Toolkit for economic growth

Paulo Brito
pbrito@iseg.ulisboa.pt

24.2.2021
(revised 25.2.2021)

Topics

1. Production functions: properties
2. Production functions and input incomes
3. Production functions and types of inputs
4. Production functions and technology bias
5. Extensions: continuum of inputs, multisector economies
6. Production function and input accumulation and dynamics
7. Types of growth dynamics and the production function

Production function: properties

Production function

- Production function

$$y = F(\mathbf{x}) \equiv F(x_1, \ldots, x_n)$$

- $y =$ output of one good (can be used in consumption and/or investment)
- $\mathbf{x} = (x_1, \ldots, x_n)$ bundle of inputs, vector assuming a discrete index set
- x_i input of good i in production (intermediate good or final good)
- $F(\cdot)$ production function formalizes the **technology**: properties of the transformation of inputs into outputs
- relevant properties:
  - general properties: increasing, concave and homogeneous
  - at the input level: necessity, marginal variation, substitutability/complementarity

Production: general properties

- Definition: F **is weakly increasing** if $\mathbf{x}^* \geq \mathbf{x} \Rightarrow F(\mathbf{x}^*) \geq F(\mathbf{x})$
- Meaning: Increase in the quantity of inputs leads to an increase in production (bigger bundle increases output)
- Derivative:

$$\frac{\partial F(\mathbf{x})}{\partial x_i} = \lim_{\epsilon \to 0} \frac{F(\ldots, x_i + \epsilon, \ldots) - F(\ldots, x_i, \ldots)}{\epsilon}$$

- Gradient: vector of first derivatives

$$DF(\mathbf{x}) = \Big(\frac{\partial F(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial F(\mathbf{x})}{\partial x_n}\Big)^\top$$

- if $DF(\mathbf{x}) \geq 0$ then F is weakly increasing

Production: general properties

- Definition: F **is concave** if given any two bundles $\mathbf{x}$ and $\mathbf{x}^*$

$$F(\mathbf{x}) - F(\mathbf{x}^*) \leq +DF(\mathbf{x}^*) \cdot (\mathbf{x} - \mathbf{x}^*)$$

  (strictly concave if $<$)

- Meaning: increases in the quantity if inputs increases production less than linearly
- Hessian: matrix of second derivatives

$$D^2F(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 F(\mathbf{x})}{\partial x_1^2} & \dots & \frac{\partial^2 F(\mathbf{x})}{\partial x_1 \partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial^2 F(\mathbf{x})}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 F(\mathbf{x})}{\partial x_n^2} \end{pmatrix}$$

- If $F(\mathbf{x})$ is concave then $D^2F(\mathbf{x})$ is negative semi-definite.
- $D^2F(\mathbf{x})$ is negative semi-definite if the principal minors of odd order are negative and the principal minors of even order are positive

Production: general properties

- Definition: F **is homogeneous of degree** η if changing $\mathbf{x}$ to a input of λ, where λ is a positive number changes output by a input of λ^η

$$\lambda^\eta F(\mathbf{x}) - F(\lambda \mathbf{x}^*)$$

  where $\lambda \mathbf{x}^* = (\lambda x_1, \dots \lambda x_n)^\top$

- η measures the returns to scale
- Therefore, we say the production function displays
  - decreasing returns to scale if $\eta < 1$
  - constant returns to scale if $\eta = 1$
  - increasing returns to scale if $\eta > 1$

- Euler theorem,

$$\eta F(\mathbf{x}) = DF(\mathbf{x}) \cdot \mathbf{x} = \sum_{i=1}^{n} \frac{\partial F(\mathbf{x})}{\partial x_i} \, dx_i$$

- A fundamental requirement for the existence of growth is that $F(\cdot)$ is linearly homogeneous

