The Malthusian growth model

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Malthusian economics

- Popular definition of **Malthusian** (Malthus (1798)): population grows exponentially and food grows linearly
- This would lead either to **catastrophe** or to the existence of a natural (not nice) **endogenous stabilization mechanism**, in the absence of "moral restraint"
- The existence of that endogenous mechanism, relating population and wages, is consistent with the economic history in pre-industrial W. Europe, in particular after the Black Death (1346-1353)

Wages and population in historical data

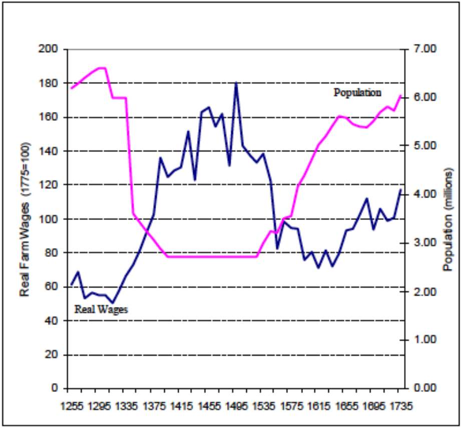


Figure 5: Population and Real Wages in England, 1250-1750 CE
(Source: Clark, 2005)

Malthusian theory

- We will see that the existence of **decreasing marginal returns** to labor is a necessary (although not sufficient) condition.
- The idea that the existence of a fixed resource and decreasing returns to production implies that **growth processes eventually stop** is present in most Classical economists (Quesnay, Smith, Ricardo, Marx) and, possibly, in modern ecologists.
- But it was Thomas Malthus who stated it more clearly in An Essay on the Principle of Population (1798) and systematically gathered data to sustain it.
- We next provide a modern view of the theory

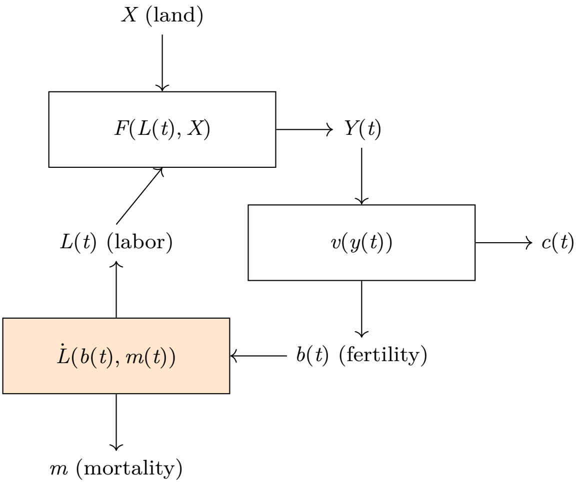
A modern view on the Malthusian model

The general idea:

- It presents the joint dynamics of production and population growth
- For pre-industrial societies: there are two main factors of production labor and land
- Labor is the **reproducible** factor of production (no capital accumulation, no R&D)
- The basic dynamic mechanism is: increase in income leads to increase in population and in labor supply; this increases aggregate income, but income per capita does not increase at the same pace, leading eventually to a steady state (positive extensive effect but negative intensive effect).
- **Decreasing marginal returns for the reproducible factor** is the main driving force behind the non-existence of growth in the long run.
- Even with exogenous technical progress there is no growth. The conditions for the existence of long run growth are very specific (learning-by-doing)

