Foundations of Financial Economics
Two period financial markets

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Topics

- Financial assets
- Financial market
- State prices
- Portfolios
- Absence of arbitrage opportunities
- Complete financial markets
- Arbitrage pricing
- Equity premium
- Equity premium puzzle
Assume again that there is complete information regarding time $t = 0$, but incomplete information regarding time $t = 1$.

Until now we have (mostly) assumed that the probability distribution is given by nature.

But next we start assuming that information regarding time $t = 1$ is provided by the financial market.

A financial market is defined by a collection of contingent claims.

The general idea: under some conditions, we can extract an implicit probability distribution of the states of nature from the structure of the financial market.
A financial asset or contract or contingent claim: is defined by the price and payoff pair \((S_j, V_j)\) (for asset \(j\)):

- for an observed price, \(S_j\), paid at time \(t = 0\)
- a contingent payoff, \(V_j\), will be received at time \(t = 1\),

\[
V_j = (V_{j1}, \ldots, V_{js}, \ldots, V_{jN})^\top
\]
Financial assets

Returns and rates of return

- Return $R_j$ and rate of return $r_j$ of asset $j$ are related as
  \[ R_j = 1 + r_j, \]

- In the 2-period case: it is the payoff/price ratio
  \[
  R_j = \frac{V_j}{S_j} = \left( \begin{array}{c}
  \frac{V_{j,1}}{S_j} \\
  \cdots \\
  \frac{V_{j,s}}{S_j} \\
  \cdots \\
  \frac{V_{j,N}}{S_j}
  \end{array} \right) = \left( \begin{array}{c}
  1 + r_{j,1} \\
  \cdots \\
  1 + r_{j,s} \\
  \cdots \\
  1 + r_{j,N}
  \end{array} \right)
  \]
Financial assets

Timing, information and flow of funds

Two alternative ways of representing (net) income flows:

▶ as a price-payoff sequence

\[
\begin{pmatrix}
V_{j,1} \\
\ldots \\
V_{j,s} \\
\ldots \\
V_{j,N}
\end{pmatrix}
\]

\[-S_j \quad 0 \quad 1\]

▶ as an investment-return sequence

\[
\begin{pmatrix}
R_{j,1} \\
\ldots \\
R_{j,s} \\
\ldots \\
R_{j,N}
\end{pmatrix}
\]

\[-1 \quad 0 \quad 1\]
Financial assets

Statistics

- From the information, at time \( t = 0 \), on the probabilities for the states of nature at time \( t = 1 \) we can compute:

- **Expected return** for asset \( j \) at time \( t = 1 \), from the information at time \( t = 0 \)

  \[
  \mathbb{E}[R_j] = \sum_{s=1}^{N} \pi_s R_{j,s} = \pi_1 R_{j,1} + \ldots \pi_s R_{j,s} + \ldots + \pi_N R_{j,N}
  \]

- **Variance of the return** for asset \( j \) at time \( t = 1 \), from the information at time \( t = 0 \)

  \[
  \text{Var}(R_j) = \sum_{s=1}^{N} \pi_s (R_{j,s} - \mathbb{E}[R_j])^2 = \pi_1 (R_{j,1} - \mathbb{E}[R_j])^2 + \ldots \pi_N (R_{j,N} - \mathbb{E}[R_j])^2
  \]

- **Observation**: an useful relationship

  \[
  \text{Var}(R) = \mathbb{E}[R^2] - (\mathbb{E}[R])^2
  \]
Classification of assets

Risk classification

As regards **risk**:

- risk-less or risk-free asset: $V^\top = (v, \ldots, v)^\top$

- risky asset: $V^\top = (v_1, \ldots, v_N)^\top$ with at least two different elements, i.e. there are at least two elements, $v_i$ and $v_j$ such that $v_i \neq v_j$ for $i \neq j$
Classification of assets

Types of assets

Particular types of assets as regards the income flows:

- **one period bonds** with unit facial value:

  \[ S = \frac{1}{1 + i}, \quad V^\top = (1, 1, \ldots, 1)^\top \]

- **deposits**:

  \[ S = 1, \quad V^\top = (1 + i, 1 + i, \ldots, 1 + i)^\top \]

- **equity**: the payoff are dividends

  \[ S_e, \quad V_e^\top = (d_1, \ldots, d_N)^\top \]
Classification of assets

Types of assets

Derivatives:

- forward contract on an underlying asset with offered price $p$:
  \[ S_f, \ V_f^\top = (v_1 - p, \ldots, v_N - p)^\top \]

