# Economic Growth:
# Problem set 2: Malthusian models

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Disclaimer: This problem set is provided as a help to self study, in an open academic spirit of providing potentially interesting problems. Solving them is not mandatory, however it is advisable because exams' questions will be in a large part similar to some of them. The instructor does not commit himself to provide the solutions of them all, but is available to help solving specific difficulties arising in efforts to actually solving them. Not all questions have been completely verified.

## Malthusian growth theory

1. Assume that the representative consumer solves the problem: $\max_{c,b}\{u(c,b):\ c+\rho b \le y\}$ where $c$ is consumption, $b$ is the birth rate, $\rho$ is the cost of raising children and $y$ is income. Assume that the utility function is

$$u(c,b) = \frac{(cb^{\phi})^{1-\theta}}{1-\theta},\ \theta > 0,\ \phi > 0$$

and the aggregate production function is Cobb-Douglas $Y = X^{\alpha}L^{1-\alpha}$, with $0 < \alpha < 1$, where $X$ is the stock of land. Population growth is $\dot{L}/L = b - m$, where the mortality rate, $m$, is constant and exogenous, and $L(0)$ is given.

   (a) Prove that the fertility rate is $b = (\phi/(1+\phi))(y/\rho)$, and determine the dynamic equation for population growth.

   (b) Determine the steady state population level and study their dynamic behaviour.

(c) What is the effect of an increase in $m$ and $\rho$ ?

(d) Supply an economic intuition for the the previous results.

2. Assume that the representative consumer solves the problem: $\max_{c,b}\{u(c,b) : c + \rho b \leq y\}$ where $c$ is consumption, $b$ is the birth rate, $\rho$ is the cost of raising children and $y$ is income. Assume that the utility function is

$$u(c,b) = \frac{(cb^{\phi})^{1-\theta}}{1-\theta}, \ \theta > 0, \ \phi > 0$$

and the aggregate production function is Cobb-Douglas $Y = AX^{\alpha}L^{1-\alpha}$, with $0 < \alpha < 1$, where $A$ is the total factor productivity, $X$ is the stock of land. Population growth is $\dot{L}/L = b - m$, where the mortality rate, $m$, is constant and exogenous, and $L(0)$ is given. Assume that the total factor productivity grows with a constant rate such that $\alpha m > \gamma > 0$

(a) The per capita GDP in efficiency units is $y = Y/(LA)$. Write a differential equation for $y$.

(b) Study, qualitatively, the solution for that differential equation.

(c) Does long run growth exists for this economy ? Supply an economic intuition.

3. Let the dynamics of population be described by the differential equation $\dot{L} = (b - m)L$, where $L$ is total population. Assume that both the birth and the mortality rates are functions of per capita GDP, $y$, where $b = \beta y$ and $m = \mu/y$, for $\beta > 0$ and $\mu > 0$. The technology for this economy is represented by the production function $Y = X^{\alpha}L^{1-\alpha}$ where $0 < \alpha < 1$, and $X$ denoted the land endowment. The initial population is given, $L(0) = L_0$.

(a) Write an equation for the population dynamics.

(b) Determine the steady state level of population and study their dynamic behaviour.

(c) What are the effects of a reduction in the mortality rate parameter $\mu$, on population and the per capita GDP ?

(d) Does long run growth exists for this economy ? Provide an economic intuition.

4. Assume that the representative consumer solves the problem: $\max_{c,b}\{u(c,b) : c + \rho b \leq y\}$ where $c$ is consumption, $b$ is the birth rate, $\rho$ is the cost of raising children and $y$ is per capita income. Assume that the utility function is

$$u(c,b) = \ln(c) + \phi \ln(b), \ \phi > 0$$

and the aggregate production function is Cobb-Douglas $Y = (AX)^{\alpha} L^{1-\alpha}$, with $0 < \alpha < 1$, where $X$ is the stock of land, $A$ is land-specific productivity and $L$ is population. Population growth is $\dot{L}/L = b - m$, where the mortality rate, $m$, is constant and exogenous, and $L(0) = L_0 > 0$ is given. Land productivity grows at a rate $\gamma > 0$.

(a) Defining $\ell \equiv L/A$, obtain a differential equation for $\ell$.

(b) Study the qualitative dynamics of the model. Provide an intuition for your results.

(c) Derive the growth facts (long run growth rate, long run per capita output and transition dynamics). What are the effects of an increase in $\gamma$ ?