Assumptions

- **Production**:
  - production uses two factors: labor and land
  - the production function has constant returns to scale
  - the only reproducible factor is labor, and it faces decreasing marginal returns
- **Population**:
  - fertility is endogenous and mortality is exogenous
- **Farmers**:
  - households are land-owners
  - they choose among consumption and child-rearing
  - there is no saving



where $v(y) = \max\{u(c, b) : c + pb \leq y\}$ and $y = \frac{Y}{L}$

The model

Production

- **Production function**: we assume a Cobb-Douglas production

$$Y(t) = (AX)^{\alpha} L(t)^{1-\alpha},\ 0 < \alpha < 1$$

  where: A productivity, X stock of land, L labor input

- Property 1: constant returns to scale

$$(\lambda AX)^{\alpha} (\lambda L)^{1-\alpha} = \lambda Y$$

- implication: the Euler theorem holds

$$Y = \frac{\partial Y}{\partial L} L + \frac{\partial Y}{\partial X} X$$

The model

Production technology

- Property 2: positive marginal returns for labor and land

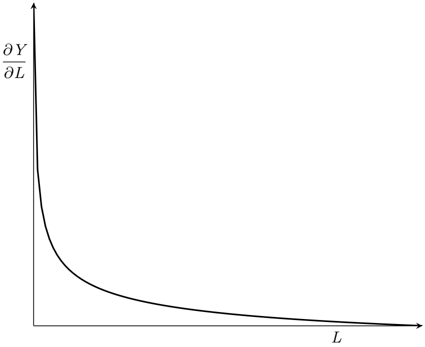
$$\frac{\partial Y}{\partial L} = (1-\alpha)\frac{Y}{L} > 0, \ \frac{\partial Y}{\partial X} = \alpha\frac{Y}{X} > 0$$

- Property 3: Inada production function: $\lim_{L\to 0} \dfrac{\partial Y}{\partial L} = \infty$ and $\lim_{L\to\infty} \dfrac{\partial Y}{\partial L} = 0$
- Implication: no bias in technical change (why ?)

$$MRS_{L,X} = \frac{(1-\alpha)\,X}{\alpha\,L}$$

The model

Inada property regarding the marginal productivity of labor



The model

Production technology

- Property 4: decreasing marginal returns for both factors

$$\frac{\partial^2 Y}{\partial L^2} = -\alpha(1-\alpha)\frac{Y}{L^2} < 0, \frac{\partial^2 Y}{\partial X^2} = -\alpha(1-\alpha)\frac{Y}{X^2} < 0$$

- Property 5: the two factors are gross (or Edgeworth) complements

$$\frac{\partial^2 Y}{\partial X \partial L} = \alpha(1-\alpha)\frac{Y}{L\,X} > 0$$

- Property 6: the technology is concave (but not strictly concave)

$$\frac{\partial^2 Y}{\partial L^2}\frac{\partial^2 Y}{\partial X^2} - \left(\frac{\partial^2 Y}{\partial X \partial L}\right)^2 = 0$$

- Implication: (1) the AU elasticities are constant

$$\varepsilon_{LL} = \alpha,\ \varepsilon_{XX} = 1-\alpha,\ \varepsilon_{LX} = -\alpha$$

- and (2) we already known that the elasticity of substitution is equal to one: $ES_{LX} = 1$

The model

Production efficiency

- Optimal allocation of factors, or production efficiency, in a market economy:

$$\max_{L,X}\{Y(L,X) - wL - RX\}$$

where w is the wage rate and R are is land rent

- and competitive markets lead to

$$w(L,X) = \frac{\partial Y}{\partial L} = (1-\alpha)\frac{Y}{L} > 0$$
$$R(L,X) = \frac{\partial Y}{\partial X} = \alpha\frac{Y}{X} > 0$$

- Factors are Hicksian substitutables

$w_L < 0, w_X > 0$ wages decrease with labor and increase with land;

$R_X < 0, R_L > 0$ rents decrease with land and increase with labor.

