Foundations of Financial Economics
Deterministic two-period GE asset pricing

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Functions of finance

- There are several functions of finance:
  - intertemporal allocation of resources which may or may not be time-dependent (consumption smoothing)
  - inter-state of nature allocation of resources which are uncertain (insurance)
  - financing investment (increase in the resource capacity)
  - matching timing profiles of expenditures and incomes of different agents
  - matching uncertainty profiles of different agents
  - information revelation and pooling
  - distribution of income and wealth

- In this course, we will be mainly concerned with the first two functions.

- In this lecture, we deal only with the first in the most simple framework: two period perfect information models.
Topics covered

- Interest rates, asset pricing and the intertemporal allocation problem under perfect information
- Under several economic environments, defined by
  - Fundamentals: preferences and technology
  - Market structures: Arrow-Debreu securities, financial assets
Syllabus

- Intertemporal consumption preferences
- General equilibrium in a representative agent Arrow-Debreu economy
- General equilibrium in an heterogeneous Arrow-Debreu economy
- General equilibrium in a frictionless finance economy
- General equilibrium in a finance economy with frictions: heterogeneous market participation
Intertemporal choice

Intertemporal utility function

- We index variables by time. In the simplest case, we have $T = \{0, 1\}$
- We consider the sequences $\{c_0, c_1\}$ where $c_t$, for $t = 0, 1$ is consumption in period $t$
- Sequences $\{c_0, c_1\}$ are ranked by means of an intertemporal utility functional, $U(\{c_0, c_1\})$
- The optimum is a sequence for which $u$ is maximum
Intertemporal choice

Intertemporal utility function

- In this simple model, instead of dealing with sequences \( \{c_0, c_1\} \)
  we consider as a vector of real non-negative numbers
  \( \mathbf{c} = (c_0, c_1) \in \mathbb{R}_+^2 \)

- Therefore the **intertemporal utility function** (IUF) can be seen as a mapping \( U : \mathbb{R}_+^2 \rightarrow \mathbb{R} \),

  \[ u = U(c_0, c_1) \]

  where \( u \) is a number allowing to rank vectors

- Behavioral assumptions can be imposed on the structure of \( U(\cdot) \)

- General assumption: the intertemporal utility function is **continuous** and **differentiable**
Intertemporal utility

Main properties

▶ **Positive marginal utility**: increase in consumption in any period increases utility

\[
U_0 \equiv \frac{\partial U(c_0, c_1)}{\partial c_0} > 0, \quad U_1 \equiv \frac{\partial U(c_0, c_1)}{\partial c_1} > 0
\]

▶ **Stationary**: the temporal utility functions are independent of time (but there can be discounting)

▶ **Impatience**: preference for consumption today rather than in the future

▶ **Intertemporal properties**: let

\[
U(c_0, c_1) = V(u(c_0), v(c_0, c_1))
\]

*V* is called the Koopmans aggregator

▶ intertemporal independence if \( v_{c_0} = 0 \)

▶ intertemporal substitution if \( v_{c_0} < 0 \)

▶ intertemporal complementarity (addiction) if \( v_{c_0} > 0 \)

▶ How can we identify those properties?
Intertemporal utility

Intertemporal marginal rate of substitution

First, we use the substitution concepts in an intertemporal perspective:

- the **intertemporal marginal rate of substitution** is defined as

\[
IMRS_{0,1}(c_0, c_1) = -\frac{dc_1}{dc_0}
\]

- **Intuition**: how much we are willing to sacrifice consumption at \( t = 1 \) (tomorrow) in order to increase one unit of consumption at \( t = 0 \) (today)

- Taking total differentials to the utility function \( U(c_0, c_1) \) such that \( dU = 0 \)

\[
U_0(c_0, c_1)dc_0 + U_1(c_0, c_1)dc_1 = 0
\]

then

\[
IMRS_{0,1}(c_0, c_1) = \frac{U_0(c_0, c_1)}{U_1(c_0, c_1)} \bigg|_{U=\text{constant}}
\]
Intertemporal utility

