The Malthusian growth model

Paulo Brito
pbrito@iseg.ulisboa.pt

9.3.2022 (revised)
Malthusian economics

▶ Popular definition of **Malthusian** (Malthus (1798)): population grows exponentially and food grows linearly

▶ This would lead either to **catastrophe** or to the existence of a natural (not nice) **endogenous stabilization mechanism**, in the absence of ”moral restraint”

▶ The existence of that endogenous mechanism, relating population and wages, is consistent with the economic history in pre-industrial W. Europe, in particular after the **Black Death** (1346-1353)
Wages and population in historical data

Figure 5: Population and Real Wages in England, 1250-1750 CE
(Source: Clark, 2005)
Malthusian theory

- We will see that the existence of **decreasing marginal returns** to labor is a necessary (although not sufficient) condition.

- The idea that the existence of a fixed resource and decreasing returns to production implies that **growth processes eventually stop** is present in most Classical economists (Quesnay, Smith, Ricardo, Marx) and, possibly, in modern ecologists.

- But it was **Thomas Malthus** who stated it more clearly in *An Essay on the Principle of Population* (1798) and systematically gathered data to sustain it.

- We next provide a modern view of the theory
A modern view on the Malthusian model

The general idea:

▶ It presents the joint dynamics of production and population growth
▶ For pre-industrial societies: there are two main factors of production labor and land
▶ Labor is the **reproducible** factor of production (no capital accumulation, no R&D)
▶ The basic dynamic mechanism is: increase in income leads to increase in population and in labor supply; this increases aggregate income, but income per capita does not increase at the same pace, leading eventually to a steady state (positive extensive effect but negative intensive effect).
▶ **Decreasing marginal returns for the reproducible factor** is the main driving force behind the non-existence of growth in the long run.
▶ Even with exogenous technical progress there is no growth. The conditions for the existence of long run growth are very specific (learning-by-doing)
Assumptions

- **Production:**
  - production uses two factors: labor and land
  - the production function has constant returns to scale
  - the only reproducible factor is labor, and it faces decreasing marginal returns

- **Population:**
  - fertility is endogenous and mortality is exogenous

- **Farmers:**
  - households are land-owners
  - they choose among consumption and child-rearing
  - there is no saving
\[ F(L(t), X) \rightarrow Y(t) \]

\[ v(y(t)) \rightarrow c(t) \]

\[ \dot{L}(b(t), m(t)) \]

\[ m \text{ (mortality)} \]

where \( v(y) = \max\{u(c, b) : c + pb \leq y\} \) and \( y = \frac{Y}{L} \)
The model
Production

▶ **Production function**: we assume a Cobb-Douglas production

\[ Y(t) = (AX)^\alpha L(t)^{1-\alpha}, \ 0 < \alpha < 1 \]

where: \( A \) productivity, \( X \) stock of land, \( L \) labor input

▶ Property 1: constant returns to scale

\[ (\lambda AX)^\alpha (\lambda L)^{1-\alpha} = \lambda Y \]

▶ implication: the Euler theorem holds

\[ Y = \frac{\partial Y}{\partial L} L + \frac{\partial Y}{\partial X} X \]
The model
Production technology

- Property 2: positive marginal returns for labor and land

\[
\frac{\partial Y}{\partial L} = (1 - \alpha) \frac{Y}{L} > 0, \quad \frac{\partial Y}{\partial X} = \alpha \frac{Y}{X} > 0
\]

- Property 3: Inada production function: \( \lim_{L \to 0} \frac{\partial Y}{\partial L} = \infty \) and \( \lim_{L \to \infty} \frac{\partial Y}{\partial L} = 0 \)

- Implication: no bias in technical change (why ?)

\[
MRS_{L,X} = \frac{(1 - \alpha) X}{\alpha L}
\]
The model

Inada property regarding the marginal productivity of labor

\[
\frac{\partial Y}{\partial L}
\]
The model

Production technology

▶ Property 4: decreasing marginal returns for both factors

$$\frac{\partial^2 Y}{\partial L^2} = -\alpha(1 - \alpha) \frac{Y}{L^2} < 0, \quad \frac{\partial^2 Y}{\partial X^2} = -\alpha(1 - \alpha) \frac{Y}{X^2} < 0$$

