Toolkit for economic growth

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Topics

1. Production functions: properties
2. Production functions and input prices
3. Production functions and types of inputs
4. Production functions and technology bias
5. Extensions: continuum of inputs, multi-sector economies
6. Production functions, input accumulation and dynamics
7. Types of growth dynamics and the production function

1. Production function: properties

Production function

- Production function:

$$y = F(\mathbf{x}) \equiv F(x_1, \dots, x_n)$$

- $F(\cdot)$ formalizes a **technology**: a transformation of **inputs** into **outputs**;
- $y =$ output of one good (can be used in consumption and/or investment)
- $\mathbf{x} = (x_1, \dots, x_n)$ bundle of inputs, it is a vector if we assume a discrete index set $\{1, \dots, n\}$
- $x_i =$ quantity of input i in production (intermediate good or final good)
- Most important general features of the technology: **marginal variations**, and **homogeneity**
  - at the input level (x_i): necessity, marginal variation, substitutability/complementarity
  - at the bundle level $(\mathbf{x})$t: increasing/constant/decreasing returns to scale

Production: general properties

- Definition: F **is weakly increasing** if $\mathbf{x}^* \geq \mathbf{x} \Rightarrow F(\mathbf{x}^*) \geq F(\mathbf{x})$

  Meaning: a general ncrease in the quantity of inputs leads to an increase in production (bigger bundle increases output)

  Derivative:

  $$F_i(\mathbf{x}) \equiv \frac{\partial F(\mathbf{x})}{\partial x_i} = \lim_{\epsilon \to 0} \frac{F(\ldots, x_i + \epsilon, \ldots) - F(\ldots, x_i, \ldots)}{\epsilon}$$

  Gradient: vector of first derivatives

  $$DF(\mathbf{x}) = \left(\frac{\partial F(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial F(\mathbf{x})}{\partial x_n} \right)^\top$$

- if $DF(\mathbf{x}) \geq 0$ then F is weakly increasing

Production: general properties

- Definition: F **is concave** if given any two bundles $\mathbf{x}$ and $\mathbf{x}^*$

$$F(\mathbf{x}) - F(\mathbf{x}^*) \leq DF(\mathbf{x}^*) \cdot (\mathbf{x} - \mathbf{x}^*)$$

  (strictly concave if $<$)

  Meaning: increases in the quantity of inputs increases production less than linearly

  Hessian: matrix of second derivatives

$$D^2F(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 F(\mathbf{x})}{\partial x_1^2} & \dots & \frac{\partial^2 F(\mathbf{x})}{\partial x_1 \partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial^2 F(\mathbf{x})}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 F(\mathbf{x})}{\partial x_n^2} \end{pmatrix}$$

- If $F(\mathbf{x})$ is concave then $D^2F(\mathbf{x})$ is negative semi-definite:

  that is: the principal minors of odd order are negative and the principal minors of even order are positive

Production: general properties

- Definition: F **is homogeneous of degree** η if changing $\mathbf{x}$ by a factor λ, where λ is a positive number, changes output by a factor of λ^η

$$\lambda^\eta F(\mathbf{x}) = F(\lambda \mathbf{x}^*)$$

  where $\lambda \mathbf{x}^* = (\lambda x_1, \ldots \lambda x_n)^\top$

  η measures the **returns to scale**

- We say the production function displays
  - **decreasing returns to scale** (DRS) if $\eta < 1$
  - **constant returns to scale** (CRS) if $\eta = 1$
  - **increasing returns to scale** (IRS) if $\eta > 1$

- Euler's theorem,

$$\eta F(\mathbf{x}) = DF(\mathbf{x}) \cdot \mathbf{x} = \sum_{i=1}^{n} \frac{\partial F(\mathbf{x})}{\partial x_i} x_i$$

- A fundamental requirement for the existence of growth is that $F(\cdot)$ is linearly homogeneous, that is $\eta = 1$

Production function: specific properties

- Necessity: input x_i is **necessary** if output is equal to zero if it is not used in production

$$x_i = 0 \Rightarrow F(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots x_n) = 0$$

- Variation in production: measured by the differential

$$dy = DF(\mathbf{x}) \cdot d\mathbf{x} = \sum_{i=1}^{n} F_i(\mathbf{x})\, dx_i$$