- european call option with exercise price $p$:
  \[ S_c, \ V_c^\top = (\max\{v_1 - p, 0\}, \ldots, \max\{v_N - p, 0\})^\top \]

- european put option with exercise price $p$:
  \[ S_p, \ V_p^\top = (\max\{p - v_1, 0\}, \ldots, \max\{p - v_N, 0\})^\top \]
Financial market

**Definition:** A financial market is a collection of $K$ traded assets. It can be characterized by the structure of prices and payoffs of all $K$ assets: i.e by the pair $(S, V)$:

- vector of observed **prices**, at time $t = 0$ (we use row vectors for prices)
  \[ S = (S_1, \ldots, S_K), \]

- and a matrix of contingent (i.e., uncertain) **payoffs**, at time $t = 1$
  \[ V = \begin{pmatrix} V_{11} & \cdots & V_{K1} \\ \vdots & \ddots & \vdots \\ V_{1N} & \cdots & V_{KN} \end{pmatrix} \]

- The participants observe $S$ and know $V$, but they do not know which payoff they will get for every asset (i.e, which line of $V$ will be realized)
Financial market

The payoff matrix contains the following information:

- Each column represents the payoff for asset \( j = 1, \ldots, K \)

\[
V = \begin{pmatrix}
V_{11} & \cdots & V_{j1} & \cdots & V_{K1} \\
\vdots & & \vdots & & \vdots \\
V_{1s} & \cdots & V_{js} & \cdots & V_{Ks} \\
\vdots & & \vdots & & \vdots \\
V_{1N} & \cdots & V_{jN} & \cdots & V_{KN}
\end{pmatrix}
\]

- Each row represents the outcomes (in terms of payoffs) for state of nature \( s = 1, \ldots, N \)

\[
V = \begin{pmatrix}
V_{11} & \cdots & V_{j1} & \cdots & V_{K1} \\
\vdots & & \vdots & & \vdots \\
V_{1s} & \cdots & V_{js} & \cdots & V_{Ks} \\
\vdots & & \vdots & & \vdots \\
V_{1N} & \cdots & V_{jN} & \cdots & V_{KN}
\end{pmatrix}
\]
Equivalently, we can characterize a financial market by the matrix of returns

\[
\begin{pmatrix}
R_{11} & \cdots & R_{K1} \\
\vdots & \ddots & \vdots \\
R_{1N} & \cdots & R_{KN}
\end{pmatrix}
\]

where \( R_{j,s} = \frac{V_{j,s}}{S_j} = 1 + r_{j,s} \)

Therefore

\[
\begin{pmatrix}
R_{11} & \cdots & R_{K1} \\
\vdots & \ddots & \vdots \\
R_{1N} & \cdots & R_{KN}
\end{pmatrix} = \begin{pmatrix} V \end{pmatrix} \begin{pmatrix} \text{diag}(S) \end{pmatrix}^{-1}
\]

where

\[
\begin{pmatrix}
S_1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & S_j & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & S_K
\end{pmatrix}
\]
State prices

**Implicit state valuation:** the modern approach to financial economics characterizes asset markets by the prices of the states of nature *implicit* in the relationship between present prices $S$ and future payoffs $V$, i.e.

$$Q := \{x : S = xV\}$$

- if $Q$ is unique it is a vector

$$Q = (q_1, \ldots, q_N)_{1 \times N}$$

satisfying

$$S_{1 \times K} = Q_{1 \times N} V_{N \times K} \iff S^\top_{K \times 1} = V^\top_{K \times N} Q^\top_{N \times 1}$$

- **Definition:** $Q$ is a state price vector if it is positive, i.e. $q_s > 0$ for all $s \in \{1, \ldots, N\}$
Arbitrage opportunities

**Definition:** if there is at least one \( q_s \leq 0, \ s = 1, \ldots, \ N \) then we say *there are arbitrage opportunities*

**Definition:** we say *there are no arbitrage opportunities* if \( q_s > 0, \) for all \( s = 1, \ldots, \ N \)

**Intuition:**

- existence of arbitrage (opportunities) means there are free or negatively valued states of nature
- absence of arbitrage means every state of nature is costly and therefore, positively priced.

**Proposition 1**

*Given \((S, V)\), there are no arbitrage opportunities if and only if \(Q\) is a vector of state prices.*
Completeness of a financial market

**Definition:** if $Q$ is unique, then we say markets are **complete**

**Definition:** if $Q$ is not unique then we say markets are **incomplete**

Intuition:

- **completeness:** there is an **unique** valuation of the states of nature.