Production function: specific properties

- Necessity: input x_i is **necessary** if $x_i = 0 \Rightarrow f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots x_n) = 0$ (zero output if it is not used)
- Variation in production: measured by the differential

$$dy = DF(\mathbf{x}) \cdot \mathbf{x} = \sum_{i=1}^{n} F_i(\mathbf{x})\, dx_i$$

- Marginal product (or productivity):

$$MP_i = F_i = \frac{\partial F(\mathbf{x})}{\partial x_i}$$

  meaning: variation in production if input i in increased by one unit

$$dy = \sum_{i=1}^{n} F_i(\mathbf{x})\, dx_i = F_i, \text{if } d\mathbf{x} = (0, \ldots, 0, 1, 0, \ldots, 0, \ldots 0)$$

- we say input i is **productive** if $F_i > 0$.

Production function: specific properties

- If $F(\mathbf{x})$ is strictly increasing then all inputs are productive.
- **Uzawa property** if $MP_i \in (0, \infty)$

$$\lim_{x_i \to 0} F_i(\mathbf{x}) = +\infty, \text{ and } \lim_{x_i \to \infty} F_i(\mathbf{x}) = 0$$

- A technology is non-Uzawa if there are bounds (superior or inferior) in the MP of any input (or $MP =$ constant)

Production function: technology

- Change in the marginal product:

$$MP_{ij} = F_{ij} = \frac{\partial^2 F(\mathbf{x})}{\partial x_i \, \partial j}, \text{ for any pair, } i.j = 1, \ldots, n$$

- The marginal product for input i, MP_i is
  - decreasing if $MP_{ii} < 0$
  - constant if $MP_{ii} = 0$
  - increasing if $MP_{ii} > 0$
- If the technology is concave (among all other conditions) we have:

$$F_{ii}(\mathbf{x}) \leq 0 \text{ for all } i = 1, \ldots, n \text{ (decreasing MP)}$$

$$F_{ii}(\mathbf{x}) \, F_{jj}(\mathbf{x}) - F_{ij}(\mathbf{x})^2 \geq 0, \text{ for all pairs } i \neq j = 1, \ldots, n$$

- If a technology is concave then the MP_i, for all i, are non-increasing.

Production function: technology

- Allen-Uzawa elasticities

$$\text{(own elasticities) } \epsilon_{ii} = -\frac{F_{ij}(\mathbf{x})x_j}{F_i(\mathbf{x})},\ \ i = 1, \ldots, n$$

$$\text{(crossed elasticities) } \epsilon_{ij} = -\frac{F_{ij}(\mathbf{x})x_j}{F_i(\mathbf{x})},\ \ i, j = 1, \ldots, n$$

- Then regarding "own" elasticities
  - If MP_i is constant then the "own" elasticity is equal to zero
  - If MP_i is decreasing then the "own" elasticity is positive
- For "crossed" elasticities, we say there is **gross or Edgeworth**:
  - substitutability if $\epsilon_{ij} > 0$ $(F_{ij} < 0)$
  - independence if $\epsilon_{ij} = 0$ $(F_{ij} = 0)$
  - complementarity if $\epsilon_{ij} < 0$ $(F_{ij} > 0)$
- do not confuse with Hicksian substitutability which is evaluated from the change of demand as regards input prices

Production function: technology

- Compensated changes in two inputs: variations in inputs i and j such that the output is constant

$$dy = F_i(\mathbf{x})\,dx_i + F_j(\mathbf{x})\,dx_j = 0$$

- **Marginal rate of substitution** between inputs i and j: compensated change in j for a unit change in i

$$MRS_{ij} \equiv -\left.\frac{dx_j}{dx_i}\right|_{dy=0}$$

- Then: it is equal to the ratio of marginal products

$$MRS_{ij} = \frac{F_i(\mathbf{x})}{F_j(\mathbf{x})},\ i \neq j = 1, \ldots, n$$

Production function: technology

- Elasticity of substitution between inputs i and j

$$ES_{ij} \equiv \left. \frac{d \ln (x_j / x_i)}{d \ln MRS_{ij}(\mathbf{x})} \right|_{dy=0}$$

