The Solow growth model

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Background

- Most European nations were industrialized in the dawn of the XX century, and the main driver of growth was the accumulation of capital (both physical and financial)
- After the WWII: definition of the ideia of the GDP and first Statistics Agencies to measure it (see a nice history of the concept Coyle (2014)
- First "stylized facts" (covering a short time span) appeared: v.g. Kaldor's stylized facts
- The Solow (1956) paper tried to explain some of those facts
- At a time in which the "Keynesian" model (ISLM) was the state of the art
- Most economic growth theory and empirics takes this models as a reference point.
- Robert Solow was awarded the Nobel Prize in 1987

Kaldor's stylized facts (1963)

Fact K1 per capita GDP (y) grows along time, and its rate of growth shows no decreasing tendency (debatable: for mature countries);

Fact K2 the stock of capital (K) grows along time;

Fact K3 r (r.o.r of capital) is roughly constant (debatable: it shows a slightly downward tendency for most developing countries);

Fact K4 the ratio K/Y is roughly constant;

Fact K5 the shares of capital and labor in the aggregate income are approximately constant (debatable: this is not the case after the early 1980's) ;

Fact K6 the growth rate of the gdp per capita (y) varies substantially across countries.

Solow (1956) model

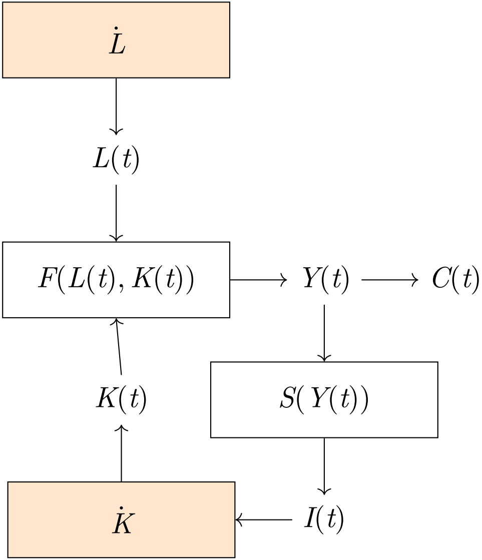
Structure of the economy

- Environment:
  - closed economy producing a single composite good
  - there is only one reproducible factor: capital
  - there are no idle factors (no unemployment)
- Population:
  - exogenous
- Growth engine: capital accumulation

Solow (1956) model

Assumptions

- Production:
  - production uses two factors: labor and physical capital
  - production technology: neoclassical (increasing, concave, Inada, CRS)
- Households: add-hoc behaviour
  - inelastic supply labor (labor supply independent from wage)
  - ad-hoc savings (mechanically proportional to income)
  - static expectations (no anticipations)
- There is macroeconomic consistency (market clearing), but not necessarily microeconomic consistency (decisions on labor supply, consumption, savings and asset transactions are not necessarily consistent)


$\dot{L}$
$L(t)$
$F(L(t), K(t))$
$Y(t)$
$C(t)$
$K(t)$
$S(Y(t))$
$\dot{K}$
$I(t)$

Solow model

The model: production technology

- **Neo-classical production function** (see slide toolkit)

$$Y(t) = F(A, K(t), L(t)) = AK(t)^{\alpha}L(t)^{1-\alpha},\ 0 < \alpha < 1$$

  where: A productivity, K stock of capital, L labor input

- properties
  - constant returns to scale (CRS)
  - increasing in both factors: $\nabla F(K, L) = (F_K, F_L)^{\top} > \mathbf{0}$
  - concave in (K, L)
  - Inada

$$\lim_{K \to 0} F_K(K, L) = \lim_{L \to 0} F_K(K, L) = +\infty$$

$$\lim_{K \to \infty} F_K(K, L) = \lim_{L \to \infty} F_K(K, L) = 0$$

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The model: factor demand and distribution

Implications:

1. Inverse factor demand functions
   - the demand K is such that the rate of return of capital equals the marginal productivity of capital

$$r(t) = F_K(K, L) = \alpha \frac{Y(t)}{K(t)}$$

   - the demand L is such that the wage rate equals the marginal productivity of labor

$$w(t) = F_L(K, L) = (1-\alpha) \frac{Y(t)}{L(t)}$$

2. from CRS and Euler's theorem the distribution of income is

$$Y(t) = r(t)K(t) + w(t)L(t)$$

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The model: factor dynamics

- **Population growth**

$$\dot{N}(t) = nN(t)$$

n is the exogenous rate of growth

- **No unemployment** (or demand and supply of labor)

$$L(t) = N(t)F$$

- **Capital accumulation**

$$\dot{K} = I(t) - \delta K(t)$$

net investment =gross investment - capital depreciation

$\delta > 0$ rate of depreciation of capital

Solow model: labour market

Consumption and investment

- **"Keynesian" consumption function**

$$C(t) = (1 - s)\,Y(t)$$

  $0 < s < 1$ is the marginal propensity to consume

- **savings decisions**

$$S(t) = s\,Y(t)$$

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Macroeconomic equilibrium

- **Equilibrium in the product market**

$$Y(t) = C(t) + I(t)$$

  aggregate supply = aggregate demand

- By Walras's law we could also "close the model" by using the **equilibrium in the capital market**

$$S(t) = I(t)$$

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GDP per capita

- The per capita GDP is

$$y(t) \equiv \frac{Y(t)}{N(t)}$$

- taking log-derivatives w.r.t time we have

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{N}}{N} \Leftrightarrow g(t) = g_Y(t) - n(t)$$

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The model: the rate of growth

- The per capita GDP is

$$y(t) \equiv \frac{Y(t)}{N(t)} = A\left(\frac{K(t)}{N(t)}\right)^{\alpha} = Ak(t)^{\alpha}$$

defining the capital intensity by

$$k \equiv \frac{K}{L} = \frac{K}{N}$$

- Then

$$\boxed{g(t) = \frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} = \alpha g_k(t)}$$

- the rate of growth is a linear function of the rate of growth of the capital intensity
- but the ratio between the two is less than one

$$\frac{g(y)}{g_k(t)} = \alpha \in (0,1)$$

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The capital accumulation equation

- the dynamic equations of the model are

$$\begin{cases} \dot{K} & = sAK^{\alpha}N^{1-\alpha} - \delta K \\ \dot{N} & = nN \end{cases}$$

- using the definition of capital intensity, k, we obtain the famous **Solow growth equation**

$$\boxed{\begin{cases} \dot{k} = sAk^{\alpha} - (n+\delta)k & \text{for } t \geq 0 \\ k(0) = k_0, & \text{for } t = 0 \end{cases}}$$

Solow model

The growth equation

- As

$$\dot{y} = \alpha \frac{\dot{k}}{k} y$$

- the dynamics equation for the per capita GDP is

$$\boxed{\begin{cases} \dot{y} = \alpha \left(sA^{\frac{1}{\alpha}} y(t)^{1-\frac{1}{\alpha}} - (n+\delta) \right) y(t) & \text{for } t \geq 0 \\ y(0) = y_0 = Ak_0^{\alpha}, & \text{for } t = 0 \end{cases}}$$

- We can solve the model for k or for y (or both)

Solow model

Solving the model

There are **several approaches** for solving the model, i.e, finding a function $k(t)$ (or $y(t)$)

1. We can solve it by **linearization** in the neighborhood of the steady state(s)
2. Sometimes, we can solve it **explicitly** (because it is a Bernoulli ODE)
3. We can solve it **numerically** (see python notebook)
4. It is always a good idea to have a **geometric** illustration of the model (if it has a low dimension)

Solow model

First method: linear approximation of k

- Write the Solow accumulation equation as

$$\dot{k} = G(k) = s\,A\,k^{\alpha} - (n+\delta)k$$

- We start by determining the **steady state(s)**:
  $k^{*} = \{k \geq 0 : G(k) = 0\} = \{0, \bar{k}\}$ where

$$\bar{k} = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

- We consider the positive steady state $\bar{k}$, and take the variations $\Delta k(t) = k(t) - \bar{k}$
- We performing a first-order Taylor approximation in the neighborhood of $\bar{k}$

$$\frac{d\Delta k(t)}{dt} = \frac{dG}{dk}(\bar{k})\,\Delta k(t)$$

Solow model

First method: linear approximation of k

- The approximated (linearized) capital accumulation equation is

$$\dot{k} = \lambda \left(k(t) - \bar{k}\right)$$

where the coefficient is

$$\lambda = \frac{dG}{dk}(\bar{k}) = \alpha\, s\, A\, \bar{k}^{\alpha-1} - (n+\delta) = -(1-\alpha)\,(n+\delta) < 0$$

- Given $k(0) = k_0$ is known, then **the approximate solution** is

$$\boxed{k(t) = \bar{k} + \left(k_0 - \bar{k}\right) e^{\lambda\, t},\ \text{for } t \in [0, \infty)}$$

