Economic Growth Theory:

Problem set 5: AK and Romer models

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AK models

1. Consider an AK model with a CARA (constant absolute risk aversion) utility function. That is, consider the model:

$$\max_{(C(t))_{t\geq 0}} \int_0^\infty -\frac{e^{-\theta C(t)}}{\theta} e^{-\rho t} dt,$$

where ρ and θ are strictly positive, subject to the restriction $\dot{K} = AK(t) - C(t)$, given $K(0) = K_0 > \frac{A-\rho}{\theta A^2}$ and $\lim_{t\to\infty} e^{-At} K(t) \geq 0$.

   (a) determine the first order conditions, as a differential equation system in (C, K);

   (b) prove that the solution of the system is $C(t) = C(0) + \frac{A-\rho}{\theta} t$; $K(t) = K_0 + \frac{A-\rho}{A\theta} t$, if $C(0) = AK_0 - \frac{A-\rho}{\theta A}$;

   (c) will this model display a balanced growth path ? Discuss the properties of the model.

2. Consider a centralized economy where the supply of labor is endogenous: the representative consumer has the utility function

$$\max_{(C(t),L(t))_{t\geq 0}} \int_0^\infty \left(\ln(C(t)) - \varphi \ln(L(t))\right) e^{-\rho t} dt$$

where C is consumption, L is the labour effort in terms of time dedicated to production, the rate of time preference is positive, $\rho > 0$, and φ weights the preference for leisure as compared to consumption. Assume that $\varphi > 1$. The production function is $Y = AKL$. Therefore, the budget constraint for the economy is $\dot{K} = AKL - C$. Let $\lim_{t\to\infty} K(t)e^{-\rho t} = 0$:

(a) determine the optimality conditions as an initial-terminal value problem in (C, K);

(b) discuss the verfication of the necessary conditions for the existence of a balanced growth path;

(c) specify the model in detrended variables;

(d) determine the long run growth rate and the level variables along the BGP (hint: solve the system for $c(t)/k(t)$);

(e) discuss the implications of changes in parameters A and φ on the growth characteristics of the economy.

3. Consider a centralized economy model in which the representative consumer has the utility functional

$$\max_{(C(t))_{t\geq 0}} \int_0^\infty \ln(C(t))e^{-\rho t}dt,$$

where $\rho > 0$, subject to the restriction $\dot{K} = AK(t) - C(t)$, given $K(0) = K_0 > 0$ and $\lim_{t\to\infty} e^{-At}K(t) \geq 0$.

(a) determine the optimality conditions as initial-terminal value problem in (C, K);

(b) discuss the verification of the necessary conditions for the existence of a balanced growth path;

(c) specify the model in detrended variables, and determine the long-run (endogenous) growth rate ;

(d) prove that the solution for the detrended variables is $k(t) = K_0$ and $c(t) = \rho K_0$;

(e) Discuss the growth properties of the model, and, in particular, the implications of changes in parameter A.

4. Consider a centralized economy model in which the central planner's problem is

$$\max_{(C(t))_{t\geq 0}} \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta} L(t)\, e^{-\rho t} dt,$$

subject to the restriction $\dot{\mathbf{K}} = A\mathbf{K}(t) - C(t)L(t) - \delta\mathbf{K}(t)$, given $\mathbf{K}(0) = K_0 > 0$ and $\lim_{t\to\infty} e^{-At}\mathbf{K}(t) \geq 0$, where $\mathbf{K}(t) \equiv K(t)L(t)$ is the aggregate capital stock and $L(t)$ is total population, $C(t)$ is per-capita consumption level, and $K(t)$ is the per-capita capital stock. We assume that population grows exogenously as $L(t) = e^{nt}$, where n is the growth rate of the population. Consider the following assumptions over the parameters: $\theta > 0$ and $0 < \rho < (\theta - 1)(A - \delta) - \theta\, n$ where A is the TFP and δ is the depreciation rate of capital.

