Economic Growth Theory:

Problem set 6:

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Expanson of varieties models

1. Households' utility function $U = \int_0^\infty \ln(C(t))e^{-\rho t}dt$ displaying love for variety where $C(t)$ is a composite of $j \in [0, N(t)]$ varieties of products.

$$C(t) = \left(\int_0^{N(t)} c(t,j)^{\theta/(\theta-1)} dj \right)^{(\theta-1)/\theta}, \ \theta > 1$$

   There is one firm which produces each variety. Any firm has two activities: production and R&D. Production $y(j,t) = h(j,t)L_y(j,t)$ where $h(j,t)$ and $L_y(j,t)$ denote the productivity per worker and the number of workers employed in manufacturing, respectively, in the product of good j. From the R&D activity an increase in the stock of human capital results, and we assume that there are externalities, $\dot{h}(j) = \xi \left(h(j,t)^{1-\alpha} H(t)^\alpha \right) L_r(j,t)$, where H measures the externality and $L_r(j)$ is the number of workers in the R&D activity. We assume that the total labor force is constant, then $L = \int_0^{N(t)} L_y(j,t) + L_r(j,t) dj$. Let us assume a symmetric equilibrium in which $L_y(j) = L_y$, $L_r(j) = L_r$, and $h(j) = h$ for any j. Let us also assume that there is a central planner which maximizes the utility of the representative consumer, and internalizes the externality.

(a) If we choose L_y, and N as control variables, prove that the central planners' problem is equivalent to

$$\max_{L_y,N} \int_0^\infty \ln\left(H(t)L_y(t)N(t)^{\theta/(\theta-1)}\right) e^{-\rho t}dt$$

subject to $\dot{H} = \xi H(t)\left(LN(t)^{-1} - L_y(t)\right)$ for $H(0) = H_0$ given:

(b) write the first order conditions (as a differential equation system in (N, H);

(c) discuss the existence of a balanced growth path, write the system in detrended variables;

(d) find the long run growth rate, and discuss the effects of changes in the productivity of R& D activities (parameter ξ). Discuss your results, and the growth of the economy according to this model.

2. 1 Consider an expansion of varieties growth model for a competitive economy, with the following assumptions: (1) there is a growing mass of varieties, $j \in (0, N(t)]$, which are final consumption goods; (2) its demand is determined, at every point in time, by the representative household which tries to minimize the cost of consuming a bundle of varieties denoted by $C(t)$; (3) the change through time of the consumption bundle is determined from the solution of an intertemporal optimization problem, in which the flow of income comes from labor and the returns on equity, and savings take the form of changes in the value of firms $V(t)$: therefore $\dot{V} = w(t)L + r(t)V(t) - C(t)$; (4) labor L is in fixed quantity, and is allocated to production L_p and to research and development L_r: $L = L_p(t) + L_r(t)$; (5) the supply side is represented by an infinity of industries each one producing a variety $j \in [0, N(t)]$, in which an entrant will become a infinitely long lived monopolist; (6) the production of every variety uses labor as the only input, i.e, $y(j,t) = l(j,t)$, which implies that the aggregate allocation to production is $L_p(t) = \int_0^{N(t)} l(j,t)dj$; (7) the change in the number of varieties $\dot{N}$ can only take place if the potential entrant performs successful R&D activities, which also use labor as the only factor of production, implying that the rate of growth of varieties is $\frac{\dot{N}}{N} = \eta L_r(t)$,

where η is the productivity of R&D research labor and L_r is aggregate labor allocated to R&D; and (8) entry is determined by the free-entry condition $\eta V(t) = w(t)$, where $w(t)$ is the wage rate, and $V(t) = \int_0^{N(t)} v(j,t)dj$, where $v(j,t)$ is the value of becoming a monopolist in industry j.

From all those assumptions, we can obtain the competitive equilibrium for that economy, as depending on the path for $\big(N(t), C(t)\big)_{t\in[0,\infty)}$ which is the solution of

$$\dot{N} = \eta\, N\left(L - L_p\right),\ \text{for } L_p = C\, N^{\frac{1}{1-\theta}} \tag{1}$$

$$\dot{C} = C\left(r - \rho\right),\ \text{for } r = \eta\left(\frac{2-\theta}{\theta-1}\right) L + \eta\, L_p. \tag{2}$$

given $N(0) = n_0 > 0$, where L_p is the (endogenous) aggregate labor allocated to production and r is the real interest rate. In those equations we have the following parameters: $\theta > 1$ is the elasticity of substitution between varieties, $\eta > 0$ is the productivity of R&D labor, $\rho > 0$ is the rate of time preference, and L is the (constant and given) total population.

(a) After detrending the variables of the system, as $C(t) = c(t)e^{\gamma\, t}$, and $N(t) = n(t)e^{\gamma_n\, t}$, rewrite the system (1)-(2) in the detrended variables (n, c) (tip: find γ_n such that this system is time-independent).

(b) Find the endogenous long run growth rate, γ, the long-run ratio c/n, and the balanced growth path for C and N.

(c) The per capita output in this economy is

$$Y(t) = \frac{1}{\theta}\left(N(t)^{\frac{1}{\theta-1}} + (\theta-1)\,\frac{C(t)}{L}\right). \tag{3}$$

Find the balanced-growth path for Y. Discuss the growth features of this model, and, in particular, the growth implications for increases in θ and η.

2 Assume there is instead a central planner which tries to find a Pareto optimal growth expansion for the economy presented in Question 1. The efficient path for (C, N) can be found by solving the following problem:

$$\max_{C} \int_0^{\infty} \ln\left(C(t)\, e^{-\rho t} dt\right.$$

subject to

$$\dot{N} = \eta\, N\left(L - L_p\right),\ \text{for } L_p = C\, N^{\frac{1}{1-\theta}}$$

given $N(0) = n_0$ and $\lim_{t\to\infty} N(t)e^{-\rho t} \geq 0$.

(a) Represent the optimality conditions as a dynamic system in (N, C).

(b) After detrending the previous system, find the optimal balanced growth path for N and C.

(c) Using equation (3) find the optimal balanced-growth for per-capita output Y. Compare the growth characteristics of this centralized economy with the ones you obtained for the competitive economy in Question 1.