- **incompleteness:** there is **not an unique** valuation of the states of nature.
Completeness of a financial market
Conditions for completeness

We can classify completeness of a market by looking at the number of assets, number of states of nature and the independence of payoffs:

1. If \( K = N \) and \( \det(V) \neq 0 \) then markets are complete and all assets are independent;
2. If \( K > N \) and \( \det(V) \neq 0 \) then markets are complete and there are \( N \) independent and \( K - N \) redundant assets;
3. If \( K < N \) or \( K \geq N \) and \( \det(V) = 0 \) then markets are incomplete.

Proposition 2

Given \((S, V)\), markets are complete if and only if \( \dim(V) = N \)

\( \dim(V) = \) number of linearly independent columns (i.e., assets)
Consider a financial market \((S, V)\), such that \(K = N\) and \(\det(V) \neq 0\).

We defined state prices from \(S = QV\).

As \(K = N\) and \(\det(V) \neq 0\) then \(V^{-1}\) exists and is unique.

Therefore, we obtain uniquely

\[
Q = \underbrace{S V^{-1}}_{1 \times N, K \times N, 1 \times K}
\]

that is

\[
(q_1, \ldots, q_N) = (S_1, \ldots, S_K) \begin{pmatrix}
V_{1,1} & \cdots & V_{1,K} \\
\cdots & \cdots & \cdots \\
V_{N,1} & \cdots & V_{N,K}
\end{pmatrix}^{-1}
\]

If \(Q \gg 0\) we say there are no arbitrage opportunities.

And because \(Q\) is unique we say markets are complete.
Alternatively, because

\[ \begin{bmatrix} V^\top & Q^\top \end{bmatrix}_{K \times N \times N \times 1} = \begin{bmatrix} S^\top \end{bmatrix}_{K \times 1} \]

If \( \det(V) \neq 0 \) then

\[ Q^\top = (V^\top)^{-1} S^\top \]

that is

\[
\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \left( \begin{array}{ccc} V_{1,1} & \cdots & V_{N,1} \\ \vdots & \ddots & \vdots \\ V_{1,K} & \cdots & V_{N,K} \end{array} \right)^{-1} \begin{pmatrix} S_1 \\ \vdots \\ S_K \end{pmatrix}
\]
As $R_{js} = V_{js}/S_j$ equivalently, we can write

$$Q_{1 \times N} R_{N \times K} = 1_{1 \times K}$$

$$(q_1, \ldots, q_N) \begin{pmatrix} R_{1,1} & \cdots & R_{1,K} \\ \cdots & \cdots & \cdots \\ R_{N,1} & \cdots & R_{N,K} \end{pmatrix} = (1 \ 1 \ 1)$$

then

$$Q = 1^\top R^{-1}$$
Characterization of a financial market

No arbitrage opportunities and completeness: alternative 3

Or, alternatively,

\[ R^\top Q^\top = 1 \]

then

\[
\begin{bmatrix}
Q^\top \\
N \times 1
\end{bmatrix}
= 
\begin{bmatrix}
(R^\top)^{-1} \\
N \times K
\end{bmatrix}
\begin{bmatrix}
1 \\
K \times 1
\end{bmatrix}
\]

matricially

\[
\begin{pmatrix}
q_1 \\
\vdots \\
q_N
\end{pmatrix} =
\begin{pmatrix}
R_{1,1} & \cdots & R_{1,K} \\
\cdots & \cdots & \cdots \\
R_{N,1} & \cdots & R_{N,K}
\end{pmatrix}^{-1}
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
\]
Characterization of a financial market
Market incompleteness

Consider a financial market \((S, V)\), such that \(K < N\), and in the partition

\[
\begin{pmatrix}
V^T_{1 \times K} & V^T_{2 \times (N-K)}
\end{pmatrix}
\]

assume that \(\det(V_1) \neq 0\)

We defined state prices from \(V^T Q^T = S^T\)

But now we have

\[
\begin{pmatrix}
V_1^T & V_2^T
\end{pmatrix}
\begin{pmatrix}
Q_1^T \\
Q_2^T
\end{pmatrix} = S^T
\]

where \(Q_1\) is \((1 \times K)\) and \(Q_2\) is \((1 \times N - K)\).
Characterization of a financial market
Market incompleteness

Then

\[ V_1^T Q_1^T + V_2^T Q_2^T = S^T \]

Because \( \det(V_1) \neq 0 \) we can make

\[ Q_1^T = (V_1^T)^{-1} (S^T - V_2^T Q_2^T) \]