Intertemporal elasticity of substitution

**Intertemporal elasticity of substitution**

\[
EIS_{0,1}(c_0, c_1) = \frac{d \ln (c_1 / c_0)}{d \ln IMRS_{0,1}(c_0, c_1)} = \frac{c_0 U_0 + c_1 U_1}{c_1 U_1 \varepsilon_{00} - 2 c_0 U_0 \varepsilon_{01} + c_0 U_0 \varepsilon_{11}}
\]

where \( \varepsilon_{ij} = - \frac{U_{ij} c_j}{U_i} \) for \( i = 0, 1 \) and \( U_{ij} = \frac{\partial^2 U}{\partial c_i c_j} \)

**Intuition:** how much does the rate of growth of the ratio \( c_1 / c_0 \) changes for a one percent increase in the \( IMRS \). This provides a scale-free measure of the preferences regarding the behavioral assumptions concerning the intertemporal allocation of consumption.
Intertemporal utility
Impatience and intertemporal complementarity

- Second, assume we start from a stationary consumption process, i.e., $c_0 = c_1 = c$ a constant
- We measure impatience by the relative change in consumption at period 1 relative to period 0. We say the IUF displays impatience if

$$IMRS_{0,1}(c) = \frac{U_0(c, c)}{U_1(c, c)} > 1$$

This means that the reduction in consumption in period $t = 1$ should be bigger than the increase in consumption in period $t = 0$, $-(c_1 - c) > c_0 - c$, to keep utility constant. This means that consumption at $t = 0$ has more value than consumption at $t = 1$

- Intertemporal dependence can be determined by the Allen-Uzawa elasticity $\varepsilon_{01}$.

$$\varepsilon_{0,1}(c) = -\frac{U_{01}(c)}{U_0(c)} \begin{cases} > 0, & \text{intertemporal substitutability} \\ = 0, & \text{intertemporal independence} \\ < 0 & \text{intertemporal complementarity} \end{cases}$$
Intertemporal utility
Example: additive IUF

- **Assumption 1**: the IUF is Intertemporally additive

\[ U(c_0, c_1) = u(c_0) + \beta u(c_1), \text{ where } \beta = \frac{1}{1 + \rho} \]

where \( \beta \in (0, 1) \) is the psychological discount factor and \( \rho \) is the rate of time preference

- **Assumption 2**: \( u \) is increasing and concave \( u''(c_t) < 0 < u'(c_t), t = 0, 1 \)

- Period marginal utilities depend only on the consumption of the same period: intertemporally independence
Intertemporal utility

Case 1: additive IUF, cont

- Marginal utilities for $c_t$, $t = 0, 1$ are
  \[ U_0 = u'(c_0), \quad U_1 = \beta u'(c_1) \]

- Derivatives of marginal utilities for $c_t$, $t = 0, 1$ are
  \[ U_{00} = u''(c_0), \quad U_{01} = 0, \quad U_{11} = \beta u''(c_1) \]

- The IMRS is
  \[ \text{IMRS}_{0,1} = \frac{U_0}{U_1} = \frac{u'(c_0)}{\beta u'(c_1)} \]

  Therefore: marginal utility for period $t = 0$ is proportional to the discounted marginal utility for period $t = 1$ (from the perspective of period $t = 0$)
  \[ u'(c_0) = \beta u'(c_1) \text{IMRS}_{0,1} \]

  we will see an analogous equation again and again translating the idea of intertemporal arbitrage.
The Allen-Uzawa elasticities are

\[ \varepsilon_{00}(c_0) = -\frac{u''(c_0)c_0}{u'(c_0)}, \quad \varepsilon_{01} = 0, \quad \varepsilon_{11}(c_1) = -\frac{u''(c_1)c_1}{u'(c_1)} \]

The elasticity of intertemporal substitution between period 0 and 1 is

\[ EIS_{0,1}(c_0, c_1) = \frac{c_0 u'(c_0) + \beta c_1 u'(c_1)}{\beta c_1 u'(c_1) \varepsilon_{00}(c_0) + c_0 u'(c_0) \varepsilon_{11}(c_1)} \]
Intertemporal utility
Case 1: additive IUF, cont