- then

$$ES_{ij}(\mathbf{x}) = \frac{x_i\, F_i(\mathbf{x}) + x_j\, F_j(\mathbf{x})}{x_j\, F_j(\mathbf{x})\, \epsilon_{ii}(\mathbf{x}) - 2 x_i\, F_i(\mathbf{x})\, \epsilon_{ij}(\mathbf{x}) + x_i\, F_i(\mathbf{x})\, \epsilon_{jj}(\mathbf{x})}$$

Production function: the benchmark case

The benchmark production function is the **generalized mean**

$$y = M_\sigma(\mathbf{x}) = \left(\sum_{i=1}^{n} \alpha_i \, x_i^\sigma \right)^{\frac{1}{\sigma}}, \; \sigma \in [-\infty, \infty]$$

where $0 \leq \alpha_i \leq 1$ is the share of input i and

$$\sum_{i=1}^{n} \alpha_i = 1$$

- We readily see that it is an homogeneous function, for any value of σ

$$M_\sigma(\lambda \mathbf{x}) = \lambda M_\sigma(\mathbf{x})$$

thus it displays a constant returns to scale technology.

Production function: the benchmark case

For different values of σ economists call different names

- Linear case: if $\sigma = 1$
- Cobb-Douglas if $\sigma = 0$. Proof:
  $\ln M_0 = \lim_{\sigma\to 0} \dfrac{\ln \sum_{i=1}^{n} \alpha_i x_i^{\sigma}}{\sigma} = \lim_{\sigma\to 0} \dfrac{\sum \alpha_i x_i^{\sigma} \ln(x_i)}{\sum \alpha_i x_i^{\sigma}} =$
  $\sum \alpha_i \ln(x_i)$
- Constant elasticity of substitution (CES) if $\sigma < 1$ and is finite
- $M_\infty = x_{max} = \max\{x_1, \dots x_n\}$
  Proof $\lim_{\sigma\to\infty} M_\sigma = x_{\max} \lim_{\sigma\to\infty} \sum_i \alpha_i \left(\dfrac{x_i}{x_{\max}}\right)^{\sigma} \Big)^{\frac{1}{\sigma}} =$
  $x_{\max} \lim_{\sigma\to\infty} \text{const}^{\frac{1}{\sigma}} = x_{\max}$ where $const \in (0, 1)$
- Leontieff if $M_{-\infty}(\mathbf{x}) = x_{min} = \min\{x_1, \dots x_n\}$
  Proof: $M_{-\infty}(\mathbf{x}) = \dfrac{1}{M_\infty(1/\mathbf{x})}$

Production function: the benchmark case

Applying our previous concepts we find:

- Marginal product of input i: all inputs are productive

$$MP_i(\mathbf{x}) = F_i(\mathbf{x}) = \alpha_i \left(\frac{F(\mathbf{x})}{x_i} \right)^{1-\sigma} \geq 0$$

- Allen-Uzawa elasticities
  - input i has **decreasing** MP_i **if** $\sigma \leq 1$

$$\epsilon_{ii} = (1-\sigma) \left(1 - \frac{\alpha_i x_i^{\sigma}}{F(\mathbf{x})^{\sigma}} \right) \geq 0 \text{ if } \sigma < 1$$

  - inputs i and j are
    - gross substitutes if $\sigma > 1$
    - gross complements if $\sigma < 1$

    because

$$\epsilon_{ij} = (\sigma - 1) \frac{\alpha_j x_j^{\sigma}}{F(\mathbf{x})^{\sigma}}$$

Production function: the benchmark case

Applying our previous concepts we find:

- Marginal rate of substitution depends only on the quantities of the two inputs

$$MRS_{ij}(\mathbf{x}) = \frac{\alpha_i}{\alpha_j}\left(\frac{x_j}{x_i}\right)^{1-\sigma}$$