▶ Property 5: the two factors are gross (or Edgeworth) complements

$$\frac{\partial^2 Y}{\partial X \partial L} = \alpha(1 - \alpha) \frac{Y}{LX} > 0$$

▶ Property 6: the technology is concave (but not strictly concave)

$$\frac{\partial^2 Y}{\partial L^2} \frac{\partial^2 Y}{\partial X^2} - \left( \frac{\partial^2 Y}{\partial X \partial L} \right)^2 = 0$$

▶ Implication: (1) the AU elasticities are constant

$$\varepsilon_{LL} = \alpha, \varepsilon_{XX} = 1 - \alpha, \varepsilon_{LX} = -\alpha$$

▶ and (2) we already known that the elasticity of substitution is equal to one:

$$ES_{LX} = 1$$
The model
Production efficiency

- Optimal allocation of factors, or production efficiency, in a market economy:

\[
\max_{L,X} \{ Y(L, X) - wL - RX \}
\]

where \( w \) is the wage rate and \( R \) are is land rent

- and competitive markets lead to

\[
w(L, X) = \frac{\partial Y}{\partial L} = (1 - \alpha) \frac{Y}{L} > 0
\]
\[
R(L, X) = \frac{\partial Y}{\partial X} = \alpha \frac{Y}{X} > 0
\]

- Factors are Hicksian substitutables

\( w_L < 0, w_X > 0 \) wages decrease with labor and increase with land;
\( R_X < 0, R_L > 0 \) rents decrease with land and increase with labor.
Farmers’ problem
Endogenous rate of population growth

- There are $L$ farmers; who receive (per capita) income from farming and decide which part to consume and which part to allocate to raising offspring, by deciding the number of offspring (Beckerian model)

- **Household’s (farmer’s) static problem** (for every $t \geq 0$)

  $\max_{c(t), b(t)} \{ c(t)^{1-\psi} b(t)^\psi : c(t) + p b(t) = y(t) \}$

  $0 < \psi < 1$ (relative) love for children, $1/\psi = "moral restraint"$
  $p > 0$ relative cost of raising children

- solution

  $c(t) = (1 - \psi)y(t)$ (consumption increases with income)

  $b(t) = \frac{\psi}{p} y(t)$ (number of children increases with income)
Population dynamics

- Population growth

\[
\dot{L} \equiv \frac{dL(t)}{dt} = (b(t) - m)L(t)
\]

- where the fertility rate is endogenous: \( b(t) = \frac{\psi}{p}y(t) \)
- the mortality rate is exogenous: \( m \) is given
- the initial level of population is assumed to be given by number \( L_0 \)

\[
L(t)\Big|_{t=0} = L(0) = L_0
\]
The Malthusian model
Endogenous rate of population growth

Then

$$\dot{L} = \left( \frac{\psi}{p} y(t) - m \right) L(t), \text{ for } t \in [0, \infty)$$

$L(0) = L_0$ given

where the per capita GDP is

$$y(t) \equiv \frac{Y(t)}{L(t)} = \left( \frac{AX}{L(t)} \right)^\alpha$$
Detour

Per-capita rate of growth arithmetics

- taking log-derivatives w.r.t time we have

\[ \frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} \]

- that we denote by

\[ g(t) = g_Y(t) - n(t) \]

- as the per capita GDP is

\[ y(t) \equiv \frac{Y(t)}{L(t)} = \left( \frac{AX}{L(t)} \right)^{\alpha} \]

- Then: **the rate of growth is exactly negatively correlated to the rate of growth of population**

\[ g(t) = \frac{\dot{y}}{y} = -\alpha \frac{\dot{L}}{L} \]
Take home

Remember:

▶ We are interested in what this model can tell us about economic growth
▶ For us growth is related to the dynamics of GDP per capita

\[ y(t) = \frac{Y(t)}{L(t)} \]