For a stationary consumption path \( \{c, c\} \) we find:

- The IMRS is independent from \( c \) and

\[
 IMRS_{0,1}(c) = \frac{1}{\beta} = 1 + \rho > 1
\]

this means that the IUF displays **impatience**, and this effect is captured by time discounting

- It displays **intertemporal dependence** because

\[
 \varepsilon_{0,1}(c) = 0
\]

- The IES is

\[
 IES_{0,1}(c) = -\frac{u'(c)}{u''(c)c} > 0
\]
Intertemporal utility

Case 1: example

- Utility function (generalized logarithm)
  \[ u(c) = \frac{c^{1-\zeta} - 1}{1 - \zeta} \]

- if \( \zeta = 1 \) we have \( u(c) = \ln(c) \) (Prove this)

- Derivatives
  \[
  U_0 = c_0^{-\zeta}, \quad U_1 = \beta c_1^{-\zeta}, \quad U_{00} = -\zeta c_0^{-\zeta-1}, \quad U_{01} = 0, \quad U_{11} = -\zeta c_1^{-\zeta-1}
  \]

- The IMRS is
  \[
  IMRS_{0,1} = \frac{1}{\beta \left( \frac{c_1}{c_0} \right)^\zeta}
  \]

- The UA elasticities are constant \( \varepsilon_{00} = \varepsilon_{11} = \zeta \)

- The IES is also constant
  \[
  EIS_{0,1} = \frac{1}{\zeta}
  \]

  This is why it is usually to call \( \zeta \) the inverse of the elasticity of intertemporal substitution.
Case 2: The Uzawa and Epstein-Hynes case

\[ U(c_0, c_1) = u(c_0) + b(c_0)v(c_1) \]

the discount factor is endogenous i.e. \( \beta = b(c) \) with \( b'(c) < 0 \) (rich people are more patient)

The crossed AU elasticity is for a stationary sequence is

\[ \varepsilon_{0,1}(c) = -\frac{b'(c)v'(c)c}{u'(c) + b'(c)v(c)} \]

displays intertemporal dependence
Case 3: Habit formation

\[
U(c_0, c_1) = u(c_0) + \beta v(c_0, c_1).
\]

where \( v_{c_0}(c_0, c_1) < 0. \)

The crossed AU elasticity is for a stationary sequence is

\[
\varepsilon_{0,1}(c) = -\frac{\beta v_{c_0,c_1}(c)c}{u'(c) + \beta v_{c_0}(c)c}
\]

can display intertemporal substitutability, independence or complementarity depending on the relationship between time discounting and the relative importance of habits, i.e., the magnitude of \( v_{c_0}(c) \)
Intertemporal utility

Case 3: habit formation example

▶ IUF

\[ U(c_0, c_1) = \ln(c_0) + \beta \ln\left(\frac{c_1}{c_0}\right)^\zeta, \quad \zeta > 0 \]

▶ Derivatives

\[ U_0 = \frac{1 - \beta \zeta}{c_0}, \quad U_1 = \frac{\beta \zeta}{c_1}, \quad U_{00} = -\frac{1 - \beta}{c_0^2}, \quad U_{01} = 0, \quad U_{11} = -\frac{\beta \zeta}{c_1^2} \]

▶ The IMRS is

\[ IMRS_{0,1}(c_0, c_1) = \left(\frac{1 - \beta \zeta}{\beta \zeta}\right) \frac{c_1}{c_0} \]

▶ The UA elasticities are constant

\[ \varepsilon_{00} = \varepsilon_{11} = 1, \quad \varepsilon_{01} = 0 \]

▶ The IES is also constant

\[ EIS_{0,1}(c_0, c_1) = 1 \]

for any \((c_0, c_1)\)
For a stationary sequence $c_0 = c_1 = c$

- The IMRS

$$IMRS_{0,1}(c) = \frac{1 - \beta \zeta}{\beta \zeta}$$

the utility displays impatience if $\zeta < \frac{1}{2\beta} = \frac{1 + \rho}{2}$. Intuition: there is impatience (according to the above definition) if the weight of past consumption is not too strong.