There are only \( K \) independent prices: i.e, \( Q \) is indeterminate and the degree of indeterminacy is \( N - K \),

As \( Q \) is not uniquely determined (i.e, we can fix arbitrarilty \( N - K \) state prices) the market is **incomplete**
Consider a financial market \((S, V)\), such that \(K > N\), partition

\[
\begin{pmatrix}
V \\
N \times K
\end{pmatrix} = \begin{pmatrix}
V_1 \\
(N \times N)
\end{pmatrix} \begin{pmatrix}
V_2 \\
(K - N \times N)
\end{pmatrix}
\]

and assume that \(\det (V_1) \neq 0\)

We defined state prices from \(V^\top Q^\top = S^\top\)

But now we have

\[
\begin{pmatrix}
V_1^\top \\
V_2^\top
\end{pmatrix} (Q^\top) = \begin{pmatrix}
S_1^\top \\
S_2^\top
\end{pmatrix}
\]

where \(S^1\) is \((1 \times N)\) and \(S^2\) is \((1 \times K - N)\).
Characterization of a financial market
Market completeness with redundant assets

- Then

\[
\begin{aligned}
V_1^T Q^T &= S_1^T \\
V_2^T Q^T &= S_2^T
\end{aligned}
\]

- Because \( \det(V_1) \neq 0 \) we can determine \( Q \) uniquely

\[ Q^T = (V_1^T)^{-1} S_1^T \]

- Which implies that the prices of the remaining \( K - N \) assets can be obtained from the prices \( S_1 \)

\[ S_2^T = V_2^T Q^T = V_2^T ((V_1^T)^{-1} S_1^T) \]

- There are \( K - N \) redundant assets (i.e., they do not add new information on \( Q \))

- As \( Q \) is uniquely determined the market is complete
Characterization of a financial market

Suggestion on how to apply this theory

▸ First: obtain $R^T$ which is a $(K \times N)$ matrix

▸ Second: check the number of assets, $K$ and the number of states of nature $N$
  ▸ if $N < K$ then markets are incomplete
  ▸ if $N \geq K$ check the number of independent rows $n$ of $R$
    ▸ if $\dim(V) < N$ then markets are incomplete
    ▸ if $\dim(V) \geq N$ then markets are complete

▸ Third: obtaining the state prices $Q$
  ▸ if $K = N$ and $\det R \neq 0$ compute $Q^T = (R^T)^{-1}1$
  ▸ if $K > N$ and $\tilde{R}$ a $N \times N$ partition of $R$ has $\det \tilde{R} \neq 0$ then compute $Q^T = (\tilde{R}^T)^{-1}1$
  ▸ if $K = N$ but $\det R = 0$, then, solve the independent equations in $R^T Q^T = 1$
  ▸ if $K < N$ solve the independent equations in $R^T Q^T = 1$. 
Example 1

Let $S = (1, 1)$ and $V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Then

$$R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad R^\top = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Therefore

$$Q^\top = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then markets are complete but there are arbitrage opportunities.
Example 2

Let $S = (1, 1)$ and $V = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

then

$$R = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = R^\top$$

Therefore

$$Q^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$$

markets are complete and there are no arbitrage opportunities
Example 3

Let \( S = (1, 1, 2) \) and \( V = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \)

then \( R = \begin{pmatrix} 2 & 1 & \frac{3}{2} \\ 1 & 2 & \frac{3}{2} \end{pmatrix} \) and \( R^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \)

we pick all the combinations of two assets

\[
Q^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}
\]

\[
Q^\top = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}
\]

and

\[
Q^\top = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{4}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}
\]

then markets are complete, there are no arbitrage opportunities and there is one redundant asset
Example 4

Let $S = (1, 2)$ and $V = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

then $R = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ and $R^\top = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ two assets have the same returns

then $R^\top Q^\top = 1^\top$ becomes

\[
\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

has an infinite number of solutions

\[
Q^\top = \begin{pmatrix} (1 - k)/2 \\ k \end{pmatrix}
\]

for an arbitrary $k$, then markets are incomplete and if $0 < k < 1$ there are no arbitrage opportunities;
Example 5

Let \( S = (1) \) and \( V = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \)

then

\[ R = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \]

then we also have

\[ (q_1, q_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \]

or

\[ 2q_1 + q_2 = 1 \]

has an infinite number of solutions \( Q = ((1 - k)/2, k) \), for any \( k \), then markets are incomplete and if \( 0 < k < 1 \) there are no arbitrage opportunities
Implicit market probabilities: risk-free probabilities

- Let $Q = (q_1, \ldots, q_N)$ be a state-price vector.
- If there are no arbitrage opportunities then $q_s > 0$, i.e. $Q \gg 0$.
- A **Radon-Nikodym derivative** is defined as

$$\pi_s^Q = q_s / \bar{q}$$

where $\bar{q} = \sum_{s=1}^{N} q_s$.

