Economic Growth Theory:
Problem set 4: Ramsey models
Solutions

Paulo Brito

Universidade de Lisboa

Email: pbrito@iseg.ulisboa.pt

16.3.2022

Problem

Consider a version of the Ramsey model with constant population where the objective utility functional for the central planner is:

$$\max_{c} \int_0^\infty \ln\left(c(t) - \bar{c}\right) e^{-\rho t}\, dt,$$

where $\rho > 0$ and $\bar{c} > \frac{\rho}{\alpha}$ is a minimum level of consumption, subject to

$$\dot{k} = Ak(t)^\alpha - c(t),\; 0 < \alpha < 1$$

where c and k are the per capita consumption and capital stock. We also assume that $k(0) = k_0$ is given and that the stock of capital is bounded.

1. Apply the Pontryiagin's principle and determine the optimality conditions as a dynamic system in (c, k).

2. Draw the phase diagram.

3. Determine the steady states and study their local stability properties.

4. Find an approximate solution to the problem in the neighborhood of the steady state associated with a maximum consumption.

5. Determine the effects of a permanent increase in productivity, A.

Solution

1. The MHDS

$$
\begin{aligned}
\dot{c} &= (c - \bar{c})\,(r(k) - \rho) \\
\dot{k} &= Ak^{\alpha} - c \\
0 &= \lim_{t \to \infty} \frac{k(t)}{c(t) - \bar{c}} e^{-\rho t} \\
K(0) &= K_0
\end{aligned}
$$

3 Steady states: corner steady state $\left(\bar{c}, \left(\frac{\bar{c}}{A}\right)^{\frac{1}{\alpha}}\right)$ interior steady state

$$
(c^*, k^*) = \left(A\left(\frac{\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}}, \left(\frac{\alpha A}{\rho}\right)^{\frac{1}{1-\alpha}}\right)
$$

satisfying $c^* > \bar{c}$. Local dynamics for the interior steady state: eigenvalues of the Jacobian

$$
\lambda_{s,u} = \frac{\rho}{2} \pm \left(\left(\frac{\rho}{2}\right)^2 - D\right)^{\frac{1}{2}}
$$

where $D = -(\bar{c} - \bar{c})^2(1-\alpha)\rho(k^*)^{-1} < 0$. It is a saddle point.

4 Approximate solution in the neighborhood of (c^*, k^*):

$$
\begin{aligned}
c(t) &= c^* + \lambda_u(k_0 - k^*)e^{\lambda_s t} \\
k(t) &= k^* + (k_0 - k^*)e^{\lambda_s t}
\end{aligned}
$$

5 Effects of a permanent unit increase in A. Asymptotically consumption and capital increase by

$$\frac{\partial c^*}{\partial A} = \frac{\rho}{\alpha(1-\alpha)}\frac{k^*}{A} > 0$$
$$\frac{\partial k^*}{\partial A} = \frac{1}{1-\alpha}\frac{k^*}{A} > 0$$

At time $t = 0$ consumption jumps by

$$\frac{\partial c(0)}{\partial A} = \frac{\partial c^*}{\partial A} - \lambda_u \frac{\partial k^*}{\partial A} = \left(\frac{(1-\alpha)\rho + \alpha\lambda_s}{\alpha(1-\alpha)}\right)\frac{k^*}{A}$$

which is ambiguous.