# Economic Growth Theory:
# Problem set 3: Ramsey models

Paulo Brito
Universidade de Lisboa
Email: pbrito@iseg.ulisboa.pt

17.3.2021

## Ramsey models

1. Consider a version of the Ramsey model with increasing population $\dot{N} = nN$ and $N(0) = 1$. The objective utility functional for the central planner is:

$$\max_{c} \int_0^{\infty} \ln\left(c(t)\right) e^{nt} e^{-\rho t} dt$$

where $\rho > n > 0$, subject to

$$\dot{k} = Ak^{\alpha} - c(t) - nk(t),\ 0 < \alpha < 1$$

where $c$ and $k$ are the percapita consumption and capital stock. Let $k(0)$ be given.

   (a) Apply the Pontryiagin's principle and determine the optimality conditions as a dynamic system in $(c, k)$.

   (b) Determine the steady state, and study their stability properties and draw the phase diagram;

   (c) Discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics.

(d) Determine the effects of an increase in productivity, $A$.

2. Consider a version of the Ramsey model with increasing population $\dot{N} = nN$ and $N(0) = 1$. The objective utility functional for the central planner is:

$$\max_{c} \int_0^{\infty} \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{nt} e^{-\rho t} dt$$

where $\rho > n > 0$ and $\theta > 0$, subject to

$$\dot{k} = Ak^{\alpha} - c(t) - nk(t),\ 0 < \alpha < 1$$

where $c$ and $k$ are the percapita consumption and capital stock. Let $k(0)$ be given.

(a) Apply the Pontryiagin's principle and determine the optimality conditions as a dynamic system in $(c, k)$.

(b) Determine the steady state, and study their stability properties and draw the phase diagram;

(c) Discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics.

(d) Determine the effects of an increase in productivity, $A$.

3. Consider a version of the Ramsey model with increasing population $\dot{N} = nN$ and $N(0) = 1$. The objective utility functional for the central planner is:

$$\max_{c} -\frac{1}{\eta} \int_0^{\infty} e^{-\eta c(t)} e^{nt} e^{-\rho t} dt$$

where $\rho > n > 0$ and $\eta > 0$, subject to

$$\dot{k} = Ak^{\alpha} - c(t) - nk(t),\ 0 < \alpha < 1$$

where $c$ and $k$ are the per capita consumption and capital stock. Let $k(0)$ be given.

(a) Apply the Pontryiagin's principle and determine the optimality conditions as a dynamic system in $(c, k)$.

(b) Determine the steady state, and study their stability properties and draw the phase diagram;

(c) Discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics.

(d) Determine the effects of an increase in productivity, $A$.

4. Consider a version of the Ramsey model with increasing population $\dot{N} = nN$ and $N(0) = 1$. The objective utility functional for the central planner is:

$$\max_{c} \int_0^{\infty} c(t)e^{nt}e^{-\rho t}dt$$

where $\rho > n > 0$, subject to

$$\dot{k} = Ak^{\alpha} - c(t) - nk(t), \; 0 < \alpha < 1$$

where $c$ and $k$ are the per capita consumption and capital stock. Let $k(0)$ be given.

(a) Determine the optimal path for the capital stock and consumption [1] Under which conditions are those paths feasible ?

(b) Under conditions that ensure feasibility, discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics.

5. Consider a version of the Ramsey model with constant population where the objective utility functional for the central planner is:

$$\max_{c} \int_0^{\infty} ln(c(t) - \bar{c})e^{-\rho t}dt$$

where $\rho > 0$ and $\bar{c} > 0$ is a minimum level of consumption, subject to

$$\dot{k} = Ak^{\alpha} - c(t) - \delta k(t), \; 0 < \alpha < 1, \; \delta > 0$$

where $c$ and $k$ are the per capita consumption and capital stock. Let $k(0)$ be given.

(a) Apply the Pontryiagin's principle and determine the optimality conditions as a dynamic system in $(c, k)$.

[1] Transform the problem into a calculus of variations problem $\max \int_0^{\infty} F(x(t), \dot{x})e^{-\delta t}dt$. The optimal path $[x(t)]_{t \geq 0}$ should verify the dynamic optimality condition is given by the Euler equation $F_x + \delta F_{\dot{x}} - F_{\dot{x}x}\dot{x} - F_{\dot{x}\dot{x}}\ddot{x} = 0$ and the initial and terminal conditions.

(b) Draw the phase diagram.

(c) Determine the steady states and study their local stability properties.

(d) Discuss the properties of the solution to the planner's problem.

(e) What are the effects of an increase in productivity, $A$ ?

6. Consider a version of the Ramsey model with constant population where the objective utility functional for the central planner is:

$$\max_{c} \int_0^{\infty} ln(c(t) - \bar{c})e^{-\rho t}dt,$$

where $\rho > 0$ and $\bar{c} > \frac{\rho}{\alpha}$ is a minimum level of consumption, subject to

$$\dot{k} = Ak(t)^{\alpha} - c(t),\ 0 < \alpha < 1$$

where $c$ and $k$ are the per capita consumption and capital stock. We also assume that $k(0) = k_0$ is given and that the stock of capital is bounded.

(a) Apply the Pontryiagin's principle and determine the optimality conditions as a dynamic system in $(c, k)$.

(b) Draw the phase diagram.

(c) Determine the steady states and study their local stability properties.

(d) Find an approximate solution to the problem in the neighborhood of the steady state associated with a maximum consumption.

(e) Determine the effects of a permanent increase in productivity, $A$.

7. Consider a version of the Ramsey model with constant population where the objective utility functional for the central planner is:

$$\max_{C} \int_0^{\infty} \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t}dt$$

where $\rho > 0$ and $\theta \geq 1$, subject to

$$\dot{K} = A(t)^{1-\alpha}K(t)^{\alpha} - C(t) - \delta K(t),\ 0 < \alpha < 1,\ \delta > 0$$

where $C$ and $K$ denote (aggregate) consumption and capital stock. Let $K(0)$ be given. There labor-augmenting technical progress where labour productivity $A(t)$ grows at the rate $\mu$ and $A(0) = 1$.

(a) Recast the variables in efficiency terms by setting $c(t) \equiv C(t)/A(t)$ and $k(t) \equiv K(t)/A(t)$. Prove that the Ramsey problem can be equivalently defined as

$$\max_{c} \int_{0}^{\infty} \frac{c(t)^{1-\theta}}{1-\theta} e^{-\tilde{\rho}t} dt$$

where $\tilde{\rho} = \rho + \mu(\theta - 1)$

$$\dot{k} = k(t)^{\alpha} - c(t) - (\delta + \mu)k(t).$$

(b) Apply the Pontryiagin's principle and determine the optimality conditions as a dynamic system in $(c, k)$.

(c) Draw the phase diagram.

(d) Determine the steady states and study their local stability properties.

(e) Characterize the growth properties of the solution to the Ramsey problem.