- Elasticity of substitution is constant, for any pair of inputs

$$ES_{ij}(\mathbf{x}) = \frac{1}{1-\sigma}$$

then i and j are

- gross substitutes implies $ES_{ij}(\mathbf{x}) > 0$
- gross complements implies $ES_{ij}(\mathbf{x}) < 0$

Production function and input prices

Problem for a competitive firm

- **Assumption**: competitive product and input markets
- Total cost:

$$C(\mathbf{x},\mathbf{w}) = \mathbf{w}\cdot\mathbf{x} = \sum_{i=1}^{n} w_i\, x_i$$

- Marginal cost of input i

$$MC_i = \frac{\partial C(\mathbf{x},\mathbf{w})}{\partial x_i}$$

- Return, assuming an unit cost for the product :

$$RT = 1 \times y = F(\mathbf{x})$$

- Profit: equal to total return minus total cost

$$\pi(\mathbf{x},\mathbf{w}) = F(\mathbf{x}) - C(\mathbf{x},\mathbf{w}) = F(x_1,\ldots,x_n) - \sum_{i=1}^{n} w_i\, x_i$$

Input demand functions

- The firm's primal problem (simpler problem) is to maximize the profit by choosing a vector of inputs

$$\pi^*(\mathbf{w}) = \max_{\mathbf{x}} \pi(\mathbf{x}, \mathbf{w}) \text{ s.t. } F(\mathbf{x}) \leq y$$

  where y is output
- We write the Lagrangean

$$L(\mathbf{x}, \lambda) = F(\mathbf{x}) - C(\mathbf{x}, \mathbf{w}) + \lambda \left(y - F(\mathbf{x})\right)$$

  where λ is the Lagrange multiplier
- Optimum conditions, for an interior solution are

$$\begin{cases} (1-\lambda)\, F_i(\mathbf{x}) = w_i, & \text{for } i = 1, \ldots, n \\ F(\mathbf{x}) = y \end{cases}$$

- If there is an interior solution we will find the (Hicksian) demand functions for all inputs

$$x^* = X_i(\mathbf{w}, y)$$

  as functions of input prices and output.

Input demand functions

For the benchmark case, the Inada conditions guarantee existence an duniqueness:

- solving the optimality condition we find

$$x_i = \left(\frac{\alpha_i (1 - \lambda)}{w_i} \right)^{\frac{1}{1-\sigma}} F(\mathbf{x})$$

- substituting in the constraint yields

$$(1 - \lambda)^{\frac{1}{1-\sigma}} = \frac{1}{P(\mathbf{w})}$$

where $P(\mathbf{w})$ is a producer price index

$$P(\mathbf{w}) \equiv \left(\sum_{i=1}^{n} \alpha_i \left(\frac{w_i}{\alpha_i} \right)^{\frac{\sigma}{\sigma-1}} \right)^{\frac{1}{\sigma}}$$

Input demand functions

- The optimal demand (Hicksian) functions are

$$x_i^* = X_i(\mathbf{w}, y) = \left(\frac{w_i}{\alpha_i}\right)^{\frac{1}{\sigma-1}} \frac{y}{P(\mathbf{w})}$$

functions of the real demand, where the deflator is a producer price index

- Comparative statics for prices: elasticities
  - for "own price" changes

$$\frac{\partial X_i}{\partial w_i}\frac{w_i}{X_i} = \frac{1}{\sigma - 1}\left(1 - \frac{\alpha_i\left(\frac{w_i}{\alpha_i}\right)^{\frac{\sigma}{\sigma-1}}}{P(\mathbf{w})^\sigma}\right), \text{ for } i = 1, \dots, n$$

  - for "crossed price" changes

$$\frac{\partial X_i}{\partial w_j}\frac{w_i}{X_i} = -\frac{1}{\sigma - 1}\frac{\alpha_i\left(\frac{w_i}{\alpha_i}\right)^{\frac{\sigma}{\sigma-1}}}{P(\mathbf{w})^\sigma}, \text{ for } i \neq j = 1, \dots, n$$