- As $\varepsilon_{01} = 0 = 0$ the model displays intertemporal independence (but this is special to this example).
Next we will address the determination of the interest rate in two-period general equilibrium models under perfect information (i.e., certainty)

We consider two (equivalent) approaches and models
- a micro-economic approach: Arrow-Debreu simultaneous equilibrium economy
- a finance (or macro-finance) approach: a finance sequential equilibrium economy

For each model we proceed in two steps:
- present and solve the consumer problem in each economy
- we define and determine the general equilibrium
A consumer has an asset (resource, endowment) in positive amount \((w > 0)\) which allows for a sequence of consumption in two periods, \(\{c_0, c_1\}\), today \(c_0\) and in the future \(c_1\).

There is a market for **forward contracts** allowing for contracting today for delivery in the future, at a price set today, \(q > 0\). We take the price paid today as a *numéraire* and all the variables are denominated at today’s price.

The value of the consumption sequence is assessed by an intertemporal utility function: \(U(c_0, c_1)\).

The budget constraint, referring to payments made today, is

\[
c_0 + qc_1 \leq w
\]
Formally, the intertemporal problem for the consumer is

\[ v(w) = \max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + q c_1 \leq w \} \]

The (interior) optimum \((c_0^*, c_1^*)\) satisfies the conditions

\[
\begin{cases}
q U_0(c_0^*, c_1^*) = U_1(c_0^*, c_1^*) \quad \text{(optimality condition)} \\
c_0^* + pc_1^* = w \quad \text{(budget constraint)}
\end{cases}
\]
At the optimum: the IMRS is equal to the relative price (internal = external valuation)

\[ IMRS^*_{0,1} = IMRS_{0,1}(c_0^*, c_1^*) = \frac{U_0(c_0^*, c_1^*)}{U_1(c_0^*, c_1^*)} = \frac{1}{q} \]

Intuition: at the optimum increasing one euro of consumption tomorrow should be matched by a reduction in \(1/q\) euro of consumption today, i.e. \(dc_0^* = -qdc_1^*\)

Therefore \(q\) is an **intertemporal relative price**: i.e., is an opportunity cost for changing the sequence of consumption through time.
Arrow Debreu model

Assumptions

▶ Now, we go from a microeconomic to a macroeconomic perspective

H1  Assume there is perfect information: deterministic general equilibrium

H2  Assume all agents are equal: representative agent economy

H3  Assume that there is an exogenous sequence of resources sustaining consumption: endowment economy

H4  Assume that all trade is done at time $t = 0$: an Arrow-Debreu economy

▶ We want to determine (endogenously) the price $q$: the Arrow-Debreu price
Arrow Debreu model

The setup

- Assume that the resource of the economy takes the form of a flow of non-durable goods, that can be collected both at time $t = 0$ and $t = 1$, $\{y_0, y_1\}$.

- Again, assume that trade can only take place at time $t = 0$, this means that the price for contracts for delivery at time $t = 1$ has to be set at time $t = 0$. We call $q$ the **Arrow-Debreu price**

- From this, wealth at time $t = 0$ is equal to the present value of the flow of endowments

  $$w = y_0 + qy_1$$
Arrow Debreu model

The setup: continuation

- All the participants have **perfect information** on prices and endowments referring to period $t = 1$ and solve a problem similar to our previous micro-economic problem;
- At every period, total consumption must be equal to total endowment;
- **Representative agent economy**: we assume that all consumers solve the same problem (same utility function and same endowments);
- What is the equilibrium forward price $q$?
General equilibrium in this economy is defined by \((c_0^*, c_1^*, q^*)\) such that

- the consumer solves the problem

\[
\max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + q c_1 \leq y_0 + q y_1 \}
\]