- **Marginal product (or productivity)**:

$$MP_i = F_i = \frac{\partial F(\mathbf{x})}{\partial x_i}$$

meaning: variation in production if input i in increased by one unit

$$dy = \sum_{i=1}^{n} F_i(\mathbf{x})\, dx_i = F_i, \text{if } d\mathbf{x} = (0, \ldots, 0, 1, 0, \ldots, 0, \ldots 0)$$

Production function: specific properties

- We say input i is **productive** if $F_i(\mathbf{x}) > 0$ for a particular value of $\mathbf{x}$
- If $F(\mathbf{x})$ is strictly increasing then all inputs are productive.
- **Inada property** if $MP_i \in (0, \infty)$

$$\lim_{x_i \to 0} F_i(\mathbf{x}) = +\infty, \text{ and } \lim_{x_i \to \infty} F_i(\mathbf{x}) = 0$$

- A technology is non-Inada if there are bounds (superior or inferior) in the MP of any input (or $MP =$ constant)
- Productiveness can be a **global** (as in Inada case) or a **local** property: in the second case, it is possible to have bundles $\mathbf{x}'$ or $\mathbf{x}''$ such that $F_i(\mathbf{x}') = 0$ or $F_i(\mathbf{x}'') < 0$ (saturation and congestion)

Production function: technology

- Change in the marginal product:

$$MP_{ij} = F_{ij} = \frac{\partial^2 F(\mathbf{x})}{\partial x_i \, \partial j}, \text{ for any pair, } i.j = 1, \ldots, n$$

- The **marginal product** for input i, MP_i is
  - **decreasing** if $MP_{ii} < 0$
  - **constant** if $MP_{ii} = 0$
  - **increasing** if $MP_{ii} > 0$

  Do not confuse with marginal returns to scale

- If the $F(\cdot)$ is concave (among all other conditions) we have:

$$F_{ii}(\mathbf{x}) \leq 0 \text{ for all } i = 1, \ldots, n \text{ (decreasing MP)}$$

$$F_{ii}(\mathbf{x}) \, F_{jj}(\mathbf{x}) - F_{ij}(\mathbf{x})^2 \geq 0, \text{ for all pairs } i \neq j = 1, \ldots, n$$

- If a technology is concave then the MP_i, for all i, are non-increasing.

Production function: technology

- Allen-Uzawa elasticities

$$\text{(own elasticities) } \epsilon_{ii} = -\frac{F_{ii}(\mathbf{x})x_i}{F_i(\mathbf{x})},\ \ i = 1, \ldots, n$$

$$\text{(crossed elasticities) } \epsilon_{ij} = -\frac{F_{ij}(\mathbf{x})x_j}{F_i(\mathbf{x})},\ \ i, j = 1, \ldots, n$$

- Then regarding "own" elasticities
  - If MP_i is constant then the "own" elasticity is equal to zero
  - If MP_i is decreasing then the "own" elasticity is positive
- **Gross or Edgeworth elasticities**:
  - inputs i and j are **substitutes** if $\epsilon_{ij} > 0$ $(F_{ij} < 0)$
  - inputs i and j are **independent** if $\epsilon_{ij} = 0$ $(F_{ij} = 0)$
  - inputs i and j are **complementar** if $\epsilon_{ij} < 0$ $(F_{ij} > 0)$
- do not confuse with Hicksian substitutability which is evaluated from the change of demand as regards input prices

Production function: technology

- **Compensated changes** in two inputs: variations in inputs i and j such that the output is constant

$$dy = F_i(\mathbf{x})\,dx_i + F_j(\mathbf{x})\,dx_j = 0$$

- **Marginal rate of substitution** between inputs i and j: is the compensated change in j for a unit change in i

$$MRS_{ij} \equiv -\left.\frac{dx_j}{dx_i}\right|_{dy=0}$$

- Then: the MRS is equal to the ratio of marginal products

$$MRS_{ij} = \frac{F_i(\mathbf{x})}{F_j(\mathbf{x})},\ \ i \neq j = 1, \ldots, n$$

Production function: technology

- Elasticity of substitution between inputs i and j is defined by

$$ES_{ij} \equiv \left. \frac{d \ln (x_j / x_i)}{d \ln MRS_{ij}(\mathbf{x})} \right|_{dy=0}$$

  meaning it the rate of growth in the (compensated) ratio x_j/x_i relative to the rate of growth of the MRS_{ij}