(a) Write the central planners's problem in terms of per-capita variables.

(b) Determine the optimality conditions as an initial-terminal value problem in the per-capita variables (C, K).

(c) Specify the model in (per-capita) detrended variables, and determine the long-run (endogenous) growth rate.

(d) Prove that the solution for the detrended variables is $k(t) = K_0$ and $c(t) = \beta K_0$, where $\beta \equiv \dfrac{(\theta-1)(A-\delta) - \theta n + \rho}{\theta}$.

(e) Discuss the growth properties of the model. What are the implications of an increase in the growth rate of population n ?

Solution

(a) The problem in detrended variables

$$\max_{C} \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta}\, e^{-(\rho-n)t} dt$$

subject to

$$\dot{K} = (A - \delta - n)\, K - C$$

$$K(0) = K_0$$

$$\lim_{t\to\infty} K(t) e^{-(A-n)t} \geq 0$$

(b) MHDS system in (K, C) is

$$\dot{K} = (A - \delta - n)K - C$$
$$\dot{C} = \frac{C}{\theta}(A - \delta - \rho)$$
$$K(0) = K_0 \text{ given}$$
$$0 = \lim_{t\to\infty} C(t)^{-\theta}K(t).e^{-(\rho-n)t}$$

(c) The MHDS system in detrended variables, (k, c), is

$$\dot{k} = (A - \delta - n - \gamma)k - c$$
$$\dot{c} = \frac{c}{\theta}(A - \delta - \rho - \theta\gamma)$$
$$k(0) = k_0 \text{ given}$$
$$0 = \lim_{t\to\infty} c(t)^{-\theta}k(t).e^{-\beta t}$$

where $\beta = \frac{(\theta-1)(A-\delta)-\rho}{\theta} - n$. At the steady state, we find the long-run endogenous growth rate $\bar{\gamma} = \frac{A-\delta-\rho}{\theta}$

(d) Substituting $\gamma = \bar{\gamma}$ in the the detrended MHDS we have the transition dynamics along the BGP

$$\dot{k} = \beta k - c$$
$$\dot{c} = 0$$
$$k(0) = k_0 \text{ given}$$
$$0 = \lim_{t\to\infty} c(t)^{-\theta}k(t)\, e^{-\beta t}$$

The first two ODE's have solutions $c(t) = c(0)$, which is unknown, and $k(t) = \frac{c(0)}{\beta} + \left(K_0 - \frac{c(0)}{\beta}\right)e^{\beta t}$. Defining $z(t) \equiv c(t)^{-\theta}k(t)$ the transversality condition is $\lim_{t\to\infty} z(t)e^{-\beta t} = 0$. Substituting the above solutions we find that it holds if and only if $c(0) = \beta K_0$. Therefore, the BGP is

$$\bar{K}(t) = \bar{k}e^{\bar{\gamma}t} = K_0e^{\bar{\gamma}t},\ \bar{Y}(t) = AK_0e^{\bar{\gamma}t},\ \bar{C}(t) = \beta K_0e^{\bar{\gamma}t}.$$

There is long run growth and there is no transitional dynamics.

(e) As $\bar{\gamma} = \dfrac{A - \delta - \rho}{\theta}$ and $\bar{y} = AK_0$ then n has no effects in the long run rate of growth and on the long run level of the GDP. However as $\bar{c} = \beta(n)\, AK_0$ and $\beta'(n) < 0$, then the increase in n has a dilution effect: it decreases the per capita consumption along the BGP.

5. Consider an economy in which physical and human capital are perfect substitutes in production and investment. We denote aggregate physical and human capital by K^a and H^a, respectively, and aggregate total capital by $W^a = K^a + H^a$. The production function is $Y^a = AW^a$, where Y^a is aggregate output, and the accumulation equation is $\dot{W}^a = Y^a - C^a$. We assume that total population follows the equation $N(t) = e^{nt}$ with $n > 0$. Consider a centralized economy model in where the central planner has the utility functional

$$\max_{(C^a(t))_{t\geq 0}} \int_0^\infty \frac{C^a(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt,$$

where $\theta > 1$ and $\rho > 0$, given $W^a(0) = W_0 > 0$ and $\lim_{t\to\infty} e^{-At} W^a(t) \geq 0$.