- markets clear

\[
c_0 = y_0,
\]
\[
c_1 = y_1
\]
Arrow Debreu model
General equilibrium for a representative agent economy

- General equilibrium conditions: \((c_0, c_1, q)\) is determined from

\[
\begin{align*}
qU_0(c_0, c_1) &= U_1(c_0, c_1) \quad \text{(micro: optimality condition)} \\
c_0 + qc_1 &= y_0 + qy_1 \quad \text{(micro: budget constraint)} \\
c_0 &= y_0 \quad \text{(aggregate: market clearing for } t = 0) \\
c_1 &= y_1 \quad \text{(aggregate: market clearing for } t = 1) 
\end{align*}
\]

- There are only three independent conditions (Walras’s law)

\[
\begin{align*}
qU_0(c_0, c_1) &= U_1(c_0, c_1) \\
c_0^* &= y_0 \\
c_1^* &= y_1 
\end{align*}
\]

- In a representative agent economy there is no trade (consumption is equal to the endowment)
Then the equilibrium AD price is

\[ q^* = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)} \]

We call \( m = IMRS_{0,1} \) the discount factor.

Equivalently: the (deterministic) equilibrium discount factor is

\[ m^*(y_0, y_1) = \frac{1}{q^*} = \frac{U_0(y_0, y_1)}{U_1(y_0, y_1)} \]

The AD price (discount factor) depends on the present and future endowments.

We need more structure on preferences to get explicit results.
For an intertemporally additive utility function

\[ q^*(y_0, y_1) = \beta \frac{u'(y_1)}{u'(y_0)} \]

- concavity of \( u(\cdot) \), i.e., \( u''(c) < 0 \), implies

\[ \frac{\partial q^*(y_0, y_1)}{\partial y_0} = -\beta \frac{u'(y_1)u''(y_0)}{(u'(y_0))^2} > 0 \]

and

\[ \frac{\partial q^*(y_0, y_1)}{\partial y_1} = \beta \frac{u''(y_1)}{u'(y_0)} < 0 \]

- The discount factor \( m(y_0, y_1) \) decreases (increases) with \( y_0 \) (\( y_1 \))
Arrow Debreu model
AD price and utility functions

Example: if \( u(c) = \ln(c) \) then

\[
q^*(y_0, y_1) = \beta \frac{y_0}{y_1} = \frac{\beta}{1 + \gamma}
\]

or, if we set \( y_1 = (1 + \gamma)y_0 \) where \( \gamma \) is the rate of growth
Arrow Debreu model
AD price and utility functions

For the habit formation utility function

\[ q^*(y_0, y_1) = \beta \frac{v_{c_1}(y_0, y_1)}{u'(y_0) + \beta v_{c_0}(y_0, y_1)} \]

Example: setting \( U(c_0, c_1) = \ln(c_0) + \beta \ln \left( \left( \frac{c_1}{c_0} \right)^\zeta \right) \) displaying intertemporal substitution then

\[ q^* = \frac{\beta \zeta}{y_1} \left( \frac{1}{y_0} - \beta \zeta \frac{1}{y_0} \right)^{-1} = \frac{\beta \zeta y_0}{(1 - \beta \zeta) y_1} = \frac{\beta \zeta}{(1 - \beta \zeta)(1 + \gamma)} \]

has the same properties if \( \beta \zeta < 1 \)
Arrow Debreu model
Assumptions

- The previous model is more general than it looks
  H1 idem
  H2 Assume heterogeneity in endowments
  H3 idem
  H4 idem
- What are the consequences for the equilibrium $q$
Assume there are two agents with the same preferences
Assume that their endowments are different ($y_i^t$ is the endowment of agent $i$ at time $t$)

\[ y^1 = \{y^1_0, y^1_1\}, \quad y^2 = \{y^2_0, y^2_1\} \]

and we assume $y^1 \neq y^2$

The flow of total endowments of the economy are

\[ y_0 = y^1_0 + y^2_0 \]
\[ y_1 = y^1_1 + y^2_1 \]

