Economic Growth Theory: Problem set 1: Malthus models Solutions

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Problem

Assume that the representative consumer solves the problem: $\max_{c,b}\{u(c,b) :\ c + \rho b \leq y\}$ where c is consumption, b is the birth rate, ρ is the cost of raising children and y is per capita income. Assume that the utility function is

$$u(c,b) = \ln(c) + \phi \ln(b),\ \phi > 0$$

and the aggregate production function is Cobb-Douglas $Y = (AX)^{\alpha} L^{1-\alpha}$, with $0 < \alpha < 1$, where X is the stock of land, A is land-specific productivity and L is population. Population growth is $\dot{L}/L = b - m$, where the mortality rate, m, is constant and exogenous, and $L(0) = L_0 > 0$ is given. Land productivity grows at a rate $\gamma > 0$.

1. Defining $\ell \equiv L/A$, obtain a differential equation for ℓ.

2. Study the qualitative dynamics of the model. Provide an intuition for your results.

3. Derive the growth facts (long run growth rate, long run per capita output and transition dynamics). What are the effects of an increase in γ ?

Solution

1. $\dot{\ell} = \ell\,(\psi(X/\ell)^{\alpha} - (m+\gamma))$ for $\psi \equiv \frac{\phi}{\rho(1+\phi)}$;

2. Steady states $\ell^* = \{\,0, \bar{\ell}\}$ with $\bar{\ell} = \left(\frac{\psi}{m+\gamma}\right)^{\frac{1}{\alpha}} X$, local dynamics $\frac{\partial \dot{\ell}}{\partial \ell}(\bar{\ell}) = -\alpha(m+\gamma)$. Solving explicitly we have

$$\ell(t) = \left(\bar{\ell}^{\alpha} + (\ell(0)^{\alpha} - \bar{\ell}^{\alpha})e^{-\alpha(m+\gamma)t}\right)^{\frac{1}{\alpha}}$$

   The conclusions are the same: if $\ell(0) > 0$ then ℓ converges to the steady state $\bar{\ell}$.

3. Growth facts: as

$$y(t) = \frac{X^{\alpha}}{\bar{\ell}^{\alpha} + (\ell(0)^{\alpha} - \bar{\ell}^{\alpha})e^{-\alpha(m+\gamma)t}}$$

   then $\lim_{t\to\infty} y(t) = \dfrac{m+\gamma}{\psi}$: no long run growth, there is transitional dynamics and the steady state level of output per capita dependes on m, γ and parameters associated to consumer problem (ψ)