- then

$$ES_{ij}(\mathbf{x}) = \frac{x_i\, F_i(\mathbf{x}) + x_j\, F_j(\mathbf{x})}{x_j\, F_j(\mathbf{x})\, \epsilon_{ii}(\mathbf{x}) - 2x_i\, F_i(\mathbf{x})\, \epsilon_{ij}(\mathbf{x}) + x_i\, F_i(\mathbf{x})\, \epsilon_{jj}(\mathbf{x})}$$

Production function: the benchmark case

The benchmark production function is the **generalized mean**

$$y = F(\mathbf{x}) = M_\sigma(\mathbf{x}) = \left(\sum_{i=1}^{n} \alpha_i\, x_i^\sigma\right)^{\frac{1}{\sigma}},\ \mathbf{x} \in \mathbb{R}_+^n$$

where $\sigma \in [-\infty, \infty]$ is a parameter, and α_i is the share of input i satisfying

$$0 \leq \alpha_i \leq 1$$
$$\sum_{i=1}^{n} \alpha_i = 1$$

- We readily see that it is an homogeneous function, for any value of σ

$$M_\sigma(\lambda\mathbf{x}) = \lambda M_\sigma(\mathbf{x})$$

thus it represents a **constant returns to scale technology**.

Production function: the benchmark case

Applying our previous concepts we find:

- Marginal product of input i: all inputs are productive

$$MP_i(\mathbf{x}) = F_i(\mathbf{x}) = \alpha_i \left(\frac{F(\mathbf{x})}{x_i} \right)^{1-\sigma} \geq 0$$

- Allen-Uzawa elasticities
  - input i has **decreasing** MP_i **if** $\sigma \leq 1$

$$\epsilon_{ii} = (1-\sigma)\left(1 - \frac{\alpha_i x_i^\sigma}{F(\mathbf{x})^\sigma}\right) \geq 0 \text{ if } \sigma < 1$$

  - inputs i and j are
    - gross substitutes if $\sigma > 1$
    - gross complements if $\sigma < 1$

  because

$$\epsilon_{ij} = (\sigma - 1)\,\frac{\alpha_j x_j^\sigma}{F(\mathbf{x})^\sigma}$$

Production function: the benchmark case

Applying our previous concepts we find:

- Marginal rate of substitution depends only on the quantities of the two inputs

$$MRS_{ij}(\mathbf{x}) = \frac{\alpha_i}{\alpha_j}\left(\frac{x_j}{x_i}\right)^{1-\sigma}$$

- Elasticity of substitution is constant, for any pair of inputs

$$ES_{ij}(\mathbf{x}) = \frac{1}{1-\sigma}$$

  then i and j are

  - gross substitutes implies $ES_{ij}(\mathbf{x}) > 0$
  - gross complements implies $ES_{ij}(\mathbf{x}) < 0$

Benchmark production function: examples

For different values of σ economists call different names

1. Linear production function: if $\sigma = 1$

$$F(\mathbf{x}) = M_1 = \sum_i \alpha_i \, x_i$$

2. Cobb-Douglas production function if $\sigma = 0$

$$F(\mathbf{x}) = \prod_i x_i^{\alpha_i} \text{ for } \sum_i \alpha_i = 1$$

Proof:

$$\ln M_0 = \lim_{\sigma \to 0} \frac{\ln \sum_{i=1}^n \alpha_i \, x_i^\sigma}{\sigma} = \lim_{\sigma \to 0} \frac{\partial \Big(\ln \sum_{i=1}^n \alpha_i \, x_i^\sigma \Big)}{\partial \sigma} =$$

$$\lim_{\sigma \to 0} \frac{\sum_{i=1}^n \alpha_i \, x_i^\sigma \, \ln(x_i)}{\sum_{i=1}^n \alpha_i \, x_i^\sigma} = \sum_i \alpha_i \ln(x_i)$$

Benchmark production function: examples

3. Constant elasticity of substitution (CES)

$$F(\mathbf{x}) = M_\sigma(\mathbf{x}), \text{ if } \sigma < 1 \text{ and is finite}$$