The general equilibrium is now
General equilibrium in this economy is defined by the allocations \((c_{0}^{*}, c_{1}^{*}, c_{0}^{*}, c_{1}^{*})\) and the price \(q^{*}\) such that

- consumer \(i \in \{1, 2\}\) solves the problem

\[
\max_{c_{0}^{i}, c_{1}^{i}} \{ U(c_{0}^{i}, c_{1}^{i}) : c_{0}^{i} + q c_{1}^{i} \leq y_{0}^{i} + q y_{1}^{i} \}, \text{ for } i = 1, 2
\]

- market clearing hold for \(t = 0, 1,\)

\[
c_{0} = y_{0}, \quad c_{1} = y_{1}
\]

where \(c_{t} = c_{t}^{1} + c_{t}^{2}\) for \(t = 1, 2\) and \(y_{t} = y_{t}^{1} + y_{t}^{2}\)
General equilibrium conditions (considering that the Walras’ law holds)

\[
\begin{align*}
q U_0(c_0^1, c_1^1) &= U_1(c_0^1, c_1^1) \quad \text{(optimality condition for agent 1)} \\
q U_0(c_0^2, c_1^2) &= U_1(c_0^2, c_1^2) \quad \text{(optimality condition for agent 2)} \\
c_t &= y_t \quad \text{(market clearing for period } t = 1, 2) \\
c_t &= c_t^1 + c_t^2 \quad \text{(aggregation of consumption for } t) \\
y_t &= y_t^1 + y_t^2 \quad \text{(aggregation of endowment for } t)
\end{align*}
\]

In this case there can be trade, because \( c_t^1 - y_t^1 = y_t^2 - c_t^2 \) can be different from zero, but the budget constraint should hold for every agent. (check this !)
Because we assumed homogeneity in preferences $U(., .)$ is the same for both consumers.

Therefore, it also holds for the aggregate consumption

$$qU_0(c_0, c_1) = U_1(c_0, c_1)$$

that is

$$qU_0(c_0^1 + c_0^2, c_1^1 + c_2^1) = U_1(c_0, c_1)$$
Using the market clearing conditions we have again

\[ q^* = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)} = \frac{U_1(y_0^1 + y_0^2, y_1^1 + y_1^2)}{U_0(y_0^1 + y_0^2, y_1^1 + y_1^2)} \]

Conclusion: if agents are **homogeneous as regards preferences** but are **heterogeneous as regards endowments** the **distribution of income between agents** has **no influence** the AD price. It is only determined by the aggregate endowment.

If there is heterogeneity in preferences, this result **will not hold** in general.
Finance economy model

Assumptions

- Now we change the market structure

H1 idem

H2 Assume a representative agent economy

H3 idem

H4 Assume a sequence of asset markets

- What is the equilibrium asset price
Finance economy model

The economy

- Assume there is a spot market for the good opening at every period $t = 0$ and $t = 1$;
- There is an asset (that can be seen as a durable good) that agents can lend and borrow at period $t = 0$ paying or receiving an interest income at period $t = 1$. The asset is in non-negative net supply at the beginning to period $t = 0$ and there is a market for the asset at time $t = 0$.
- We still assume that the agent receives a flow of endowments $y = \{y_0, y_1\}$ The agent can consume the totality if the income, or not, at the end of period 1
- Every agent has now a **sequence of budget constraints** (because trade in the good market can take place at period 1)
Finance economy model

Micro-economic problem in the finance economy

- The problem

\[
\max_{c_0, c_1, a_1, a_2} U(c_0, c_1) = u(c_0) + \beta u(c_1):
\]

- subject to

\[
\begin{align*}
    c_0 + a_1 &= y_0 + a_0 \\
    c_1 &= y_1 + (1 + r)a_1 - a_2
\end{align*}
\]

where \( a_0 \) is the level of the asset at beginning of period 0 and \( a_1 \) and \( a_2 \) are the levels at the end of period 0 and 1, and \( r \) is the real interest rate.