▶ we want to know the implications for:

▶ the rate of growth of GDP \( g(t) = \frac{\dot{y}(t)}{y(t)} \)
▶ the steady state level of GDP \( \bar{y} \)
▶ and the dynamics: i.e. separating \( g(t) \) into transition and long-run components
Solving the Malthusian model

There are two approaches to solving the model

- Approach 0: do a geometric representation of the solution of the model (phase diagram)
- Approach 1: solve the differential equation for $L$ and substitute in $y$ to get the dynamics of growth
- Approach 2: obtain a differential equation for $y$ and solve it
The model

The phase diagram: geometric intuition

Figure: Phase diagram for $\dot{L} = m \left( \left( \frac{\bar{L}}{L} \right)^\alpha - 1 \right) L$
Solving the Malthusian model

Approach 1: solving for $L$

- If we substitute $y$ in the dynamic equation for $L$ we have the initial value problem

$$\begin{cases}
\dot{L} = \left( \frac{\psi}{p} \left( \frac{AX}{L(t)} \right)^\alpha - m \right) L(t), & t \geq 0 \\
L(0) = L_0 \text{ given} & t = 0
\end{cases}$$

- we can solve it to get

$$L(t) = \left( \bar{L}^\alpha + \left( L_0^\alpha - \bar{L}^\alpha \right) e^{-m\alpha t} \right)^\frac{1}{\alpha}$$

where the steady state population is

$$\bar{L} = \left( \frac{\psi}{mp} \right)^\frac{1}{\alpha} AX$$
Solving the Malthusian model

Approach 2: solving for \( y \) directly

\[ \frac{\dot{y}}{y} = -\alpha \frac{\dot{L}}{L} \]

we obtain the dynamic equation for the GDP per capita

\[ \dot{y} = -\alpha \left( \frac{\psi}{p} y(t) - m \right) y(t) \tag{1} \]

together with the initial value

\[ y(0) = y_0 = (AX)^\alpha L_0^{1-\alpha} \]
Solving the Malthusian model

Explicit solution for $y$

- Equation (1) has two steady states $y^* = \{0, \bar{y}\}$ where

\[
\bar{y} = \frac{mp}{\psi}
\]

- we can re-write the growth equation as

\[
\dot{y} = \alpha \frac{\psi}{p} (\bar{y} - y(t)) y(t)
\]
Explicit solution for $y$

This is a Bernoulli differential equation. Therefore, it has an explicit solution:

$$y(t) = \left[ \frac{1}{\bar{y}} + \left( \frac{1}{y(0)} - \frac{1}{\bar{y}} \right) e^{-\alpha m t} \right]^{-1}, \text{ for } 0 \leq t < \infty$$

$$= \frac{\bar{y}}{1 + \left( \frac{\bar{y}}{y_0} - 1 \right) e^{-\alpha m t}}$$

satisfies \( \lim_{t \to \infty} y(t) = \bar{y} \)
Explicit solution for $g$

- the GDP growth rate is

$$g(t) = \frac{dy(t)}{dt} = \alpha m \frac{(\bar{y} - y_0) e^{-\alpha m t}}{y_0 + (\bar{y} - y_0) e^{-\alpha m t}}, \text{ for } 0 \leq t < \infty$$

- if $y_0 \neq \bar{y}$ then $g(0) = \alpha m (\bar{y} - y_0) / \bar{y}$ positive or negative depending on $\bar{y} - y_0$ and $\lim_{t \to \infty} g(t) = 0$. 
Malthusian model

Properties

1. there is **no long run growth**, because $\lim_{t \to \infty} g(t) = 0$
2. the **long run level** of GDP per capita is

   $\bar{y} = \frac{mp}{\psi}$

   increases with the mortality rate, the cost or rearing children and the "moral restraint" (no productivity effects)

3. there is **only transitional dynamics** (i.e., adjustments towards the steady state):
   