4. $M_\infty = x_{\max} = \max\{x_1, \ldots x_n\}$
   Proof
   $\lim_{\sigma\to\infty} M_\sigma = x_{\max} \lim_{\sigma\to\infty} \sum_i \left(\alpha_i \left(\frac{x_i}{x_{\max}}\right)^\sigma\right)^{\frac{1}{\sigma}} =$
   $x_{\max} \lim_{\sigma\to\infty} \text{const}^{\frac{1}{\sigma}} = x_{\max}$ where $const \in (0,1)$
5. production function if $\sigma = -\infty$

$$F(\mathbf{x}) = M_{-\infty}(\mathbf{x}) = x_{min} = \min\{x_1, \ldots x_n\}$$

Proof:

$$M_{-\infty}(\mathbf{x}) = \frac{1}{M_\infty(1/\mathbf{x})}$$

2. Production function and input prices

Problem for a competitive firm

- **Assumption**: competitive product and input markets
- Total cost:

$$C(\mathbf{x}, \mathbf{w}) = \mathbf{w} \cdot \mathbf{x} = \sum_{i=1}^{n} w_i \, x_i$$

- Marginal cost of input i

$$MC_i = \frac{\partial C(\mathbf{x}, \mathbf{w})}{\partial x_i}$$

- Return, assuming an unit cost for the product :

$$RT = 1 \times y = F(\mathbf{x})$$

- Profit: equal to total return minus total cost

$$\pi(\mathbf{x}, \mathbf{w}) = F(\mathbf{x}) - C(\mathbf{x}, \mathbf{w}) = F(x_1, \ldots, x_n) - \sum_{i=1}^{n} w_i \, x_i$$

Input demand functions

- The firm's primal problem (simpler problem) is to maximize the profit by choosing a vector of inputs

$$\pi^*(\mathbf{w}) = \max_{\mathbf{x}} \pi(\mathbf{x}, \mathbf{w}) \text{ s.t. } F(\mathbf{x}) \leq y$$

  where y is output

- To solve it: we write the Lagrangean

$$L(\mathbf{x}, \lambda) = F(\mathbf{x}) - C(\mathbf{x}, \mathbf{w}) + \lambda\,(y - F(\mathbf{x}))$$

  where λ is the Lagrange multiplier

- Optimum conditions, for an interior solution are

$$\begin{cases} (1-\lambda)\,F_i(\mathbf{x}) = w_i, & \text{for } i = 1, \ldots, n \\ F(\mathbf{x}) = y \end{cases}$$

- If there is an interior solution we will find the (Hicksian) demand functions for all inputs

$$x^* = X_i(\mathbf{w}, y)$$

  as functions of input prices and output.

Input demand functions

For the **benchmark case**, the Inada conditions guarantee existence and uniqueness of solutions to the producer problem:

- solving the optimality condition we find **demand for input** i

$$x_i = \left(\frac{\alpha_i (1 - \lambda)}{w_i} \right)^{\frac{1}{1-\sigma}} F(\mathbf{x})$$

- substituting in the constraint yields

$$(1 - \lambda)^{\frac{1}{1-\sigma}} = \frac{1}{P(\mathbf{w})}$$

where $P(\mathbf{w})$ is a producer price index

$$P(\mathbf{w}) \equiv \left(\sum_{i=1}^{n} \alpha_i \left(\frac{w_i}{\alpha_i} \right)^{\frac{\sigma}{\sigma-1}} \right)^{\frac{1}{\sigma}}$$

Input demand functions

- The **optimal demand (Hicksian) functions** are

$$x_i^* = X_i(\mathbf{w}, y) = \left(\frac{w_i}{\alpha_i}\right)^{\frac{1}{\sigma-1}} \frac{y}{P(\mathbf{w})}, \text{ for } i = 1, \dots, n$$

  are proportional to the real output, where the deflator is the producer price index

- Comparative statics for prices: elasticities
  - for "own price" changes

$$\frac{\partial X_i}{\partial w_i}\frac{w_i}{X_i} = \frac{1}{\sigma-1}\left(1 - \frac{\alpha_i\left(\frac{w_i}{\alpha_i}\right)^{\frac{\sigma}{\sigma-1}}}{P(\mathbf{w})^{\sigma}}\right), \text{ for } i = 1, \dots, n$$