- other constraints:

\[
c_0 \geq 0, \ c_1 \geq 0, \ a_1 \text{free}, \ a_2 \geq 0
\]

- Next, we prove that, it will never be optimal to have \( a_2 > 0 \)
Finance economy model
Optimality of $a_2 = 0$

- Substitute $c_0$ and $c_1$ in the utility function, assume that $\beta > 0$ and $r$ is finite, and consider the constraint for $a_2$

$$
\max \{ u(y_0 + a_0 - a_1) + \beta u(y_1 + (1 + r)a_1 - a_2) : a_2 \geq 0 \}
$$

- The first order conditions are

$$
\begin{align*}
    u'(c_0) &= \beta(1 + r)u'(c_1) \\
    \beta u'(c_1) &= \lambda \\
    \lambda a_2 &= 0, \; \lambda \geq 0, \; a_2 \geq 0
\end{align*}
$$

- We have $a_2 > 0$ if and only if $\lambda = 0$, but in this case either there is satiation or $c_1 \to \infty$ and $c_0 \to \infty$. But this is only possible if $a_0 \to \infty$. Therefore we should have $a_2 = 0$ and $\lambda > 0$. 

Taking $a_2 = 0$ and assuming $a_1$ is free (i.e., the consumer can borrow or lend freely) we can eliminate $a_1$ in the sequence of budget constraints, to get

$$c_0 + mc_1 = a_0 + y_0 + my_1$$

where $m$ is the market discount factor

$$m \equiv \frac{1}{1+r} \equiv \frac{1}{R}$$

This implies that the sequence of budget constraints is equivalent to an intertemporal budget constraint formally similar to the constraint in the Arrow-Debreu economy.

$$c_0 + mc_1 = y_0 + mc_1$$
Finance economy without frictions

General equilibrium for a representative agent finance economy

General equilibrium in this economy is defined by \((c_0^*, c_1^*, m^*)\) such that

- the consumer solves the problem

\[
\max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + mc_1 = a_0 + y_0 + mc_1 \}
\]

- market clearing hold

\[
c_0 = y_0, \quad c_1 = y_1
\]
Finance economy without frictions
General equilibrium for a representative agent finance economy

- The equilibrium equations are (from Walras’s law)

\[ mu'(c_0) = \beta u'(c_1) \]
\[ c_0 = a_0 + y_0 \]
\[ c_1 = y_1 \]

- The equilibrium discount factor is

\[ m^* = m(a_0, y_0, y_1) = \beta \frac{u'(y_1)}{u'(a_0 + y_0)} \]

- Because \( R = \frac{1}{m} \) and \( \beta = \frac{1}{1 + \rho} \) where \( \rho \) is the psychological discount factor
The equilibrium asset return (recall)

\[ R^* = 1 + r^* = (1 + \rho) \frac{u'(a_0 + y_0)}{u'(y_1)} \]

But \( R^* = R(a_0, y_0, y_1) \), with partial derivatives

\[ \frac{\partial R}{\partial a_0} = \frac{\partial R}{\partial y_0} = (1 + \rho) \frac{u''(a_0 + y_0)}{u'(y_1)} < 0 \]

\[ \frac{\partial R}{\partial y_1} = -(1 + \rho) \frac{u''(y_1)u'(a_0 + y_0)}{(u'(y_1))^2} > 0 \]

There are two main effects:

- a direct effect: high \( y_0 \) or \( a_0 \) reduce the interest rate
- an anticipation effect: high \( y_1 \) increases the interest rate
A simple finance economy

Assumptions

▶ Now we introduce heterogeneity

H1 idem
H2 Assume agents face financing constraints
H3 idem
H4 idem

▶ What is the equilibrium asset price
Finance economy with heterogeneity

Heterogenous participation

- Assume there are two agents in the economy: agent $b$ is a borrower and agent $l$ is a lender, the only one that has positive assets at time 0 ($a^l_0 > 0, a^b_0 = 0$)
- To simplify, assume agent $b$ is the only one that receives the flow of endowments $\{y_0, y_1\}$ and agent $b$ can only earn interest income
- Assume there are no constraints in the credit market
- Assume that agents have homogeneous preferences
Finance economy with heterogeneity