   ▶ if the initial GDP is small, $y(0) < \bar{y}$, then there is an increase in time of the GDP $g(t) > 0$
   ▶ if the initial GDP $y(0)$ is large, $y(0) > \bar{y}$, then there is an decrease in time of the GDP $g(t) < 0$
Malthusian model
Mechanics of the model

- if $y(0)$ is large so is the wage rate $w(0) = (1 - \alpha)y(0)$
- this implies that the initial fertility rate is higher, $b(0) = \frac{\psi}{p}y(0)$
- population increases, which increases output,
- but decreases the rate of growth of GDP

$$g(t) = -\alpha n(t)$$

because there are decreasing marginal returns due to the fact that $X$ is fixed.
Malthusian model

Trajectories: $y$, $L$ and $w$

Figure: Parameter values: $\alpha = 2/3$, $m = 0.03$, $\psi = 0.01$, $p = 10$, $A = 1$, $X = 100$, and $y(0) < \bar{y}$
Malthusian model

Exponential increase in land productivity

Can increases in land-productivity generate long-run growth?

▶ now \( Y(t) = (A(t) X)^{\alpha} L(t)^{1-\alpha} \)

▶ where \( \dot{A} = g_A A, \ g_A > 0 \)

▶ Taking logarithmic derivatives of the production function, this implies

\[
\frac{\dot{y}}{y} = \alpha \frac{\dot{A}}{A} + (1 - \alpha) \frac{\dot{L}}{L} - \frac{\dot{L}}{L} = \alpha \left( m + g_A - \frac{\psi}{p} y(t) \right)
\]

▶ there is no increase in the long run growth rate (why ?)

\[
\lim_{t \to \infty} g(t) = 0
\]

▶ there is an increase in GDP level

\[
\bar{y} = \frac{(g_A + m)p}{\psi}
\]
Malthusian model and land productivity

Phase diagram for an increase in the rate of growth of $A$
Malthusian model
Exponential increase in labor productivity

Can an increase in the productivity of labor generate long-run growth?

- now \( Y(t) = X^\alpha (h(t) L(t))^{1-\alpha} \)
- where \( \dot{h} = gh, \; gh > 0 \)
- Taking logarithmic derivatives of the production function, this implies

\[
\frac{\dot{y}}{y} = (1 - \alpha) \left( \frac{\dot{h}}{h} + \frac{\dot{L}}{L} \right) - \frac{\dot{L}}{L} = \alpha \left[ \frac{(1 - \alpha)}{\alpha} gh + m - \frac{\psi}{p} y(t) \right] y(t)
\]

there is no increase in the long run growth rate (why?)

\[
\lim_{t \to \infty} g(t) = 0
\]

there is an increase in GDP level

\[
\bar{y} = \frac{((1 - \alpha)gh + \alpha m)p}{\alpha \psi}
\]
Malthusian model
Learning by doing

Can learning by doing generate long-run growth?

- now $Y(t) = (A(t)X)^{\alpha}L(t)^{1-\alpha}$
- learning-by-doing: past production generates knowledge which increases land productivity
- Formally: $A(t) = \beta \int_{-\infty}^{t} e^{-\mu(t-s)} A(s)y(s) ds$
  where $\beta$ reproduction of knowledge, $\mu$ rate of oblivion
- taking derivatives for $t$ (Leibniz formula)

$$\dot{A} \equiv \frac{dA(t)}{dt} = (\beta y(t) - \mu) A(t)$$

- the dynamic equation for per-capita GDP becomes

$$\frac{\dot{y}}{y} = \left( \beta - \alpha \frac{\psi}{p} \right) y(t) + \alpha m - \mu$$
Malthusian model

Learning by doing: continuation

- If we assume $\beta = \alpha \frac{\psi}{p}$ (meaning $\varepsilon_{LL} \times \frac{b}{y}$) then
  
  $$\dot{y} = (\alpha m - \mu)y$$

- there is **long run growth** if $\alpha m > \mu$ because
  
  $$g(t) = \alpha m - \mu > 0, \text{ for all } t > 0$$

- the GDP level is exogenous
  
  $$y(t) = y_0 e^{(\alpha m - \mu)t}$$
Conclusions

- The existence of decreasing marginal returns to the reproducible factor of production (labor, $L$) implies that the Malthusian model does not feature long-run growth: **there is only transitional dynamics** (if initial population is too high, wages will be too low, which generates a fall in fertility and therefore a decrease in population until population is constant).