Farmers' problem

Endogenous rate of population growth

- There are L farmers; who receive (percapita) income from farming and decide which part to consume and which part to allocate to raising offspring, by deciding the number of offspring (Beckerian model)
- **Household's (farmer's) static problem** (for every $t \geq 0$)

$$\max_{c(t),b(t)} \{c(t)^{1-\psi}\, b(t)^{\psi} :\ c(t) + p\, b(t) = y(t)\}$$

$0 < \psi < 1$ (relative) love for children, $1/\psi =$ "moral restraint"
$p > 0$ relative cost of raising children

- solution

$$c(t) = (1-\psi)y(t) \text{ (consumption increases with income)}$$

$$b(t) = \frac{\psi}{p}y(t) \text{ (number of children increases with income)}$$

Population dynamics

- **Population growth**

$$\dot{L} \equiv \frac{dL(t)}{dt} = (b(t) - m)L(t)$$

- where the fertility rate is endogenous: $b(t) = \frac{\psi}{p} y(t)$
- the mortality rate is exogenous: m is given
- the initial level of population is assumed to be given by number L_0

$$L(t)|_{t=0} = L(0) = L_0$$

The Malthusian model

Endogenous rate of population growth

- Then

$$\dot{L} = \left(\frac{\psi}{p} y(t) - m\right) L(t), \text{ for } t \in [0, \infty)$$
$$L(0) = L_0 \text{given}$$

- where the per capita GDP is

$$y(t) \equiv \frac{Y(t)}{L(t)} = \left(\frac{AX}{L(t)}\right)^{\alpha}$$

Detour

Per-capita rate of growth arithmetics

- taking log-derivatives w.r.t time we have

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L}$$

- that we denote by

$$g(t) = g_Y(t) - n(t)$$

- as the per capita GDP is

$$y(t) \equiv \frac{Y(t)}{L(t)} = \left(\frac{AX}{L(t)}\right)^{\alpha}$$

- Then: **the rate of growth is exactly negatively correlated to the rate of growth of population**

$$g(t) = \frac{\dot{y}}{y} = -\alpha \frac{\dot{L}}{L}$$

Take home

Remember:

- We are interested in what this model can tell us about economic growth
- For us growth is related to the dynamics of GDP per capita

$$y(t) = \frac{Y(t)}{L(t)}$$

- we want to know the implications for:
  - the rate of growth of GDP $g(t) = \frac{\dot{y}(t)}{y(t)}$
  - the steady state level of GDP $\bar{y}$
  - and the dynamics: i.e. separating $g(t)$ into transition and long-run components

Solving the Malthusian model

There are two approaches to **solving the model**

- Approach 0: do a geometric representation of the solution of the model (phase diagram)
- Approach 1: solve the differential equation for L and substitute in y to get the dynamics of growth
- Approach 2: obtain a differential equation for y and solve it

The model

The phase diagram: geometric intuition

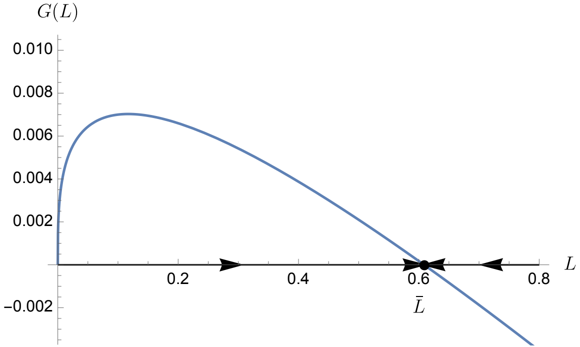


Figure: Phase diagram for $\dot{L} = m\left(\left(\frac{\bar{L}}{L}\right)^{\alpha} - 1\right)L$

Solving the Malthusian model

Approach 1: solving for L

- If we substitute y in the dynamic equation for L we have the initial value problem

$$\begin{cases} \dot{L} = \left(\frac{\psi}{p} \left(\frac{A\,X}{L(t)} \right)^{\alpha} - m \right) L(t), & t \geq 0 \\ L(0) = L_0 \text{given} & t = 0 \end{cases}$$