Input demand functions

- if $\sigma < 1$ then
$$\frac{\partial X_i}{\partial w_i}\frac{w_i}{X_i} < 0, \text{ and } \frac{\partial X_i}{\partial w_j}\frac{w_j}{X_i} > 0$$
the demand **reduces** with the "own" price and **increases** with any "crossed" price, meaning that inputs i and any other input are **substitutable in the Hicksian sense** (recall they were gross complements in the Edgeworth sense
- if $\sigma > 1$ then
$$\frac{\partial X_i}{\partial w_i}\frac{w_i}{X_i} > 0, \text{ and } \frac{\partial X_i}{\partial w_j}\frac{w_j}{X_i} < 0$$
the demand **increases** with the "own" price and **reduces** with any "crossed" price, meaning that inputs i and any other input are **complementary in the Hicksian sense** (recall they were gross substitutable in the Edgeworth sense.

MRS and input prices

- For the benchmark case, setting $F_i(\mathbf{x}) = w_i$ we find

$$x_i^*(\mathbf{w}) = \left(\frac{w_i}{\alpha_i}\right)^{\frac{1}{\sigma-1}} F(\mathbf{x}^*)$$

- Then we have a relationship between factor demands and relative prices,

$$MRS_{ij} = \frac{F_i(\mathbf{x}^*)}{F_j(\mathbf{x}^*)} = \frac{\alpha_i}{\alpha_j}\left(\frac{x_j^*}{x_i^*}\right)^{1-\sigma} = \frac{w_i}{w_j}$$

Production functions: types of inputs

Types of inputs

We can distinguish between:

- intermediary goods and factors of production
- produced inputs and non-produced inputs
- exogenous inputs and endogenous inputs
- private and aggregate inputs

Intermediate goods

- in a given production function: intermediate inputs enter as flows and factors enter as stocks in production functions

$$y = F(\mathbf{x}, \mathbf{z})$$

  where
- intermediate inputs are products of other sectors and use factors of production: example $x_i = f_i(\mathbf{z})$
- usually for the final use sector they are private goods (i.e., firms pay the full price for their use)
- they can be produced in a competitive or non-competitive market

Factors of production

- in a given production function: factors enter as stocks in production functions
$$y = F(\mathbf{x}, \mathbf{z})$$
- factors of production are usually exogenous to the firm
- but they can be exogenous or endogenous to the economy
- when factors of production are produced, their output is a flow which generates a stock-flow dynamics
$$\dot{z}_i = \frac{dz_i(t)}{dt} = G(\mathbf{z})$$
- durable goods entail necessarily a dynamic mechanism (ex: capital stock)

Factors of production

- The provision of factor of production can be internal or external to the firm
- The firm's level production function can be

$$y = f(\mathbf{z}, \mathbf{Z})$$

  where $\mathbf{z}$ are private factors and $\mathbf{Z}$ are external factors
- their use faces different incentives:
  - if the factor of production is private the firm has to pay the price w_i for its use
  - if the factor of production is an externality the firm does not have to pay for its use
- their existence introduces a distinction between production functions at the firm level $y = f(\mathbf{z}, \mathbf{Z})$ and at the aggregate level

$$y = F(\mathbf{Z}) = f(\mathbf{Z}, \mathbf{Z})$$

- then the previous properties can be different at the firm's level (related to the incentives) and at the aggregate level

Production function and technological bias

Production function in input intensity form

- Production function in efficiency form

$$y = F(\mathbf{A}, \mathbf{x}) \equiv F(A_1\, x_1, \ldots, A_n\, x_n)$$

  where A_i input augmenting index measuring the specific productivity of input i,

- In growth models:
  - A_i measures specific or aggregate productivity increases
  - $A_i(t)$, where t is time measures technical progress (or decay),
  - A_i can be exogenous or endogenous (learning-by-doing, R&D)
- Here we introduce a first take to the subject