- Therefore, if there are no arbitrage opportunities then

$$0 < \pi_s^Q < 1, \text{ and } \sum_{s=1}^{N} \pi_s^Q = 1$$

that is $\pi_s^Q$ is a **probability measure**.

- This is a probability measure implicit in the financial market $(S, V)$.

- Therefore:
  - If markets are complete the probability measure is unique.
  - If markets are incomplete the probability measure is not unique.
And asset pricing

- Using the definition for $q_s$

$$S_j = \sum_{s=1}^{N} q_s V_{sj} = \sum_{s=1}^{N} \frac{q_s}{q} V_{sj}$$

then, for any asset

$$S_j = \bar{q} \mathbb{E}_Q \left[ V_j \right]$$

- This means prices are proportional to the expected value of the payoff, using an implicit market probability.
Definition

Let $Q = (q_1, \ldots, q_N)$

Assume we have a (objective or subjective) probability distribution for the states of nature $P = (\pi_1, \ldots, \pi_N)$ such that $0 < \pi_s < 1$ and $\sum_{s=1}^{N} \pi_s = 1$

The stochastic discount factor for state $s$ is

$$m_s = \frac{q_s}{\pi_s}, \quad s = 1, \ldots, N$$

The stochastic discount factor is the random variable

$$\begin{pmatrix} m_1, \ldots, m_s, \ldots, m_N \end{pmatrix}_{1 \times N}$$
And asset pricing

- Using the definition for $q_s$

$$S_j = \sum_{s=1}^{N} q_s \ V_{sj} = \sum_{s=1}^{N} \pi_s \frac{q_s}{\pi_s} \ V_{sj}$$

then, for any asset

$$S_j = \mathbb{E} \left[ M \ V_j \right]$$

- This means that the stochastic discount factor combines both market and fundamental information.
The state price approach to asset markets

Summing up:

- The state price translates the information structure implicit in financial transactions.

- The relevant information is related to:
  - how costly is the insurance against the states of nature
  - the uniqueness of that insurance cost
  - suggests a method for pricing new assets: pricing by redundancy
Definition: A **portfolio** is a vector specifying the positions, \( \theta \), in all the assets in the market

\[
\theta^\top = (\theta_1, \ldots, \theta_K)^\top \in \mathbb{R}^K
\]

If \( \theta_j > 0 \) there is a **long** position in asset \( j \)

If \( \theta_j < 0 \) there is a **short** position in asset \( j \)
The stream of income generated by position $\theta_j$ is

\[-S_j \theta_j\]

\[
\begin{pmatrix}
\theta_j V_{j,1} \\
\vdots \\
\theta_j V_{j,s} \\
\vdots \\
\theta_j V_{j,N}
\end{pmatrix}
\]

- long position ($\theta_j > 0$): pay $S_j \theta_j$ at time $t = 0$ and receive the contingent payoff $\theta_j V_j$ at time $t = 1$
- short position ($\theta_j < 0$): receive $S_j \theta_j$ at time $t = 0$ and pay the contingent payoff $\theta_j V_j$ at time $t = 1$
Portfolios

A portfolio generates a (stochastic) stream of income \( \{ z_0^\theta, Z_1^\theta \} \)

\[
\begin{align*}
Z_0^\theta &= -C(\theta, S) = -S\theta = \sum_{j=1}^{K} S_j \theta_j \\
Z_1^\theta &= V\theta = \sum_{j=1}^{K} V_j \theta_j
\end{align*}
\]

where

\[
Z_1^\theta = \begin{pmatrix} z_{1,1}^\theta \\ \vdots \\ z_{1,s}^\theta \\ \vdots \\ z_{1,N}^\theta \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{K} V_{j,1} \theta_j \\ \vdots \\ \sum_{j=1}^{K} V_{j,s} \theta_j \\ \sum_{j=1}^{K} V_{j,N} \theta_j \end{pmatrix}
\]
The flow of income generated by a portfolio $\theta$ is:

\[ z_0^\theta \begin{pmatrix} z_{1,1}^\theta \\ \vdots \\ z_{1,s}^\theta \\ \vdots \\ z_{1,N}^\theta \end{pmatrix} \]
Proposition 3