Agents’ problems

- The **lender** problem is

\[ \max_{c_0^l, c_1^l} \{ u(c_0^l) + \beta u(c_1^l) : \ c_0^l + l^l = a_0, \ c_1^l = (1 + r)l^l \} \]

- Because \( l^l \) is free it can be simplified to

\[ \max_{l^l} \{ u(a_0 - l^l) + \beta u((1 + r)l^l) \} \]

- The optimality condition is

\[ u'(a_0 - l^l) = \beta (1 + r) u'((1 + r)l^l) \]

or equivalently

\[ u'(c_0^l) = \beta (1 + r) u'(c_1^l) \]
Finance economy with heterogeneity

Agents’ problems

- The **borrower** problem is

\[
\max_{c_0^b, c_1^b} \{ u(c_0^b) + \beta u(c_1^b) : c_0^b = y_0 + l^b, ~ c_1^l + (1 + r)l^b = y_1 \}
\]

- Because \(l^b\) is free it can be simplified to

\[
\max_{l^b} \{ u(y_0 + l^b) + \beta u(y_1 - (1 + r)l^b) \}
\]

- The optimality condition is

\[
u' (y_0 + l^b) = \beta (1 + r) u' (y_1 - (1 + r)l^b)\]

or equivalently

\[
u' (c_0^b) = \beta (1 + r) u' (c_1^b)\]
Finance economy with heterogeneity

Equilibrium equations

The equilibrium equations are

\[ u'(c_0^l) = \beta(1 + r)u'(c_1^l) \]

\[ u'(c_0^b) = \beta(1 + r)u'(c_1^b) \]

\[ c_0^l + c_0^b = y_0 + a_0 \]

\[ c_1^l + c_1^b = y_1 \]

Because preferences are homogeneous we can use the same argument as before, to get

\[ u'(y_0 + a_0) = \beta(1 + r)u'(y_1) \]
Finance economy with heterogeneity

Equilibrium interest rate

The equilibrium return is again

\[ R^* = 1 + r^* = (1 + \rho) \frac{u'(y_0 + a_0)}{u'(y_1)} \]

- Is formally similar to the representative agent economy case;
- Again, we have
  - negative liquidity effect \( R_{a_0}^* < 0 \);
  - a negative income effect, \( R_{y_0}^* < 0 \)
  - a positive anticipation effect, \( R_{y_1}^* > 0 \)
- Conclusion: without other sources of heterogeneity, limited participation has no effect on the market interest rate.
Taking the model to data

  \( R_{safe} = 1.0188 \) (average safe return) \( R_{wealth} = 1.0678 \) (average wealth return) \( \gamma = 0.0287 \) (average rate of growth)

- calibrated parameters: \( \rho = 0.02 \)

- Utility functions
  - isoelastic utility function
    \[
    U(c_0, c_1) = \frac{c_0^{1-\zeta}}{1-\zeta} + \beta \frac{c_1^{1-\zeta}}{1-\zeta}
    \]

  - habit formation:
    \[
    U(c_0, c_1) = \ln c_0 + \beta \left[ \left( \frac{c_1}{c_0} \right)^{\phi(1-\zeta)} - 1 \right]
    \]
Taking the model to data

- Additive utility: interest rate puzzle (the model over predicts the observed risk-free interest rate, for any value of the $EIS$)

- Habit formation: it is possible to find values for the parameter $\phi$, in the case $\phi \approx 0.5$ such that the model matches the observed $R$ for ”acceptable” values for $\zeta$
The previous results hold for cases in which there is
- full information (deterministic general equilibrium)
- agents have homogeneous preferences (with or without homogeneous resources)
- frictionless economy (for the case of a finance economy)

Do those results hold:
- Under imperfect information (uncertainty) ?
- Under heterogeneity in agents’ preferences ?
- Under frictions in a finance economy (ex: credit constraints ) ?