5. Assume that the representative consumer solves the problem: $\max_{c,b}\{\ln(cb^{\phi}) : c + \rho b \leq y\}$ where $c$ is consumption, $b$ is the birth rate, $\rho > 0$ is the cost of raising children, $\phi > 0$ is the love-for-children parameter, and $y$ is per capita income. The aggregate production function is CES

$$Y = \left(\alpha(AX)^{\eta} + (1-\alpha)L^{\eta}\right)^{\frac{1}{\eta}},$$

with $0 < \alpha < 1$ and $\eta > 0$, where $X$ is the stock of land, $A$ is land-specific productivity and $L$ is population. Population growth is $\dot{L}/L = b - m$, where the mortality rate, $m$, is constant and exogenous, and $L(0) = L_0 > 0$ is given. Land productivity grows at a rate $\gamma > 0$.

(a) Prove that the differential equation for the per capita product, $y \equiv Y/L$ is

$$\dot{y} = \left(1 - \alpha - y^{\eta}\right)\left(\beta y - m\right) y^{1-\eta},$$

where $\beta \equiv \frac{\phi}{\rho(1+\phi)} > 0$.

(b) Prove that if $y(0) < \min\left\{\frac{m}{\beta}, (1-\alpha)^{\frac{1}{\eta}}\right\}$ the economy will collapse, i.e. $\lim_{t\to\infty} y(t) = 0$ and if $y(0) > \min\left\{\frac{m}{\beta}, (1-\alpha)^{\frac{1}{\eta}}\right\}$ the economy will converge to $\max\left\{\frac{m}{\beta}, (1-\alpha)^{\frac{1}{\eta}}\right\}$ (Hint: draw the phase diagram)

(c) Discuss the economic intuition of those results.

6. Assume two economies $i = E, P$ which are equal in every respect, except that economy $E$ obtained an increase in its land endowment (for instance by becoming an empire). Thus $X_E > X_P$. In the two economies there is a representative farmer who solves the problem: $\max_{c,b}\{u(c,b) : c_i + \rho b_i \leq y_i\}$ where $c_i$ is consumption, $b_i$ is the birth rate, and $y_i$ is per-capita income, in country $i = E.P$, and $\rho$ is the cost of raising children. Assume that the

utility function is

$$u(c,b) = \ln\left(cb^{\phi}\right),\ \phi > 0$$

and the aggregate production function for country $i$ is Cobb-Douglas $Y_i = AX_i^{\alpha}L_i^{1-\alpha}$, with $0 < \alpha < 1$, where $A$ is the total factor productivity, $X$ is the stock of land. Population growth is $\dot{L}/L = b - m$, where the mortality rate, $m$, is constant and exogenous, and $L(0)$ is given.

(a) Write a differential equation for $y_i$.

(b) What are the growth consequences to become an empire for country $E$ ?

(c) Is that realistic ? How would you change the model in order to obtain growth effects from increasing $X_E$ ?

7. (Solve only after studying the Ramsey model)

Part 1 Consider an economy in which the aggregate production function is Cobb-Douglas $Y(t) = AX^{\beta}L(t)^{1-\beta}$, with $0 < \beta < 1$, where $A > 0$ is the total factor productivity, $X$ is the stock of land and $L(t)$ and $Y(t)$ are the level of population and the aggregate output at time $t \geq 0$. There is population growth according to the equation

$$\dot{L} = \left(b - m\right) L,$$

where the mortality rate, $m > 0$, is constant and exogenous and the fertility rate, $b$, is endogenous. The fertility rate is determined from the solution of the representative farmer problem: $\max_{c,b}\{u(c,b) :\ c + \mu b \leq y\}$ where $c$ and $y$ denote per-capita consumption and income, and $\mu$ is the unit cost of raising children. Assume that the farmer's utility function is

$$u(c,b) = \frac{\left(c\, b^{\theta}\right)^{1-\sigma} - 1}{1-\sigma},\ \theta > 0,\ \sigma > 0.$$

(a) Obtain the equation representing population dynamics.

(b) Solve (explicitly) the previous equation, assuming the initial population level $L(0) = L_0$ is given.

(c) Charaterize the behavior of per capita income in this economy.

Part 2 In the previous economy, consider that there is a benevolent monarch which wants to find an optimal path for population by using aggregate consumption, $C$, as a control variable. The utility function of the monarch is

$$\int_0^{\infty} \frac{C(t)^{1-\sigma} - 1}{1-\sigma}\, e^{-\rho t} dt$$

subject to the same law of motion for $L$, as in Part 1, such that

$$b\, L(t) = \frac{Y(t) - C(t)}{\mu}.$$

Assume again that the initial population level $L(0) = L_0$ is given and that the population is asymptotically bounded by $\lim_{t\to\infty} L(t)e^{-\rho t} \geq 0$.

(a) Obtain the MHDS (maximized Hamiltonian dynamic system) for this problem in the $(L, C)$ space, together with the initial and transversality conditions.

(b) Draw the phase diagram and discuss its properties.

(c) Charaterize the long-run behavior of per capita income in this economy. Discuss your result by comparing it with your answer to Part 1 (c).