Solow model

Second method: exact solution for k

- The explicit (exact) solution is proof

$$\boxed{k(t) = \left[\bar{k}^{1-\alpha} + \left(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\right) e^{\lambda t}\right]^{\frac{1}{1-\alpha}}, \ t \in [0, \infty)}$$

where

$$\lambda \equiv -(1-\alpha)(n+\delta) < 0$$

- The growth rate of the capital intensity is

$$g_k(t) = -(n+\delta)\left(\frac{\left(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\right) e^{\lambda t}}{\bar{k}^{1-\alpha} + \left(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\right) e^{\lambda t}}\right)$$

Solow model

Properties of the solution

1. The solution is continuous in k_0

$$k(0) = k(t|t=0) = k_0$$

2. If $k_0 > 0$, $k(t)$ converges asymptoticaly to $\bar{k}$

$$\lim_{t\to\infty} k(t) = \bar{k}$$

   independently of the initial value k_0.

3. Equivalently

$$\boxed{\lim_{t\to\infty} g_k(t) = 0 \text{ because } \lim_{t\to\infty} e^{\lambda t} = 0}$$

   Meaning: **the stock of capital converges to a steady state; there is no long-run growth**

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Mechanics of the model

- We can write Solow's equation as

$$g_k(t) = \frac{\dot{k}}{k} = \frac{s}{\alpha} r(k(t)) - (n + \delta), \text{ for any } t \geq 0$$

- low $k(0)$ means $r(0)$ is high relative to $n + \delta$
- this implies high incentive for saving and for accumulating capital
- but **capital accumulation decreases the marginal productivity of capital** because $r_k(k) = \frac{\partial r(k)}{\partial k} < 0$, which reduces progressively the incentives to accumulate capital
- this process will eliminate asymptotically the incentives to accumulate capital
- notice that in the long run capital increases just to cover $(n + \delta)$

Solow model

Mechanics

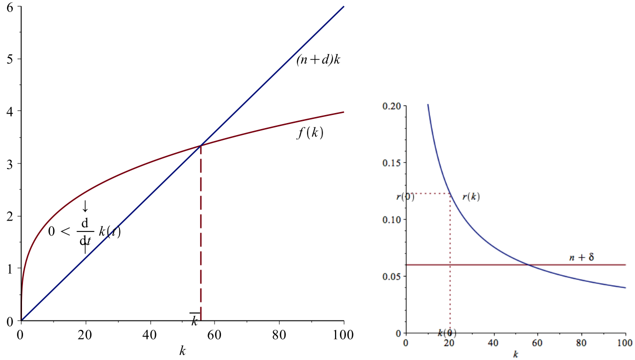

Figure: If $k(0) < \bar{k}$ ($k(0) > \bar{k}$) then capital will increase (decrease) and converge to $\bar{k}$ asymptotically

Solow model

Explicit solution for y

- Because $y(t) = Ak(t)^{\alpha}$ and

$$\bar{y} = A\bar{k}^{\alpha} = A\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

- then the GDP per capita varies along time according to

$$\boxed{y(t) = \left[\bar{y}^{\frac{1-\alpha}{\alpha}} + \left(y_0^{\frac{1-\alpha}{\alpha}} - \bar{y}^{\frac{1-\alpha}{\alpha}}\right)e^{\lambda t}\right]^{\frac{\alpha}{1-\alpha}},\ t \in [0,\infty)}$$

where

$$\lambda \equiv -(1-\alpha)(n+\delta) < 0$$

Solow model

Implications for growth

The **implication for growth** are:

- there is no long run growth, if $y(0) = y_0 > 0$ then

$$\lim_{t \to \infty} y(t) = \bar{y} \Rightarrow \lim_{t \to \infty} g(t) = 0, \text{ for any } y(0)$$

- the long run level of GDP per capita $\bar{y}$:

$$\bar{y} = A\bar{k}^{\alpha} = \left(A \left(\frac{s}{n+\delta} \right)^{\alpha} \right)^{\frac{1}{1-\alpha}}$$

therefore

$$\bar{y} = \bar{y}(\overset{+}{A}, \overset{+}{s}, \overset{-}{n}, \overset{-}{\delta})$$

The GDP level: is higher for A, and s higher, and is smaller for n and δ higher

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Implications for growth

Another **implication for growth** is:

- only transitional dynamics exists, driven by $\lambda = -(1-\alpha)(n+\delta)$, i.e. it is due to the existence of **decreasing marginal returns to the accumulating factor** k (i.e, $0 < \alpha < 1$)
- meaning:
  - the growth rate is positive if the economy is below its long run level, i.e. $y(t) < \bar{y}$, given the values of A, s, δ and n
  - the growth rate is negative if the economy is above its long run level, i.e. $y(t) > \bar{y}$, given the values of A, s, δ and n
- **Convergence issue**: countries may have different growth rates because they are differently away from their own steady state levels (absolute β-convergence) and/or because they are converging to differently steady state levels (relative β-convergence)

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Criticisms

1. A zero long-run rate of growth is **counterfactual** for industrialised economies since the Industrial Revolution (early XIX century)
2. In general, capital accumulation can display **dynamic inefficiency**, i.e $\overline{k} > k^{\text{gr}}$ where

$$k^{\text{gr}} = \text{argmax}_k\{c(k) = Ak^{\alpha} - (n+\delta)k\} = \left(\frac{\alpha A}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

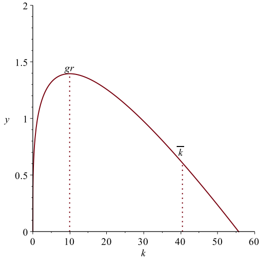

Figure: The golden rule and the steady state $\overline{k}$

Solow model

Response to criticisms

1. We can consider an extension of the Solow model with technical progress taking the form of an **increasing trend in productivity**
2. Inefficiency is related to the lack of an efficiency criterium in the decision over savings. This is the reason the **Ramsey and modern DGE** models became the benchmark in growth theory

Solow model

Extension: exogenous productivity growth

- Consider the production function

$$Y(t) = A(t)K(t)^{\alpha}L(t)^{1-\alpha},\ 0 < \alpha < 1$$

- and assume there is exogenous TFP growth

$$A(t) = A_0 e^{g_A t},\ g_A > 0$$

- What are the growth consequences ?

Solow model

Extension: exogenous productivity growth

- Because

$$y(t) = A(t)k(t)^{\alpha}$$

- then

$$g(t) = g_A + \alpha g_k(t)$$

- as $\lim_{t\to\infty} g_k(t) = 0$ then

$$\lim_{t\to\infty} g(t) = g_A > 0$$

- There is long run growth (I.e. $g(\infty) > 0$) but it has an **exogenous** nature: this model **describes** but does **not explain** long run growth.

References

- Solow (1956)
- (Acemoglu, 2009, ch. 2 and 3) , (Aghion and Howitt, 2009, ch. 1), (Barro and Sala-i-Martin, 2004, ch. 1)
- Problem set

Daron Acemoglu. *Introduction to Modern Economic Growth.* Princeton University Press, 2009.

Philippe Aghion and Peter Howitt. *The Economics of Growth.* MIT Press, 2009.

Robert J. Barro and Xavier Sala-i-Martin. *Economic Growth.* MIT Press, 2nd edition, 2004.

Diane Coyle. *GDP: A Brief but Affectionate History.* Princeton University Press, 2014.

Robert Solow. A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70(1):65–94, 1956.

Appendix

Explicit solution of the Solow model

- We can re-write the capital accumulation equation as

$$\dot{k} = (n+\delta)\left(\left(\frac{k}{\bar{k}}\right)^{\alpha-1} - 1\right)k$$

- use the transformation $z(t) = \left(\frac{k(t)}{\bar{k}}\right)^{1-\alpha}$
- then

$$\begin{aligned}\dot{z} &= (1-\alpha)z\frac{\dot{k}}{k} = \\ &= (1-\alpha)(n+\delta)\left(\frac{1}{z} - 1\right)z\end{aligned}$$

- then we get the equivalent ODE

$$\dot{z} = (1-\alpha)(n+\delta)(1-z).$$

Appendix

Continuation

- The ODE
$$\dot{z} = (1-\alpha)(n+\delta)\,(1-z)$$

- has the solution
$$z(t) = 1 + (z(0)-1)\,e^{-(1-\alpha)(n+\delta)t}$$

- then, transforming back, $k(t) = z(t)^{\frac{1}{1-\alpha}}\,\bar{k}$, we get
$$k(t) = \bar{k}\left[1 + \left(\left(\frac{k(0)}{\bar{k}}\right)^{1-\alpha} - 1\right) e^{-(1-\alpha)(n+\delta)t}\right]^{\frac{1}{1-\alpha}}$$

main.