(a) Determine the optimality conditions as an initial-terminal value problem for per capita consumption and total wealth.

(b) Discuss the verification of the necessary conditions for the existence of a balanced growth path.

(c) Specify the model in detrended variables, and determine the long-run (endogenous) growth rate.

(d) Solve the planner's problem. Determine the solution for the optimal per capita output.

(e) Discuss the growth properties of the model, and, in particular, the implications of changes in parameter A.

Growth with externalities

1. Consider a closed economy in which the population is constant and $N = 1$. In this economy a single product, which is used for consumption and investment, is produced. The technology

of production is represented by the production function $Y = AK^{\alpha}\mathbf{K}^{\beta}$, where K is the private capital stock and $\mathbf{K}$ is the aggregate capital stock, where $0 < \alpha < 1$ and $\beta > 0$. The representative consumer has the utility functional

$$\int_0^{\infty} \ln\left(C(t)\right) e^{-\rho t} dt,$$

where $\rho > 0$.

(a) Assume, first, a decentralized economy.

i. Define the general equilibrium and determine the dynamic system which represents it.

ii. For different values for β, which types of long run dynamics can we get ? Under which conditions will a BGP exists ?

iii. Assuming that there is a BGP, study the effects of an exogenous change in productivity, A.

(b) Now, assume a centralized economy.

i. Define the central planner problem, which internalizes externalities, and determine the dynamic system which represents it.

ii. For different values for β, which types of long run dynamics can we get ? Under which conditions will a BGP exists ?

iii. assuming that there is a BGP study the effects of an exogenous change in productivity, A.

iv. supply an intuition for the differences with the decentralized case.

2. Consider a centralized economy in which the representative consumer has the intertemporal utility function

$$\max_{[C]_{t\geq 0}} \int_0^{\infty} \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

where $\theta > 0$ and $\rho > 0$, and the aggregate economy constraint is

$$\dot{K} = AK(t)^{\alpha} - C(t)$$

where $0 < \alpha < 1$:

(a) Do the necessary conditions for the existence of a balanced growth path, with a positive growth rate, are verified ? Justify.

(b) Consider a decentralized economy in which the problem of the representative agent is as in (a), but assume there is an externality such that $A = A_0(K^a)^\beta$, $A_0 > 0$ and K^a is the aggregate level of capital for the economy, taken as exogenous by the consumer. Obtain a representation of the general equilibrium as a dynamical system in (K, C).

(c) Under which conditions a balanced growth path exists ? Assume, from now on, the conditions you arrived at. Find the equilibrium solution for capital, output and consumption. Extract the growth facts from this model.

(d) Is that equilibrium Pareto efficient ? If not, derive the Pareto efficient growth path.

(e) Consider again a decentralized economy in which the government has two fiscal instruments, a distortionary tax/subsidy and a non-distortionary tax/subsidy, and follows a balanced budget rule at all times. Can we design a fiscal policy, with those two requirements, such that we would have a Pareto efficient economic growth ?

Solution

This is the almost ipsis verbis in slide Romer

(a) Conditions for existence of a BGP: (1) homogeneous utility as a function of C function; (2) linear constraint in K. The second condition is not satisfied.

