Economic Growth Theory:
Problem set 3: Solow models

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Disclaimer: This problem set is provided as a help to self study, in an open academic spirit of providing potentially interesting problems. Solving them is not mandatory, however it is advisable because exams' questions will be in a large part similar to some of them. The instructor does not commit himself to provide the solutions of them all, but is available to help solving specific difficulties arising in efforts to actually solving them. Not all questions have been completely verified.

Solow growth theory

1. Consider a version of the Solow model, in which: (1) the savings function is $S(t) = sY(t)$, with $0 < s < 1$; (2) the population, L grows at a constant rate $n > 0$, $\dot{L} = nL(t)$, (3) there is no depreciation of capital, and (4) the technology is linear

$$Y(t) = AK(t)$$

   (a) Derive the accumulation equation for the detrended capital stock $k(t) \equiv K(t)/L(t)$.

   (b) Determine analyticaly the long run level for k, and discuss its economic meaning.

   (c) Will there be transitional dynamics in this model ?

   (d) Interpret the results for the properties of the model regarding the existence of a balanced growth path, the existence of transition dynamics, the existence of endogenous growth, and the effects of n over long run growth, transition, and the level effects.

2. Consider a version of the Solow model, in which: (1) the savings function is $S(t) = sY(t)$, with $0 < s < 1$; (2) the population, L grows at a constant rate $n > 0$, $\dot{L} = nL(t)$, (3) there is no depreciation of capital, and (4) the production technology is given by a Cobb-Douglas function

   (a) Derive the dynamic equation for the detrended output $y(t) \equiv Y(t)/L(t)$.

   (b) Solve the equation in (a) explicitly.

   (c) Solve the equation in (a) by approximation methods.

   (d) Supply an intuition for the results you obtained.

3. Consider a version of the Solow model, in which: (1) the savings function is $S(t) = sY(t)$, with $0 < s < 1$; (2) the population, L grows at a constant rate $n > 0$, $\dot{L} = nL(t)$, (3) there is no depreciation of capital, and (4) the technology is CES (constant elasticity of substitution)

$$Y(t) = F(K(t), L(t)) = \left(\alpha K(t)^{-\eta} + (1-\alpha)L(t)^{-\eta}\right)^{-1/\eta},\ 0 < \alpha < 1,\ \eta > -1,\ \eta \neq 0$$

   (a) Derive the accumulation equation for the detrended capital stock $k(t) \equiv K(t)/L(t)$.

   (b) Determine analytically the long run level for k, its stability properties, and discuss its economic meaning.

   (c) Study the effect of a permanent increase in n on the long run growth, transition, and the level of the product.

4. Consider a version of the Solow model, in which there are two types of labor: skilled L_s, and unskilled labor L_u. The proportion of population with each skill is constant, such that $\ell = L_u/L$ and $1 - \ell = L_s/L$, where $0 < \ell < 1$. The total population, L, grows at a constant rate $n > 0$. The technology of production involves a complementarity between capital and unskilled labor and a substitution between them and skilled labor. It is represented by the production function

$$Y(t) = (K(t) + L_u(t))^{\alpha}(AL_s(t))^{1-\alpha}$$

   where $0 < \alpha < 1$ and $A > 1$ measures the specific productivity of skilled labor. The savings function is $S(t) = sY(t)$, with $0 < s < 1$, and there is no depreciation of capital.

   (a) Derive the accumulation equation for the detrended capital stock $k(t) \equiv K(t)/L(t)$.

   (b) Prove there is a unique long run level for k. Is uniqueness related to the Inada properties , for k , of the production function ?

(c) Describe the properties of the model regarding the existence of a balanced growth path, of transition dynamics and of endogenous growth.

(d) Assume there is a permanent increase in the proportion of unskilled labour ℓ. Determine the effects over long run growth, the level effects, and the transitional dynamics. (Hint: assume that $\ell < \alpha$ and $s\alpha^{\alpha}(A(1-\alpha))^{1-\alpha} > n$).

5. Consider a version of the Solow (1956) model in which the production function is of the VES (variable elasticity of substitution) type

$$F(K,H) = AK^{\alpha}\Big(H + \alpha\beta K\Big)^{1-\alpha},\ A > 0,\ 0 < \alpha < 1,\ \beta > -1$$

where K is the stock of physical capital and H is the stock of human capital. Human capital is produced by means of a linear production function $dH(t)/dt = \gamma H(t)$, with $\gamma > 0$. The accumulation of physical capital is given by $dK(t)/dt = sF(H(t),K(t)) - \delta K(t)$ where $s > 0$ and $\delta > 0$.

(a) Does the production function satisfies the necessary conditions for the existence of a balanced growth path ? If your answer is affirmative, derive the expression for the physical capital along the BGP ?

(b) Define $k(t) \equiv K(t)/H(t)$ and determine the accumulation equation for k.

(c) Determine the stationary level k^*, by introducing assumptions over the parameters such that we have $k^* > 0$. With the same assumptions, characterise the local dynamics for $k(t)$.

(d) Supply an intuition for the decomposition of the variation of $K(t)$ between long run growth and transition.