  - for "crossed price" changes

$$\frac{\partial X_i}{\partial w_j}\frac{w_i}{X_i} = -\frac{1}{\sigma-1}\frac{\alpha_i\left(\frac{w_i}{\alpha_i}\right)^{\frac{\sigma}{\sigma-1}}}{P(\mathbf{w})^{\sigma}}, \text{ for } i \neq j = 1, \dots, n$$

Input demand functions

- if $\sigma < 1$ (CES technology) then

$$\frac{\partial X_i}{\partial w_i}\frac{w_i}{X_i} < 0, \text{ and } \frac{\partial X_i}{\partial w_j}\frac{w_j}{X_i} > 0$$

  the demand **reduces** with the "own" price and **increases** with any "crossed" price, meaning that inputs i and any other input are **substitutable in the Hicksian sense** (recall they were gross complements in the Edgeworth sense)

- if $\sigma > 1$ then

$$\frac{\partial X_i}{\partial w_i}\frac{w_i}{X_i} > 0, \text{ and } \frac{\partial X_i}{\partial w_j}\frac{w_j}{X_i} < 0$$

  the demand **increases** with the "own" price and **reduces** with any "crossed" price, meaning that inputs i and any other input are **complementary in the Hicksian sense** (recall they were gross substitutable in the Edgeworth sense).

MRS and input prices

- For the benchmark case, setting $F_i(\mathbf{x}) = w_i$ we find

$$x_i^*(\mathbf{w}) = \left(\frac{w_i}{\alpha_i}\right)^{\frac{1}{\sigma-1}} F(\mathbf{x}^*)$$

- Then we have a relationship between factor demands and relative prices,

$$MRS_{ij} = \frac{F_i(\mathbf{x}^*)}{F_j(\mathbf{x}^*)} = \frac{\alpha_i}{\alpha_j}\left(\frac{x_j^*}{x_i^*}\right)^{1-\sigma} = \frac{w_i}{w_j}$$

3. Production functions: types of inputs

Types of inputs

The following distinction is important for economic growth:

- intermediary goods and factors of production
- produced inputs and non-produced inputs
- exogenous inputs and endogenous inputs
- private inputs and aggregate inputs

Intermediate goods

- in a given production function: intermediate inputs, $\mathbf{z}$, enter as flows and factors $\mathbf{k}$ enter as stocks in production functions

  $$y = F(\mathbf{z}, \mathbf{k})$$

  where

- intermediate inputs are products of other sectors and use factors of production: example $z_i = f_i(\mathbf{k})$
- usually for the final use sector they are private goods (i.e., firms pay the full price for their use)
- they can be produced in a competitive or non-competitive market

Factors of production

- in a given production function: factors enter as stocks in production functions
$$y = F(\mathbf{k})$$
- factors of production are usually exogenous to the firm
- but they can be exogenous or endogenous to the economy
- when **factors of production are produced**, their output is a flow which generates a stock-flow dynamics
$$\dot{k}_i = \frac{dk_i(t)}{dt} = G(\mathbf{k})$$
- durable goods entail necessarily a dynamic mechanism (ex: capital stock)

Factors of production

- The provision of factor of production can be internal or external to the firm
- The firm's level production function can be
$$y = f(\mathbf{k}, \mathbf{K})$$
where $\mathbf{k}$ are private factors and $\mathbf{K}$ are external factors
- their use faces different incentives:
  - if the factor of production is private the firm has to pay the price w_i for its use
  - if the factor of production is an externality the firm does not have to pay for its use
- their existence introduces a distinction between **production functions at the firm** level $y = f(\mathbf{k}, \mathbf{K})$ and at the **aggregate level**
$$y = F(\mathbf{K}) = f(\mathbf{K}, \mathbf{K})$$
- then the properties of the production function, can be different at the firm's level (related to the incentives) and at the aggregate level. For instance. Ex: DRS at the private level and display CRS at the aggregate level.