(b) Decentralized economy equilibrium when the consistency condition holds $K^a = K$

$$\dot{K} = AK^{\alpha+\beta} - C$$
$$\dot{C} = \frac{C}{\theta}(A\alpha K^{\alpha+\beta-1} - \rho)$$
$$K(0) = K_0 \text{given}$$
$$\lim_{t\to\infty} \frac{K(t)}{C(t)^\theta} e^{-\rho t} = 0$$

(c) $\alpha + \beta = 1$. Equilibrium solutions are $K(t) = K_0 e^{\gamma_d t}$ and $C(t) = \beta_d \theta K_0 e^{\gamma_d t}$ where $\gamma_d = \frac{\alpha A - \rho}{\theta}$ and $\beta_d = \frac{A(\alpha-\theta)-\rho}{\theta}$

(d) No. If externalities we solve the centralized problem with the constraint $\dot{K} = AK - C$. We find the solution $K(t) = K_0\, e^{\gamma_c t}$ and $C(t) = \beta_c \theta K_0\, e^{\gamma_c t}$ where $\gamma_c = \frac{A-\rho}{\theta}$ and $\beta_c = \frac{A(1-\theta)-\rho}{\theta}$.

(e) We consider again the decentralized economy with constraint $\dot{K} = (1-\tau)AK - C + G$. Now the equilibrium is, because of the budget constraint, $G = \tau AK$ represented by

$$\dot{K} = AK - C$$
$$\dot{C} = \frac{C}{\theta}(A\alpha(1-\tau) - \rho)$$
$$K(0) = K_0 \text{given}$$
$$\lim_{t\to\infty} \frac{K(t)}{C(t)^{\theta}} e^{-\rho t} = 0$$

and the growth rate is $\gamma_{taxes} = \frac{\alpha(1-\tau)A-\rho}{\theta}$. We can implement a Pareto growth, i.e. $\gamma_{taxes} = \gamma_c$ if we choose τ such that $(1-\tau)\alpha = 1$.

3. Consider an economy in which there are externalities in consumption. The representative consumer has the utility functional

$$\int_0^{+\infty} \frac{1}{1-\theta} \left(C(t)^{1-\beta} (\mathbf{C}(t)^{\beta} \right)^{1-\theta} e^{-\rho t} dt,$$

where $\rho > 0$, $\theta > 0$ and $0 < \beta < 1$, C is the private consumption and $\mathbf{C}$ is the aggregate consumption. The representative agent has the instantaneous budget constraint $\dot{K} = AK - C$, and $K(0)$ is given and K is asymptotically bounded. Assume that $A > \rho$.

(a) Write down the optimality conditions for the representative agent. Justify;

(b) Introducing the micro-macro consistency condition, determine the general equilibrium of this economy as a dynamic system in (C, K).

(c) Solve the dynamic system. Discuss the growth facts we can extract from this model ?

4. Consider an economy in which the is a government which provides a public good which generates externalities in consumption. The representative consumer has the utility function

$$\int_0^{+\infty} \frac{1}{1-\theta} \left(C(t)^{1-\beta} (G(t)^{\beta} \right)^{1-\theta} e^{-\rho t} dt,$$

where $\rho > 0$, $\theta \geq 1$ and $0 < \beta < 1$, C is the private consumption and G is the quantity of the public good. The government levies income taxes, τY and keeps a balanced budget, at all times. The representative agent has the instantaneous budget constraint $\dot{K} = (1-\tau)Y - C + G$, where $Y = AK$. $K(0)$ is given and K is asymptotically bounded.

(a) Determine the optimality conditions for the representative agent, as a system in (C, K), together with the initial and terminal conditions. Justify;

(b) Determine the differential equation representation of the general equilibrium of this economy.

(c) Is there a balanced growth path for this economy ? Under which conditions may we have a positive growth rate (keep this assumptions from now on). Write the problem in the previous question in detrended variables.

(d) Characterize the growth facts about the previous economy as regards the long run growth rate, long run level of the product and the existence of transitional dynamics. Provide an intuition.