Types of technical progress

- Consider the benchmark production function in intensity form

$$y = F(\mathbf{A}, \mathbf{x}) = \left(\sum_{i=1}^{n} \alpha_i \left(A_i\, x_i \right)^{\sigma} \right)^{\frac{1}{\sigma}}$$

- Assume the benchmark production function

$$MP_i = \alpha_i\, A_i^{\sigma} \left(\frac{F(\mathbf{A}, \mathbf{x})}{x_i} \right)^{1-\sigma}$$

$$MRS_{ij} = \frac{\alpha_i}{\alpha_j} \left(\frac{A_i}{A_j} \right)^{\sigma} \left(\frac{x_j}{x_i} \right)^{1-\sigma}$$

- The effect of the technical progress on production depends on:
  - the vector $\mathbf{A}$, in particular if it is equal for all inputs: Hicks neutral technical progress $A_i = A$ for all i
  - on the substitutability properties of the production function, if $\mathbf{A}$ is heterogeneous

Types of technical progress

There are several concepts of neutrality and bias in technical progress. Here we consider

- **neutral** technical progress: if the change in any A_i leaves the MRS_{ij} unchanged
- **biased** technical progress: if the change in any A_i changes the MRS_{ij}.
- If we consider input prices

$$MRS_{ij} = \frac{\alpha_i}{\alpha_j}\left(\frac{A_i}{A_j}\right)^{\sigma}\left(\frac{x_j}{x_i}\right)^{1-\sigma} = \frac{w_i}{w_j}$$

- In the above sense
  - if $\sigma = 0$ then the technical progress is neutral in this sense
  - if $\sigma \neq 0$ then the technical progress is biased
- However, observe that the demand functions $x_i = X_i(\mathbf{A}, \mathbf{w})$ can depend on $\mathbf{A}$. If $\sigma = 0$ we can interpret this as an income effect.

Types of technical progress

- We can also write
$$\frac{\alpha_i}{\alpha_j}\left(\frac{A_i\,x_i}{A_j\,x_j}\right)^{\sigma} = \frac{w_i\,x_i}{w_j\,x_j}$$
- Assume that the expenditures in inputs (or factor shares in national income) is constant and $\sigma \neq 0$, then:
  - if the technical progress is **neutral** the ratio of the inputs remains constant;
  - if the technical progress is biased an increase in A_i/A_j the technical progress is $i-$**saving**, i.e, there is a reduction in its quantity; the ratio of the two inputs changes.

Extensions 1: continuum of inputs

Continuum of inputs

- Dixit-Stiglitz production functions consider a continuum of inputs which makes y a functional

$$y = F[x] = \left(\int_0^N \alpha(i)\, x(i)^\sigma \, di \right)^{\frac{1}{\sigma}}$$

- How to calculate $MP(i)$?
- we use the functional derivative

$$MP(i) = \frac{\delta F[x]}{\delta x(i)} = \alpha(i) \left(\frac{F[x]}{x(i)} \right)^{1-\sigma}$$

- all the concepts of production theory can be adapted to this case.
- in particular

$$MRS(i,j) = \frac{\alpha(i)}{\alpha(j)} \left(\frac{x(j)}{x(i)} \right)^{1-\sigma}$$

and

$$ES(i,j) = \frac{1}{1-\sigma}$$

Extensions 2: multisector economies

Multisector economies

- If we consider the existence of m production sectors, in this case we have an input-output structure

$$\mathbf{y} = \mathbf{F}(\mathbf{x})$$

where now $\mathbf{x}$ is a $(m \times n)$ vector

$$\begin{pmatrix} y_1 \\ \dots \\ y_m \end{pmatrix} = \begin{pmatrix} F_1(x_{11}, \dots, x_{1n}) \\ \dots \\ F_m(x_{m1}, \dots, x_{mn}) \end{pmatrix} =$$