Assume there are arbitrage opportunities. Then there exists at least one portfolio $\vartheta$ such that

$$z_0^\vartheta = 0 \text{ and } Z_1^\vartheta > 0$$

or

$$z_0^\vartheta > 0 \text{ and } Z_1^\vartheta = 0.$$ 

(Obs. A positive vector $X > 0$ has non-negative elements, and has at least one equal to zero. A strictly positive vector $X \gg 0$ has only positive elements)

Intuition

with a zero cost we can get a positive income in at least one state of nature.
If we have an initial income, we will pay a zero cost in the future, in every state of nature.
Example 1

- We consider Example 1 again: \( S = (1, 1) \) and \( V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \)

- We can build a portfolio \( \vartheta = (\vartheta_1, \vartheta_2)^\top \) such that
  \[
  z_0^\vartheta = -S\vartheta = -(\vartheta_1 + \vartheta_2) = 0, \text{ that is}
  \]
  \[
  \vartheta_1 = -x, \vartheta_2 = x
  \]

- The return at time \( t = 1 \) is
  \[
  Z_1^\vartheta = V\vartheta = \begin{pmatrix} -x + x \\ -x + 2x \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}
  \]

  then if we take a long position in the risky asset \((x > 0)\) we have a positive income at \( t = 1 \) with a zero investment.
Proposition 4

Assume there are **no arbitrage opportunities**. A portfolio with cost, \( z_0^\theta = 0 \), generates a **nonpositive** income at time 1, \( Z_1^\theta \).

- **Non-positive** \( Z_1^\theta \) means that there is at least one state of nature, \( s \) such that \( z_{1,s}^\theta < 0 \).

**Intuition:** with a zero cost although we can get a positive income in several states of nature there is **at least one state of nature** in which there is a negative income.
Example 2

- Consider the previous Example 2: \( S = (1, 1) \) and \( V = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \)

- We can build a portfolio \((\vartheta_1, \vartheta_2)\) such that \( z_0^\vartheta = -S\vartheta = 0 \):
  \[
  \vartheta_1 = -x, \quad \vartheta_2 = x
  \]

- The return at time \( t = 1 \) is
  \[
  Z_1^\vartheta = V\vartheta = \begin{pmatrix} -2x + x \\ -x + 2x \end{pmatrix} = \begin{pmatrix} -x \\ x \end{pmatrix}
  \]

  then whatever the position, long \((x > 0)\) or short \((x < 0)\), in the risky asset we will not have positive income at \( t = 1 \) (in some states of the nature we win, but we will lose in others).
Definition: Replicating portfolio: Assume we have a contingent income \( W = (w_1, \ldots, w_N)^\top \). To the portfolio, \( \vartheta \), build using the assets in the market, generating the same (contingent) income

\[
\vartheta \equiv \{ \theta : Z_1 = V\theta = W \}
\]

we call replicating portfolio. The cost of the replicating portfolio is

\[
C(\vartheta, S) = S\vartheta
\]
Arbitrage asset pricing theory (APT)

- Deals with the determination of asset prices through replication.

- APT vs GEAP (general equilibrium asset pricing):
  - in APT we take \((S, V)\) as given \((S\) is exogenous\)
  - in GEAP we take \(V\) as given and want to determine \(S\) from the fundamentals \((S\) is endogenous\)

- Both theories have three equivalent formulations:
  - using state prices \(Q\)
  - using the stochastic discount factor \(M\)
  - using market or risk-neutral probabilities \(\pi^Q\)
Proposition 5

Assume there is a financial market \((S, V)\) such that there are no arbitrage opportunities. Then, given a new asset with payoff \(V_k\) its price can be determined by redundancy by using the expression

\[ S_k = QV_k \]

where \(Q = SV^{-1} \gg 0\) is determined from the market pair \((S, V)\).

That is

\[ S_k = \sum_{s=1}^{N} q_s V_{ks} \]
Arbitrage asset pricing

Using state prices

From what we have previously seen there are two important results

Proposition 6

If markets are complete then the value, $S_k$, is unique. If markets are incomplete then $S_k$ is not unique.