6. Consider a growth model in which technological innovation takes the form of variety expansion. Assume: (1) the final good is produced by a continuum of $N(t)$ perfectly substitutable intermediate goods (i.e., varieties); (2) each variety is produced by a monopolist, although there is free entry in the markets for intermediate goods; (3) the introduction of a new variety, and therefore of a new producer, is done after R&D activities are developed. With those assumptions, we can determine: (1) the production of the final good as a function of the number of varieties

$$Y(t) = \phi N(t),\ \text{where}\ \phi \equiv \left(A\alpha^{2\alpha}\right)^{1/(1-\alpha)}$$

where $0 < \alpha < 1$ is the elasticity of substitution among varieties and $A > 0$ is the exogenous productivity parameter, in the production of the final good; (2) the aggregate income which is generated as a result of imperfect competition in the production of intermediate goods is $X(t) = \alpha^2 \phi N(t)$, and (3) the expenditure in R&D activities is $I(t) = \eta \phi \dot{N}$, where $\eta > 0$ is the average cost of development of a new variety, in units of the final good. Assume, further, that the consumption function for the final good is $C(t) = (1-s)(Y(t) - X(t))$ where $0 < s < 1$ is the savings rate. The equilibrium condition for the final good market is $Y(t) = C(t) + I(t) + X(t)$.

(a) Derive an equation for the accumulation of varieties.

(b) Determine the solution for $Y(t)$, assuming that $N(0)$ is given. Characterize the growth dynamics for this economy. Give an economic intuition for your results

7. Assuming that the Solow model is a good representation of two economies, A and B. The economies have the same technology of production and the same demographic data, but differ as regards the initial capital intensity k and the savings rate. Let the Solow accumulation equation be

$$\dot{k}_i = s_i A k_i(t)^\alpha - n k_i(t),\ i = A, B.$$

Assume that: $k_A(0) > k_B(0)$, $1 > s_B > s_A > 0$, $A > 0$, $0 < \alpha < 1$ and $n \geq 0$

(a) Characterize the differences in the growth dynamics between the two countries.

(b) Will there be convergence ? If affirmative, which kind of convergence ?

(c) Assuming there is any form of catch up, how can we measured its timing ?

8. Assume that the Solow model is a good representation of the capital accumulation dynamics for two countries, labelled by 1 and 2, respectively. Let the economies have the same preferences and the same demographic data, but differ as regards the initial capital intensity, $k_i(0)$ and the TFP. The Solow accumulation equation would be

$$\dot{k}_i = s A_i k_i(t)^\alpha - n k_i(t),\ i = 1, 2.$$

Assume that: $k_1(0) > k_2(0)$, $A_1 < A_2$, $0 < s < 1$, $0 < \alpha < 1$ and $n \geq 0$.

(a) Characterize the differences in the growth dynamics between the two countries.

(b) Will there be convergence ? If affirmative, which kind of convergence ?

(c) Assuming there is some form of catch up, provide a measure of its timing ?

9. Consider a model in which the production function is Cobb-Douglas, $Y = AK^{\alpha}L^{1-\alpha}$, where Y, K and L denote output and capital and labour inputs, respectively. Assume that $A > 0$ and $0 < \alpha < 1$. There is full employment and population grows as $\dot{L} = n\,L$ with $n > 0$. There is no capital depreciation and the consumption function is $C = \nu\,K$ where $\nu > 0$.

    (a) Derive the dynamic equation for the per-capita output $y(t) \equiv Y(t)/L(t)$.

    (b) Determine the steady state(s), study its (their) local dynamic properties.

    (c) Interpret the growth dimensions (long run rate of growth, long run level and transitional dynamics) for this model. Give an economic interpretation.

    (d) Let the productivity parameter be time-dependent, as $A(t) = A_0\,e^{\gamma_A\,t}$, for $\gamma_A > 0$. Describe how this would change your previous conclusions.

10. 1 Assume a Solow economy in which the production technology is given by

$$Y = K^{\alpha}\,S^{\beta}\,L^{1-\alpha-\beta}$$

for $0 < \alpha < 1$ and $0 < \beta < 1 - \alpha$, where K and S denote private physical capital and public infrastructures, respectively, and L denotes the labor input. The stock of public infrastructures is assumed to be constant and the initial level for the other inputs, $K(0) = k_0 > 0$ and $L(0) = l_0 > 0$ are given. The labor input is equal to the population and grows as $\dot{L} = n\,L$, for $n > 0$. The investment expenditure is $I = \dot{K} + \delta\,K$, where $\delta > 0$ is the rate of depreciation. The consumption function is $C(t) = c\,Y(t)$, where the marginal (equal to the average) propensity to consume is $0 < c < 1$.

    (a) Defining the capital intensity by $k \equiv \frac{K}{L}$, obtain the accumulation equation for k.

    (b) Characterize analytically and geometrically the solution of the accumulation equation. Solve the linearized differential equation for k.

    (c) What will be the growth and level effects, on the per-capita output, of a permanent increase in the relative stock of public infrastructures $s \equiv \frac{S}{L}$?

    2 Consider the same production function, consumption function, and population growth as in the previous question. However, now consider that the government decides to increase and repair public infrastructures, but wants to keep a balanced budget by charging an income tax. In particular: (1) the government expenditure, G, be given by $G = \dot{S} + \delta S$, where δ is the depreciation rate of public infrastructures; and (2) the fiscal rule entails: $G = \tau\,Y$ at all times, where $\tau \in [0, 1 - c)$ is the income tax rate.

(a) In this case, the dynamics of the economy is determined from the system of coupled differential equations over (k, s)

$$\dot{k} = (1 - c - \tau)\, y(k, s) - (\delta + n)\, k,$$
$$\dot{s} = \tau\, y(k, s) - (\delta + n)\, s$$

where $y = y(k, s)$ is the per-capita product. Prove this.

(b) Find the steady state of the previous system. Characterize the dynamics of the previous model (tip: draw the phase diagram). Will there be long-run growth ?

(c) What will be the effects of increasing the tax rate, τ, over the long-run per-capita product level and the ratio s/k ? Provide an intuition for your results.