4. Production function and technological bias

Production function in input intensity form

- Production function in efficiency form

$$y = F(\mathbf{A}, \mathbf{x}) \equiv F(A_1\, x_1, \ldots, A_n\, x_n)$$

  where A_i input augmenting index measuring the specific productivity of input i,

- In growth models:
  - A_i measures specific or aggregate productivity increases
  - $A_i(t)$, where t is time measures technical progress (or decay),
  - A_i can be exogenous or endogenous (learning-by-doing, R&D)
- Here we introduce a first take on the subject

Types of technical progress

- Consider the benchmark production function in intensity form

$$y = F(\mathbf{A}, \mathbf{x}) = \left(\sum_{i=1}^{n} \alpha_i \left(A_i \, x_i \right)^{\sigma} \right)^{\frac{1}{\sigma}}$$

  A_i is the input-specific productivity

- Then the marginal product and the rate of substitution are

$$MP_i = \alpha_i \, A_i^{\sigma} \left(\frac{F(\mathbf{A}, \mathbf{x})}{x_i} \right)^{1-\sigma}$$

$$MRS_{ij} = \frac{\alpha_i}{\alpha_j} \left(\frac{A_i}{A_j} \right)^{\sigma} \left(\frac{x_j}{x_i} \right)^{1-\sigma}$$

- The effect of the technical progress on production depends on:
  - the vector $\mathbf{A}$, in particular if it is equal for all inputs: Hicks neutral technical progress $A_i = A$ for all i
  - on the substitutability properties of the production function, if $\mathbf{A}$ is heterogeneous

Types of technical progress

There are several concepts of neutrality and bias for technical progress. Here we consider

- **neutral** technical progress: if the change in any A_i leaves the MRS_{ij} unchanged
- **biased** technical progress: if the change in any A_i changes the MRS_{ij}.
- If we consider input prices

$$MRS_{ij} = \frac{\alpha_i}{\alpha_j}\left(\frac{A_i}{A_j}\right)^{\sigma}\left(\frac{x_j}{x_i}\right)^{1-\sigma} = \frac{w_i}{w_j}$$

- In the above sense
  - if $\sigma = 0$ then the technical progress is neutral in this sense
  - if $\sigma \neq 0$ then the technical progress is biased
- However, observe that the demand functions $x_i = X_i(\mathbf{A}, \mathbf{w})$ can depend on $\mathbf{A}$. If $\sigma = 0$ we can interpret this as an income effect.

Types of technical progress

- We can also write
$$\frac{\alpha_i}{\alpha_j}\left(\frac{A_i\,x_i}{A_j\,x_j}\right)^{\sigma} = \frac{w_i\,x_i}{w_j\,x_j}$$
- Assume that the expenditures in inputs (or factor shares in national income) is constant and $\sigma \neq 0$, then:
  - if the technical progress is **neutral** the ratio of the inputs remains constant;
  - if the technical progress is biased an increase in A_i/A_j the technical progress is i–**saving**, i.e, there is a reduction in its quantity; the ratio of the two inputs changes.

5.1 Extensions 1: continuum of inputs

Continuum of inputs

- Dixit-Stiglitz production functions consider a continuum of inputs which makes y a functional

$$y = F[x] = \left(\int_0^N \alpha(i)\, x(i)^\sigma \, di \right)^{\frac{1}{\sigma}}$$

- How to calculate $MP(i)$?
- we use the functional derivative

$$MP(i) = \frac{\delta F[x]}{\delta x(i)} = \alpha(i) \left(\frac{F[x]}{x(i)} \right)^{1-\sigma}$$

- all the concepts of production theory can be adapted to this case.
- in particular

$$MRS(i,j) = \frac{\alpha(i)}{\alpha(j)} \left(\frac{x(j)}{x(i)} \right)^{1-\sigma}$$

and

$$ES(i,j) = \frac{1}{1-\sigma}$$

5.2 Extensions 2: multisector economies

Multisector economies

- If we consider the existence of m production sectors, in this case we have an input-output structure

$$\mathbf{y} = \mathbf{F}(\mathbf{x})$$

where now $\mathbf{y}$ is a $(m \times 1)$ vector and $\mathbf{x}$ is a $(m \times n)$ matrix

$$\mathbf{x} = \begin{pmatrix} x_{11} & \dots & x_{1n} \\ & \dots & \\ x_{j1} & \dots & x_{jn} \\ & \dots & \\ x_{m1} & \dots & x_{mn} \end{pmatrix}$$

and

$$\begin{pmatrix} y_1 \\ \dots \\ y_m \end{pmatrix} = \begin{pmatrix} F_1(x_{11}, \dots, x_{1n}) \\ \dots \\ F_m(x_{m1}, \dots, x_{mn}) \end{pmatrix}$$