- In these economies a vector of prices is also required, $\mathbf{p} = (p_1, \dots, p_m)^\top$
- We can extend all the previous concepts component-wise, that is for every sector

Multisector economies

- We can find the optimal allocation as a solution of the problem

$$\max_{x_{11},\dots,x_{mn}} \sum_{i=1}^{n} p_i y_i(\mathbf{x}_i) - \sum_{j=1}^{n} w_j\, x_j,\ \text{st} \sum_{i=1}^{n} x_{ij} = x_j$$

- under some conditions, we can find, at the optimum a relationship as

$$y_j = F^j(\mathbf{x})$$

a supply function for every sector as a function of the aggregate input.

Production function and input accumulation and dynamics

Investment and savings

- Assume there is only one factor of production and there are no intermediate goods
$$y = f(x)$$
- Assume that the good produce is durable and can be used both for consumption and investment, then
$$y = c + \dot{x}$$
- Let savings be a function of x, $s = s(x) = y - c$
- Then we have a stock-flow dynamics where
$$\dot{x} = \frac{dx(t)}{dt} = s(x(t))$$
- The solution to this differential equation, gives
$$x(t) = x_0 + \int_0^t s\big(x(\tau)\big)\,d\tau$$

Types of growth dynamics and the production function

Growth dynamics

- From this solution we can obtain the dynamics of product from

$$y(t) = f\big(x(t)\big)$$

- We can also set directly an ODE on the GDP

$$\dot{y} \equiv \frac{dy(t)}{dt} = \mu(y)$$

- We say **there is long run growth** if the solution to this equation tends asymptotically to an exponential

$$\lim_{t\to\infty} y(t) \propto e^{\gamma t},\ \gamma > 0$$

- we will see that this requires the **technology to be linear at the aggregate level**. V.g: $y = Ak$

Growth theory: on the exponential structure of growth

- Long run growth exits for a particular mathematical structure of $\mu(y)$
  - logistic growth: $\mu(y) = \alpha y(\beta - y)$,
  - exponential growth: $\mu(y) = \gamma y$,
  - power law growth: $\mu(y) = y^{\phi}$ for $\phi > 1$,
- razor edge property of growth models: although the exponential case is very particular it this the structure underlying (almost) all growth theories

Growth theory: on the exponential structure of growth

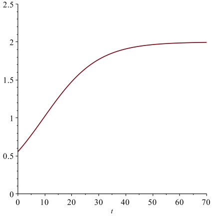

Figure: Logistic growth $\mu(y) = \alpha y(\beta - y)$

- there is short run (transition) growth
- but there is no long-run growth

Growth theory: on the exponential structure of growth

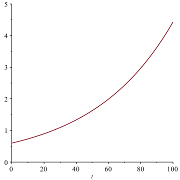

Figure: Exponential growth $\mu(y) = \gamma y$

- there is no short run (transition) growth
- but there is long-run growth
- gdp becomes infinite ($y(t) \to \infty$) in **infinite** time

Growth theory: on the exponential structure of growth

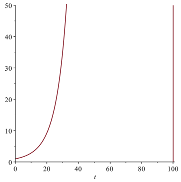

Figure: Power law growth $\mu(y) = y^{\phi}$ for $\phi > 1$

- GDP becomes infinite ($y(t) \to \infty$) in **finite** time
- and collapses afterwards

Growth and transition dynamics

- In order to have both long run growth and transition dynamics we need to have at least two durable goods

$$\dot{x}_1 = s_1(x_1, x_2)$$
$$\dot{x}_2 = s_2(x_1, x_2)$$

- We have long run growth if we can find a **balanced growth path**, i.e., a solution of type

$$x_1(t) = \phi_1(t)\, e^{\gamma t},$$
$$x_2(t) = \phi_2(t)\, e^{\gamma t},$$

  where γ is the long run growth rate and $g_i(\phi_1(t), \phi_2(t))$ are the transition components;

- This also requires a particular structure of the production functions: they should be CRS (at the aggregate level) for every sector.