- we can solve it to get

$$L(t) = \left(\bar{L}^{\alpha} + \left(L_0^{\alpha} - \bar{L}^{\alpha} \right) e^{-m\alpha\, t} \right)^{\frac{1}{\alpha}}$$

where the steady state population is

$$\bar{L} = \left(\frac{\psi}{m\,p} \right)^{\frac{1}{\alpha}} A\,X$$

Solving the Malthusian model

Approach 2: solving for y directly

- From

$$\frac{\dot{y}}{y} = -\alpha \frac{\dot{L}}{L}$$

- we obtain the dynamic equation for the GDP per capita

$$\dot{y} = -\alpha \left(\frac{\psi}{p} y(t) - m \right) y(t) \tag{1}$$

together with the initial value

$$y(0) = y_0 = (AX)^{\alpha} L_0^{1-\alpha}$$

Solving the Malthusian model

Explicit solution for y

- Equation (1) has two steady states $y^* = \{0, \bar{y}\}$ where

$$\bar{y} = \frac{mp}{\psi}$$

- we can re-write the growth equation as

$$\dot{y} = \alpha \frac{\psi}{p} \left(\bar{y} - y(t) \right) y(t)$$

Explicit solution for y

- This is a Bernoulli differential equation. Therefore, it has an explicit solution appendix

$$y(t) = \left[\frac{1}{\bar{y}} + \left(\frac{1}{y(0)} - \frac{1}{\bar{y}}\right) e^{-\alpha m t}\right]^{-1}, \text{for } 0 \leq t < \infty$$
$$= \frac{\bar{y}}{1 + \left(\frac{\bar{y}}{y_0} - 1\right) e^{-\alpha\, m\, t}}$$

- satisfies $\lim_{t\to\infty} y(t) = \bar{y}$

Explicit solution for g

- the GDP growth rate is

$$g(t) = \frac{dy(t)}{dt} = \alpha m \frac{\left(\bar{y} - y_0\right) e^{-\alpha\, m\, t}}{y_0 + \left(\bar{y} - y_0\right) e^{-\alpha\, m\, t}}, \text{ for } 0 \leq t < \infty$$

- if $y_0 \neq \bar{y}$ then $g(0) = \alpha\, m \left(\bar{y} - y_0\right)/\bar{y}$ positive or negative depending on $\bar{y} - y_0$ and $\lim_{t\to\infty} g(t) = 0$.

Malthusian model

Properties

1. there is **no long run growth**, because $\lim_{t\to\infty} g(t) = 0$
2. the **long run level** of GDP per capita is

$$\bar{y} = \frac{mp}{\psi}$$

   increases with the mortality rate, the cost or rearing children and the "moral restraint" (no productivity effects)

3. there is **only transitional dynamics** (i.e., adjustments towards the steady state):
   - if the initial GDP is small, $y(0) < \bar{y}$, then there is an increase in time of the GDP $g(t) > 0$
   - if the initial GDP $y(0)$ is large, $y(0) > \bar{y}$, then there is an decrease in time of the GDP $g(t) < 0$

Malthusian model

Mechanics of the model

- if $y(0)$ is large so is the wage rate $w(0) = (1 - \alpha)y(0)$
- this implies that the initial fertility rate is higher, $b(0) = \frac{\psi}{p}y(0)$
- population increases, which increases output,
- but decreases the rate of growth of GDP

$$g(t) = -\alpha n(t)$$

because there are decreasing marginal returns due to the fact that X is fixed.

Malthusian model

Trajectories: y, L and w

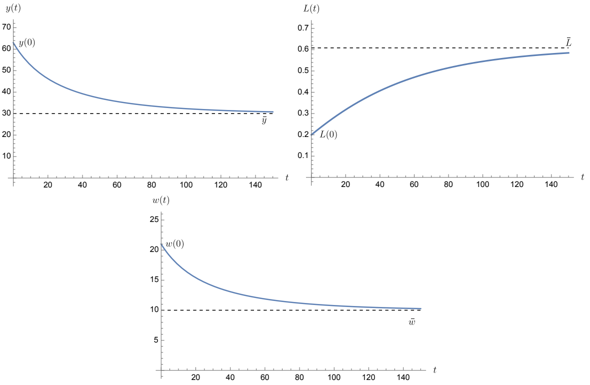


Figure: Parameter values: $\alpha = 2/3$, $m = 0.03$, $\psi = 0.01$, $p = 10$, $A = 1$, $X = 100$, and $y(0) < \bar{y}$

Malthusian model

Exponential increase in land productivity

Can increases in land-productivity generate long-run growth ?

