$k(0) = k(t|t=0) = k_0$$

- If $k_0 > 0$, $k(t)$ converges asymptoticaly to $\bar{k}$

$$\lim_{t \to \infty} k(t) = \bar{k}$$

  independently of the initial value k_0.

- Equivalently

$$\boxed{\lim_{t \to \infty} g_k(t) = 0 \text{ because } \lim_{t \to \infty} e^{\lambda t} = 0}$$

  Intuition: there is no long run growth

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Mechanics of the model

- We can write Solow's equation as

$$g_k(t) = \frac{\dot{k}}{k} = \frac{s}{\alpha} r(k(t)) - (n + \delta)$$

- low $k(0)$ means $r(0)$ is high relative to $n + \delta$
- this implies high incentive for saving and for accumulating capital
- but capital accumulation decreases the marginal productivity of capital because $r_k(k) = \frac{\partial r(k)}{\partial k} < 0$, which reduce progressively the incentives to accumulate capital
- which stops asymptotically the incentives to accumulate capital
- notice that in the long run capital increases just to cover $(n + \delta)$

Solow model

Mechanics

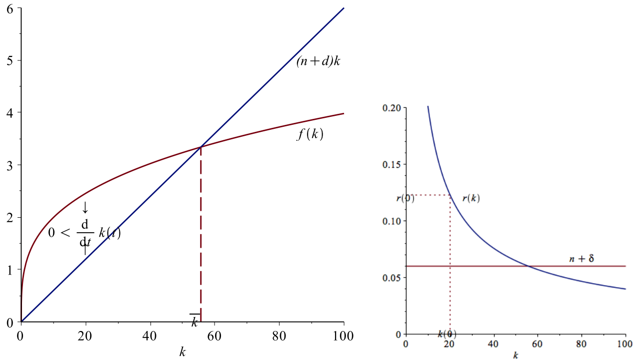

Figure: If $k(0) < \bar{k}$ ($k(0) > \bar{k}$) then capital will increase (decrease) and converge to $\bar{k}$ asymptotically

Solow model

Explicit solution for y

- Because $y(t) = Ak(t)^{\alpha}$ and

$$\bar{y} = A\bar{k}^{\alpha} = A\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

- then the GDP per capita varies along time according to

$$\boxed{y(t) = \left[\bar{y}^{\frac{1-\alpha}{\alpha}} + \left(y_0^{\frac{1-\alpha}{\alpha}} - \bar{y}^{\frac{1-\alpha}{\alpha}}\right) e^{\lambda t}\right]^{\frac{\alpha}{1-\alpha}},\ t \in [0, \infty)}$$

where

$$\lambda \equiv -(1-\alpha)(n+\delta) < 0$$

Solow model

Implications for growth

The **implication for growth** are:

- there is no long run growth, if $y(0) = y_0 > 0$ then

$$\lim_{t \to \infty} y(t) = \bar{y} \Rightarrow \lim_{t \to \infty} g(t) = 0$$

- the long run level of GDP per capita : increases with A, and s and decreases with n and δ
- only transitional dynamics exists, driven by $\lambda = -(1-\alpha)(n+\delta)$, i.e. it is due to the existence of **decreasing marginal returns to the accumulating factor** k

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Criticisms

1. A zero long-run rate of growth is **counterfactual** for industrialised economies since the Industrial Revolution
2. capital accumulation can display **dynamic inefficiency**, i.e $\bar{k} > k^{\mathrm{gr}}$ where

$$k^{\mathrm{gr}} = \mathrm{argmax}_k\{c(k) = Ak^{\alpha} - (n+\delta)k\} = \left(\frac{\alpha A}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

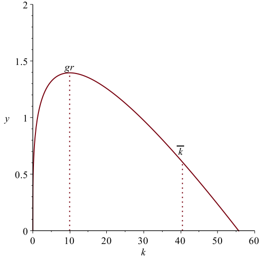

Figure: The golden rule and the steady state $\bar{k}$

Solow model

Extension: exogenous productivity growth

- Consider the production function

$$Y(t) = A(t)K(t)^{\alpha}L(t)^{1-\alpha},\ 0 < \alpha < 1$$

- and that there is exogenous TFP growth

$$A(t) = A_0 e^{g_A t},\ g_A > 0$$

- What are the growth consequences ?

Solow model

Extension: exogenous productivity growth

- Because

$$y(t) = A(t)k(t)^{\alpha}$$

- then

$$g(t) = g_A + \alpha g_k(t)$$

- as $\lim_{t\to\infty} g_k(t) = 0$ then

$$\lim_{t\to\infty} g(t) = g_A > 0$$

- There is long run growth but only of an **exogenous** nature: this describes but does not explain.

References

- Solow (1956)
- (Acemoglu, 2009, ch. 2 and 3) , (Aghion and Howitt, 2009, ch. 1), (Barro and Sala-i-Martin, 2004, ch. 1)

Daron Acemoglu. *Introduction to Modern Economic Growth*. Princeton University Press, 2009.

Philippe Aghion and Peter Howitt. *The Economics of Growth*. MIT Press, 2009.

Robert J. Barro and Xavier Sala-i-Martin. *Economic Growth*. MIT Press, 2nd edition, 2004.

Diane Coyle. *GDP: A Brief but Affectionate History*. Princeton University Press, 2014.

Robert Solow. A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70(1):65–94, 1956.

Appendix

Explicit solution of the Solow model

- We can re-write the capital accumulation equation as

$$\dot{k} = (n+\delta)\left(\left(\frac{k}{\bar{k}}\right)^{\alpha-1} - 1\right) k$$

- use the transformation $z(t) = \left(\frac{k(t)}{\bar{k}}\right)^{1-\alpha}$
- then

$$\begin{aligned}\dot{z} &= (1-\alpha) z \frac{\dot{k}}{k} = \\ &= (1-\alpha)(n+\delta)\left(\frac{1}{z} - 1\right) z\end{aligned}$$

- then we get the equivalent ODE

$$\dot{z} = (1-\alpha)(n+\delta)\,(1-z)\,.$$

Appendix

Continuation

- The ODE

$$\dot{z} = (1-\alpha)(n+\delta)\,(1-z)$$

- has the solution

$$z(t) = 1 + (z(0) - 1)\,e^{-(1-\alpha)(n+\delta)t}$$

- then, transforming back, $k(t) = z(t)^{\frac{1}{1-\alpha}}\,\bar{k}$, we get

$$k(t) = \bar{k}\left[1 + \left(\left(\frac{k(0)}{\bar{k}}\right)^{1-\alpha} - 1\right) e^{-(1-\alpha)(n+\delta)t}\right]^{\frac{1}{1-\alpha}}$$

main.

Daron Acemoglu. *Introduction to Modern Economic Growth*. Princeton University Press, 2009.

Philippe Aghion and Peter Howitt. *The Economics of Growth*. MIT Press, 2009.

Robert J. Barro and Xavier Sala-i-Martin. *Economic Growth*. MIT Press, 2nd edition, 2004.

Diane Coyle. *GDP: A Brief but Affectionate History*. Princeton University Press, 2014.

Robert Solow. A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70(1):65–94, 1956.