5. Assume an economy with a government. In this economy, the public expenditure generates two externalities: a consumption externality and a productive externality. The government finances public expenditures through a balanced budget raising taxes over total income, Y, with tax $\tau \in (0, 1)$. That is, the government budget constraint is $G(t) = \tau Y(t)$. Assume that we have a representative household, which determines jointly the consumption, savings and production activities. Therefore, the instantaneous budget constraint for teh agent is $\dot{K}(t) = (1-\tau)Y(t) - C(t) + G(t)$, where total income is $Y(t) = AK(t)^{\alpha}G(t)^{\eta}$, where A is a constant which aggregates both the labor input and the productivity parameter, $0 < \alpha < 1$, η can have any sign, and $K(0) = K_0$ is given. The intertemporal utility function is

$$\int_0^{+\infty} \ln\left([C(t)]^{1-\beta}[G(t)]^{\beta}\right) e^{-\rho t} dt,$$

where $\rho > 0$, and $0 < \beta < 1$.

(a) Determine the a dynamic system which represents the optimality conditions for the representative agent. Justify;

(b) Define and determine a representation for the general equilibrium of this economy, keeping τ as an exogenous parameter. Justify.

(c) Under which conditions can we have a BGP. Assume those conditions from now on. What would be the long run growth rate ?

(d) Determine the equilibrium solution for C, K and Y.

(e) Discuss the consequences of the government activity in this economy. Is there any policy which may make the general equilibrium Pareto optimal ?

6. Assume an economy with a government. In this economy, the public expenditure generates two externalities: a consumption externality and a productive externality. The government finances public expenditures through a balanced budget raising taxes over total income, Y, with tax $\tau \in (0,1)$. That is, the government budget constraint is $G(t) = \tau Y(t)$. Assume that we have a representative household, which determines jointly the consumption, savings and production activities. Therefore, the instantaneous budget constraint for the agent is $\dot{K}(t) = (1-\tau)Y(t) - C(t)$, where total income is $Y(t) = AK(t)^{\alpha}G(t)^{\eta}$, where A is a constant which aggregates both the labor input and the productivity parameter, $0 < \alpha < 1$, η can have any sign, and $K(0) = K_0$ is given. The intertemporal utility function is

$$\int_0^{+\infty} \frac{1}{1-\theta}\left([C(t)]^{1-\beta}[G(t)]^{\beta}\right)^{1-\theta} e^{-\rho t}dt,$$

where $\rho > 0$, $\theta > 0$ and $0 < \beta < 1$.

(a) Determine the a dynamic system which represents the optimality conditions for the representative agent. Justify;

(b) Define and determine a representation for the general equilibrium of this economy, keeping τ as an exogenous parameter. Justify.

(c) Under which conditions can we have a BGP. Assume those conditions from now on. What would be the long run growth rate ?

(d) Can we have indeterminacy in this economy ?

(e) Discuss the consequences of the government activity in this economy.

7. Assume an economy with a government in which the public expenditure generates a productive externality. The government finances public expenditures by a tax over total income. Then the government budget constraint is $G(t) = \tau Y(t)$, where $G(t)$ and $Y(t)$ are the levels of government expenditures and aggregate income and τ is the tax rate. Assume that we have a representative household, which determines jointly the consumption, savings and production activities. Therefore, the instantaneous budget constraint for the private agent is $\dot{K}(t) = (1-\tau)Y(t) - C(t) + G(t)$, where total income is equal to total output $Y(t) = AK(t)^{\alpha}G(t)^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$. The intertemporal utility functional is

$$\int_0^{+\infty} \ln\left(C(t)\right)e^{-\rho t}dt,$$

where $\rho > 0$. The initial level for the capital stock is given, $K(0) = K_0$, and the asymptotic value of the capital stock is bounded in present-value terms.

(a) Determine the optimality conditions for the private agent as a dynamic system in (C, K). Justify;

(b) Define and determine a representation for the dynamic general equilibrium (DGE) of this economy, keeping τ as an exogenous parameter. Justify.

(c) Under which conditions can we have a BGP ? Write the DGE in detrended variables assuming those conditions from now on. What would be the long run growth rate ?

(d) Determine the equilibrium solution for $Y(t)$.

(e) Discuss the consequences of an increase in the tax rate, τ, in this economy. Is there any policy which may make the general equilibrium Pareto optimal ?