Proposition 7

Assume a financial market $(S, V)$ is complete and there are $K - N$ redundant assets. Then the price of any asset is equal to the cost of their replicating portfolios built with $N$ independent assets.
Arbitrage asset pricing

Using stochastic discount factors

- If $Q$ is a state price vector and $\pi$ is the vector or probabilities, we define the stochastic discount factor (SDF) for state $s$ by

$$m_s \equiv \frac{q_s}{\pi_s}$$

**Proposition 8**

Assume there is a financial market $(S, V)$ such that there are no arbitrage opportunities. Then, given a new asset with payoff $V_k$ its price can be determined by redundancy by using the expression

$$S_k = \mathbb{E}[MV_k]$$

where $M = (m_1, \ldots, m_N)^\top$ is the stochastic discount factor.
Arbitrage asset pricing
Using stochastic discount factors

Proof: as $S_k = \sum_{s=1}^{N} q_s V_{ks}$ and using the definition of SDF we get

$$S_k = \sum_{s=1}^{N} q_s V_{ks} = \sum_{s=1}^{N} \pi_s m_s V_{ks}$$

Intuition: if there are no arbitrage opportunities the price of an asset is the expected value of the future payoffs discounted by the stochastic discount factor (which is the market discount for uncertain payoffs)
Arbitrage asset pricing
Using risk-neutral probabilities

Therefore,

\[ S_k = \sum_{s=1}^{N} q_s V_{ks} = \bar{q} \sum_{s=1}^{N} \pi_s^Q V_{ks} = \bar{q} \mathbb{E}^Q[V_k] \]

or compactly

\[ S_k = \bar{q} \mathbb{E}^Q[V_k] \]

Intuition: if there are no arbitrage opportunities there is an equivalent probability measure such that the price of an asset is proportional of the expected value of the future payoffs.

Although \( \bar{q} \) is mysterious we can calculate it a intuitive way.
Arbitrage asset pricing
Existence of risk-free asset

Proposition 9
Assume there are no arbitrage opportunities and there is a risk-free bond with price \( 1/(1 + i) \) and face value 1. Then the price of any asset verifies the relationship

\[
S_j = \frac{1}{1 + i} \mathbb{E}^Q[V_j]
\]

Exercise: prove this.
Proposition 10

Assume there are no arbitrage opportunities and there is a risk-free asset. Then there is a (market) probability measure such that the expected returns of every asset is equal to the return of the risk-free asset.

Proof. From Proposition 8 we have

\[ 1 + i = \mathbb{E}^Q[R_j], \text{ for any, } j = 1, \ldots, K \]

As this holds for any asset, therefore \( \pi^Q \) has the property

\[ 1 + i = \mathbb{E}^Q[R_1] = \ldots = \mathbb{E}^Q[R_K] \]
Application
Arbitrage and completeness with two assets

Assume that $N = 2$ and there is a risky and a risk-free asset

\[ S = \left( \frac{1}{1+i}, p \right), \quad V = \begin{pmatrix} 1 & d_1 \\ 1 & d_2 \end{pmatrix} \]

and that $r_1 < i < r_2$.

Then markets are complete and there are no arbitrage opportunities.

The return matrix is

\[ R = \begin{pmatrix} 1 & d_1/p \\ 1 & d_2/p \end{pmatrix} = \begin{pmatrix} 1 + i & 1 + r_1 \\ 1 + i & 1 + r_2 \end{pmatrix} \]
If there are risk-free assets and absence of arbitrage opportunities, then any asset can be priced by redundancy as

\[ S_k = q_1 V_{k,1} + q_2 V_{k,2} \]

where

\[ q_1 = \frac{i - r_2}{(1 + i)(r_1 - r_2)}, \quad q_2 = \frac{r_1 - i}{(1 + i)(r_1 - r_2)}. \]

Proof: assume there is a risk-free and a risky asset such that

\[ QR = 1^\top \] becomes

\[
\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 + i & 1 + i \\ 1 + r_1 & 1 + r_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

and if there are no arbitrage opportunities \( r_1 < i < r_2 \) implying \( q_1 > 0 \) and \( q_2 > 0 \).
Application

Simple Black and Scholes (1973) equation

Consider an european call option with exercise price \( p \) on an underlying asset with payoff

\[
V_{\text{underlying}} = (d_1, d_2)
\]

Then the contingent payoff of the option is

\[
V_{\text{option}} = (\max\{d_1 - p, 0\}, \max\{d_2 - p, 0\})
\]

Question: assuming there are no arbitrage opportunities and there is a risk-free asset what is the market price of the option?

