Economic Growth Theory:
Problem set 3: Solow growth models
Solutions

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Problem

Consider a version of the Solow model, in which: (1) the savings function is $S(t) = sY(t)$, with $0 < s < 1$; (2) the population, L grows at a constant rate $n > 0$, $\dot{L} = nL(t)$, (3) there is no depreciation of capital, and (4) the technology is CES (constant elasticity of substitution)

$$Y(t) = F(K(t), L(t)) = \left(\alpha K(t)^{-\eta} + (1-\alpha)L(t)^{-\eta}\right)^{-1/\eta},\ 0 < \alpha < 1,\ \eta > -1,\ \eta \neq 0$$

1. Derive the accumulation equation for the detrended capital stock $k(t) \equiv K(t)/L(t)$.
2. Determine analytically the long run level for k, its stability properties, and discuss its economic meaning.
3. Study the effect of a permanent increase in n on the long run growth, transition, and the level of the product.

Solution

1. Accumulation equation: $\dot{k} = s\left(\alpha k^{-\eta} + 1 - \alpha\right)^{-\frac{1}{\eta}} - nk$

2. Steady state: $\bar{k} = \left(\frac{(s/n)^{\eta} - \alpha}{1-\alpha}\right)^{\frac{1}{\eta}}$. Stability properties: the steady state is asymptotically stable because

$$\lambda = \left.\frac{\partial \dot{k}}{\partial k}\right|_{k=\bar{k}} = -n\left(1 - \alpha(n/s)^{\eta}\right) < 0$$

3. Effect of a shock in n: (1) no effect on the long run growth rate: $\gamma = 0$; (2) negative effect on the long run level of GDP $\bar{y} = f(\bar{k}) = \left(\alpha\bar{k}^{-\eta} + 1 - \alpha\right)^{-\frac{1}{\eta}}$ because $f'(k) > 0$ and

$$\frac{\partial \bar{k}}{\partial n} = -(s/n)^{1+\eta}\left(\frac{(s/n)^{\eta} - \alpha}{1-\alpha}\right)^{\frac{1-\eta}{\eta}} < 0$$

   (3) transition effect $\frac{\partial \lambda}{\partial n} < 0$

Problem

Assume that the Solow model is a good representation of the capital accumulation dynamics for two countries, labelled by 1 and 2, respectively. Let the economies have the same preferences and the same demographic data, but differ as regards the initial capital intensity, $k_i(0)$ and the TFP. The Solow accumulation equation would be

$$\dot{k}_i = sA_i k_i(t)^{\alpha} - nk_i(t),\ i = 1, 2.$$

Assume that: $k_1(0) > k_2(0)$, $A_1 < A_2$, $0 < s < 1$, $0 < \alpha < 1$ and $n \geq 0$.

1. Characterize the differences in the growth dynamics between the two countries.
2. Will there be convergence ? If affirmative, which kind of convergence ?
3. Assuming there is some form of catch up, provide a measure of its timing ?

Solution

1. $\gamma_1 = \gamma_2 = 0$, $\bar{y}_1 < \bar{y}_2$ and $\lambda_1 = \lambda_2$ where $\bar{y}_i = \left(A_i\left(\frac{s}{n}\right)^{\alpha}\right)^{\frac{1}{1-\alpha}}$, $\lambda_i = -(1-\alpha)n$
2. $t \approx \frac{1}{(\alpha-1)n}\ln\left(\frac{\bar{k}_2 - \bar{k}_1}{\bar{k}_2 - \bar{k}_1 + k_1(0) - k_2(0)}\right)$