- now $Y(t) = (A(t)\, X)^{\alpha} L(t)^{1-\alpha}$
- where $\dot{A} = g_A A,\ g_A > 0$
- Taking logarithmic derivatives of the production function, this implies

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{A}}{A} + (1-\alpha)\frac{\dot{L}}{L} - \frac{\dot{L}}{L} = \alpha \left(m + g_A - \frac{\psi}{p} y(t) \right)$$

- there is **no increase in the long run growth rate** (why ?)

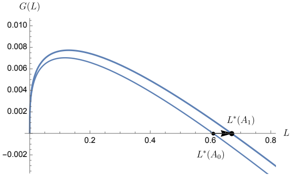
$$\lim_{t \to \infty} g(t) = 0$$

- there is **an increase in GDP level**

$$\bar{y} = \frac{(g_A + m)p}{\psi}$$

Malthusian model and land productivity

Phase diagram for an increase in the rate of growth of A



Malthusian model

Exponential increase in labor productivity

Can an increase in the productivity of labor generate long-run growth ?

- now $Y(t) = X^{\alpha}(h(t)\, L(t))^{1-\alpha}$
- where $\dot{h} = g_h h,\ g_h > 0$
- Taking logarithmic derivatives of the production function, this implies

$$\frac{\dot{y}}{y} = (1-\alpha)\left(\frac{\dot{h}}{h} + \frac{\dot{L}}{L}\right) - \frac{\dot{L}}{L} = \alpha\left[\frac{(1-\alpha)}{\alpha} g_h + m - \frac{\psi}{p} y(t)\right] y(t)$$

- there is **no increase in the long run growth rate** (why ?)

$$\lim_{t\to\infty} g(t) = 0$$

- there is **an increase in GDP level**

$$\bar{y} = \frac{((1-\alpha) g_h + \alpha m) p}{\alpha \psi}$$

Malthusian model

Learning by doing

Can learning by doing generate long-run growth ?

- now $Y(t) = (A(t)X)^{\alpha} L(t)^{1-\alpha}$
- learning-by-doing: past production generates knowledge which increases land productivity
- Formally: $A(t) = \beta \int_{-\infty}^{t} e^{-\mu(t-s)} A(s) y(s)\, ds$
  where β reproduction of knowledge, μ rate of oblivion
- taking derivatives for t (Leibniz formula)

$$\dot{A} \equiv \frac{dA(t)}{dt} = (\beta y(t) - \mu)\, A(t)$$

- the dynamic equation for per-capita GDP becomes

$$\frac{\dot{y}}{y} = \left(\beta - \alpha \frac{\psi}{p}\right) y(t) + \alpha m - \mu$$

Malthusian model

Learning by doing: continuation

- If we assume $\beta = \alpha \dfrac{\psi}{p}$ (meaning $\varepsilon_{LL} \times \frac{b}{y}$) then

$$\dot{y} = (\alpha m - \mu) y$$

- there is **long run growth** if $\alpha m > \mu$ because

$$g(t) = \alpha m - \mu > 0, \text{for all } t > 0$$

- the GDP level is exogenous

$$y(t) = y_0 \, e^{(\alpha m - \mu) t}$$

Conclusions

- The existence of decreasing marginal returns to the reproducible factor of production (labor, L) implies that the Malthusian model does not feature long-run growth: **there is only transitional dynamics** (if initial population is too high, wages will be too low, which generates a fall in fertility and therefore a decrease in population until population is constant)
- **exogenous permanent increases** in productivity will only increase the long-run GDP **level** but will **not** generate long-run growth
- however, **endogenous** increases in productivity (v.g, generated by learning-by-doing) **may** generate long run growth (but in this case there is not transition dynamics). Learning-by-doing generates a **reproduction** mechanism.