Answer: the price for a call option with exercise price \( p \) is

\[
S_{\text{option}} = \frac{(r_2 - i) \max\{d_1 - p, 0\} + (r_1 - i) \max\{d_2 - p, 0\}}{(1 + i)(r_1 - r_2)}
\]

(prove this)
Equity premium

- We call **equity premium** to the difference in the rates of return between the risky and a risk-free asset

\[ r - i = \begin{pmatrix} r_1 - i \\ r_2 - i \end{pmatrix} \]

- The **Sharpe index** is defined as

\[
\frac{\mathbb{E}[r - i]}{\sigma[r]} = \frac{\sum_{s=1}^{2} \pi_s (r_s - i)}{\sqrt{\sum_{s=1}^{2} \pi_s ((r_s - i) - \mathbb{E}[r - i])^2}}
\]

where \( \sigma[r - i] = \sqrt{\mathbb{V}[r - i]} \) is the standard deviation of the equity premium (prove that \( \sigma[r - i] = \sigma[r] \)).
Risk neutral probabilities

**Proposition 11**

*If there are no arbitrage opportunities then there is a (market) risk neutral probability distribution such that the expected value of the equity risk premium is zero,*

\[ E^Q[r - i] = 0 \]

This is the reason why \( \pi^Q_s \) are called risk neutral probabilities:

\( \mathbb{P}^Q = (\pi^Q_1, \ldots \pi^Q_N) \) is a probability measure such that the expected value of the equity premium is zero.
Proof: Let

\[ R^\top = \begin{pmatrix} 1 + i & 1 + i \\ 1 + r_1 & 1 + r_2 \end{pmatrix} \]

Then, because \( R^\top Q^\top = 1 \)

\[ \begin{cases} (1 + i)q_1 + (1 + i)q_2 = 1 \\ (1 + r_1)q_1 + (1 + r_2)q_2 = 1 \end{cases} \]

Then \( (1 + r_1)q_1 + (1 + r_2)q_2 = (1 + i)q_1 + (1 + i)q_2. \)

Therefore

\[ q_1(r_1 - i) + q_2(r_2 - i) = 0 \]
Risk neutral probabilities

- Proof (cont) Define $\bar{q} = q_1 + q_2$

\[
\frac{q_1}{\bar{q}}(r_1 - i) + \frac{q_1}{\bar{q}}(r_2 - i) = 0
\]

- Define again $\pi_s^Q = \frac{q_s}{\bar{q}}$. If there are no arbitrage opportunities, then $q_s > 0$ and $\pi_s^Q > 0$. Because $\sum_{s=1}^{2} \pi_s^Q = 1$ then $\pi_s^Q$ are probabilities.

- At last

\[
\pi_1^Q(r_1 - i) + \pi_2^Q(r_2 - i) = 0
\]
The Sharpe index and the SDF

Proposition 12
Assume there are no arbitrage opportunities and there is a risk-free asset. Then the Sharpe index verifies the relationship

\[ \frac{\mathbb{E}[r - i]}{\sigma[r]} = -\rho_{r-i,m} \left( \frac{\mathbb{E}[M]}{\sigma[M]} \right)^{-1} \]

where \( \rho_{r-i,m} \) is the coefficient of correlation between the equity premium and the stochastic discount factor.
The Sharpe index and the SDF

Proof: Using our previous definition of the stochastic discount factor $m_s = q_s/\pi_s$ then

$$\mathbb{E}[M(r - \hat{r})] = 0$$

But $\mathbb{E}[M(r - \hat{r})] = Cov[M(r - \hat{r})] + \mathbb{E}[M]\mathbb{E}[r - \hat{r}]$

Using the definition of correlation

$$\rho_{r-i,m} = \frac{Cov[M(r - \hat{r})]}{\sigma[M] \sigma[r - \hat{r}]} \in (-1, 1)$$

Then $\rho_{r-i,m} \sigma[M] \sigma[r - \hat{r}] + \mathbb{E}[M]\mathbb{E}[r - \hat{r}] = 0$
The equity premium and DGE models

▶ **Intuition:** the Sharpe index is equal symmetric to the product between coefficient of variation of the stochastic discount factor and correlation coefficient between the equity premium and the stochastic discount factor, $\rho_{r-i,m}$

▶ The coefficient of variation of the stochastic discount factor contains the aggregate market value of risk.

▶ $\mathbb{E}[M]/\sigma[M]$ can be derived from simple DSGE models (as we will see next)

▶ Observe that the Sharpe index satisfies

$$\frac{\mathbb{E}[R - R^f]}{\sigma[R]}$$

where $R^f = 1 + i$ is the risk-free return and $R$ can be taken as a return on a market index. (see http://web.stanford.edu/~wfsharpe/art/sr/SR.htm)