- **Exogenous permanent increases** in productivity will only increase the long-run GDP level but will **not** generate long-run growth.

- However, **endogenous** increases in productivity (v.g, generated by learning-by-doing) **may** generate long run growth (but in this case there is not transition dynamics). Learning-by-doing generates a **reproduction** mechanism.
Questions

- Are those conclusions robust to changes in the preferences between consumption and fertility?
- Are those conclusions robust to changes in the technology? In particular are they robust to the existence of biased technical change?
- Solving the problem set may provide answers to those questions.
References

- Original work: Malthus (1798)
- Textbook: (Galor, 2011, ch 2, 3)

Appendix
Solving a linear ODE’s

▶ The linear ODE

\[ \dot{x} = \lambda (x(t) - \bar{x}) \]

has an exact solution

\[ x(t) = \bar{x} + (x(0) - \bar{x}) e^{\lambda t} \]

where \( \lambda \) is an arbitrary constant

▶ The initial value problem

\[
\begin{aligned}
\dot{x} &= \lambda (x(t) - \bar{x}) \\
x(0) &= x_0 \text{ given}
\end{aligned}
\]

has the exact solution

\[ x(t) = \bar{x} + (x_0 - \bar{x}) e^{\lambda t} \]
Appendix

The linear and Bernoulli ODE’s

The Bernoulli equation is

\[ \dot{x} = \alpha x(t) - \beta x(t)^\eta \]

If we set \( z(t) = x(t)^{1-\eta} \) and differentiate

\[
\dot{z} = (1 - \eta) x(t)^{-\eta} \dot{x} = \\
= (1 - \eta) x(t)^{-\eta} (\alpha x(t) - \beta x^\eta) = \\
= \lambda (z(t) - \bar{z})
\]

is a linear ODE with solution with \( \lambda = (1 - \eta)\alpha \) and \( \bar{z} = \frac{\beta}{\alpha} \)

\[ z(t) = \bar{z} + (z(0) - \bar{z}) e^{\alpha(1-\eta)t} \]

transforming back by making \( x(t) = z(t)^{1\over 1-\eta} \)

\[ x(t) = \left( \frac{\beta}{\alpha} + \left( x(0)^{1-\eta} - \frac{\beta}{\alpha} \right) e^{\alpha(1-\eta)t} \right)^{1\over 1-\eta} \] (2)
Appendix
Approximating a Bernoulli ODE by a linear ODE

▶ Assume we have the Bernoulli ODE

\[
\dot{x} = G(x) \equiv \alpha x - \beta x^n = x (\alpha - \beta x^{n-1}) \tag{3}
\]

▶ The derivative of \( G(x) \) is

\[
G'(x) = \alpha - \eta \beta x^{n-1}.
\]

▶ It has two steady states, \( \beta/\alpha > 0 \): \( x^{ss} = \{0, \bar{x}\} \) for \( \bar{x} = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{n-1}} \).

▶ The derivative of \( G(x) \) evaluated at the steady state \( \bar{x} \) is

\[
\lambda \equiv G'(x) \bigg|_{x=\bar{x}} = \alpha \left( 1 - \eta \right).
\]
Appendix

Approximating a Bernoulli ODE by a linear ODE (cont.)

- We approximate the ODE (3) in the neighborhood of the steady state \( \bar{x} \) by

\[
\dot{x} = \lambda (x - \bar{x})
\]

- which has solution

\[
x(t) = \bar{x} + (x_0 - \bar{x}) e^{\lambda t} \tag{4}
\]

This approximate solution can be compared with the exact solution (2).

- For the Malthusian model we can see that they are pretty close (thick for (2) and dashed for (4))

![Graph showing a comparison between the approximate and exact solutions]