Questions

- Are those conclusions robust to changes in the preferences between consumption and fertility ?
- Are those conclusions robust to changes in the technology ? In particular are they robust to the existence of biased technical change ?
- Solving the problem set may provide answers to those questions.

References

- Original work: Malthus (1798)
- Textbook: (Galor, 2011, ch 2, 3)
- Population economics: Razin and Sadka (1995)

Oded Galor. *Unified Growth Theory*. Princeton University Press, 2011.

Thomas R. Malthus. *An Essay on the Principle of Population*. W. Pickering, 1798. 1986.

Assaf Razin and Efraim Sadka. *Population Economics*. MIT Press, 1995.

Appendix

Solving a linear ODE's

- The linear ODE

$$\dot{x} = \lambda(x(t) - \bar{x})$$

has an exact solution

$$x(t) = \bar{x} + (x(0) - \bar{x})e^{\lambda t}$$

where k is an arbitrary constant

- The initial value problem

$$\begin{cases} \dot{x} = \lambda(x(t) - \bar{x}) \\ x(0) = x_0 \text{ given} \end{cases}$$

has the exact solution

$$x(t) = \bar{x} + (x_0 - \bar{x})e^{\lambda t}$$

Appendix

The linear and Bernoulli ODE's

- The Bernoulli equation is

$$\dot{x} = \alpha x(t) - \beta x(t)^{\eta}$$

- If we set $z(t) = x(t)^{1-\eta}$ and differentiate

$$\begin{aligned}\dot{z} &= (1-\eta)x(t)^{-\eta}\dot{x} = \\ &= (1-\eta)x(t)^{-\eta}\left(\alpha x(t) - \beta x^{\eta}\right) = \\ &= \lambda\left(z(t) - \bar{z}\right)\end{aligned}$$

- is a linear ODE with solution with $\lambda = (1-\eta)\alpha$ and $\bar{z} = \dfrac{\beta}{\alpha}$

$$z(t) = \bar{z} + \left(z(0) - \bar{z}\right) e^{\alpha(1-\eta)t}$$

- transforming back by making $x(t) = z(t)^{\frac{1}{1-\eta}}$

$$x(t) = \left(\frac{\beta}{\alpha} + \left(x(0)^{1-\eta} - \frac{\beta}{\alpha}\right) e^{\alpha(1-\eta)t}\right)^{\frac{1}{1-\eta}} \tag{2}$$

Appendix

Approximating a Bernoulli ODE by a linear ODE

- Assume we have the Bernoulli ODE

$$\dot{x} = G(x) \equiv \alpha x - \beta x^{\eta} = x\left(\alpha - \beta\, x^{\eta-1}\right) \tag{3}$$

- The derivative of $G(x)$ is

$$G'(x) = \alpha - \eta\, \beta\, x^{\eta-1}.$$

- It has two steady states, $\beta/\alpha > 0$: $x^{ss} = \{0, \bar{x}\}$ for $\bar{x} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\eta-1}}$.
- The derivative of $G(x)$ evaluated at the steady state $\bar{x}$ is

$$\lambda \equiv G'(x)\Big|_{x=\bar{x}} = \alpha\,(1-\eta).$$

Appendix

Approximating a Bernoulli ODE by a linear ODE (cont.)

- We approximate the ODE (3) in the neighborhood of the steady state $\bar{x}$ by

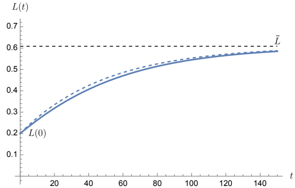
$$\dot{x} = \lambda \left(x - \bar{x} \right)$$

- which has solution

$$x(t) = \bar{x} + \left(x_0 - \bar{x} \right) e^{\lambda t} \tag{4}$$

  This approximate solution can be compared with the exact solution (2).

- For the Malthusian model we can see that they are pretty close (thick for (2) and dashed for (4))



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