- In these economies a vector of prices is also exists, $\mathbf{p} = (p_1, \dots, p_m)^\top$
- We can extend all the previous concepts component-wise, that is for every sector

Multisector economies

- We can find the optimal allocation as a solution of the problem

$$\max_{x_{11},\dots,x_{mn}} \sum_{i=1}^{n} p_i y_j(\mathbf{x}_j) - \sum_{i=1}^{n} w_i\, x_{ji}, \text{ st} \sum_{i=1}^{n} x_{ji} = x_j$$

- under some conditions, we can find, at the optimum a relationship as

$$y_j = F^j(\mathbf{x})$$

  a supply function for every sector as a function of the aggregate input.

6. Production function and input accumulation and dynamics

Investment and savings

- Assume there is only one factor of production and there are no intermediate goods
$$y = f(x)$$
- Assume that the good produce is durable and can be used both for consumption and investment, then
$$y = c + \dot{x}$$
- Let savings be a function of x, $s = s(x) = y - c$
- Then we have a stock-flow dynamics where
$$\dot{x} = \frac{dx(t)}{dt} = s(x(t))$$
- The solution to this differential equation, gives
$$x(t) = x_0 + \int_0^t s\big(x(\tau)\big)\, d\tau$$

7. Types of growth dynamics and the production function

Growth dynamics

- From this solution we can obtain the dynamics of product from

$$y(t) = f\big(x(t)\big)$$

- We can also set directly an ODE on the GDP

$$\dot{y} \equiv \frac{dy(t)}{dt} = \mu(y)$$

- We say **there is long run growth** if the solution to this equation tends asymptotically to an exponential

$$\lim_{t\to\infty} y(t) \propto e^{\gamma t},\ \gamma > 0$$

- we will see that this requires the **technology to be linear at the aggregate level**. V.g: $y = Ak$

Growth and function $\mu(y)$

- **Long run growth only exists for a particular mathematical structure of $\mu(y)$**
  - **logistic law growth: $\mu(y) = \alpha y(\beta - y)$,**
  - **exponential law growth: $\mu(y) = \gamma y$,**
  - **power law growth: $\mu(y) = y^{\phi}$ for $\phi > 1$,**
  - **linear law growth: $\mu(y) = \mu$ for $\mu > 0$,**
- **razor edge property of growth models: although the exponential case is very particular it this the structure underlying (almost) all growth theories**

Logistic growth

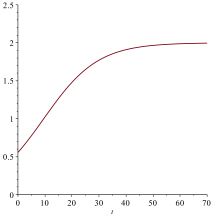

Figure: Logistic growth $\mu(y) = \alpha y(\beta - y)$

- there is short run (transition) growth
- but there is no long-run growth

Power law growth

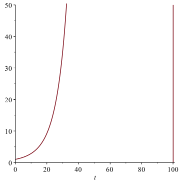

Figure: Power law growth growth $\mu(y) = y^{\phi}$ for $\phi > 0$

- GDP becomes infinite ($y(t) \to \infty$) in **infinite** time
- but the rate of growth is permanently decreasing

Exponential growth

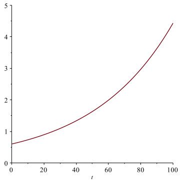

Figure: Exponential growth $\mu(y) = \gamma y$

- there is no short run (transition) growth
- but there is long-run growth
- GDP becomes infinite ($y(t) \to \infty$) in **infinite** time

Growth and transition dynamics

- In order to have both long run growth and transition dynamics we need to have at least two durable goods

$$
\begin{aligned}
\dot{x}_1 &= s_1(x_1, x_2) \\
\dot{x}_2 &= s_2(x_1, x_2)
\end{aligned}
$$

- We have long run growth if we can find a **balanced growth path**, i.e., a solution of type

$$
\begin{aligned}
x_1(t) &= \phi_1(t)\, e^{\gamma t}, \\
x_2(t) &= \phi_2(t)\, e^{\gamma t},
\end{aligned}
$$

  where γ is the long run growth rate and $g_i(\phi_1(t), \phi_2(t))$ are the transition components;

- This also requires a particular structure of the production functions: they should be CRS (at the aggregate level) for every sector.