Foundations of Financial Economics
Revisions of utility theory

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Topics of the lecture

- Marginal concepts frequent in economics
- Basic utility theory
Marginalist concepts

Value function

- Consider a number of different objects \textbf{indexed} as \( I = \{1, \ldots, i, \ldots, n\} \)
- The \textbf{quantity} of object \( i \) is denoted \( x_i \in \mathbb{R} \)
- We can represent a \textbf{bundle} of objects by the vector \( \mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n \), where
- The \textbf{value} of a bundle is given by the (at least twice-) differentiable function

\[
F = F(\mathbf{x}) = F(x_1, \ldots, x_i, \ldots, x_n)
\]

- In economics usually \( F(\cdot) \) represents is a utility or a production function
- Change in value is represented by the differential (under very weak conditions)

\[
dF = F_1 \, dx_1 + \ldots + F_i \, dx_i + \ldots = \nabla F \cdot d\mathbf{x}
\]

where \( \nabla F \) is the gradient

\[
\nabla F = (F_1, \ldots, F_i, \ldots, F_n)^\top
\]
Marginalist concepts

Marginal values: goods

- Denote the partial derivative of object $i$ by

$$F_i(x) \equiv \frac{\partial F(x)}{\partial x_i}$$

- We say object $i$ is a

$$\begin{cases} 
\text{good} & \text{if } F_i(x) > 0 \text{ for any } x \in \mathbb{R}^n \\
\text{saturated} & \text{if } F_i(x) = 0 \text{ for any } x \in \mathbb{R}^n \\
\text{bad} & \text{if } F_i(x) < 0 \text{ for any } x \in \mathbb{R}^n 
\end{cases}$$

- From now on we consider goods, i.e. $F_i > 0$ for any $i \in I$

- We call \textbf{marginal contribution} of good $i$ to the change in value brought about by $dx_i$

$$(\text{Definition}) \quad M_i \equiv \frac{dF}{dx_i}$$

- For the bundle variation $dx = (0, \ldots, 0, dx_i, 0, \ldots, 0)$ then $dF = F_i \, dX_i$ and therefore the marginal contribution is equal to the partial derivative

$$(\text{Implication}) \quad M_i = F_i$$

therefore a good has a positive marginal contribution for value
Marginalist concepts
Relative marginal changes

- Observe that \( M_i(x) = F_i(x) \) because \( F_i \) is a function of \( x \).
- If \( F \) is twice-differentiable we can calculate second-order derivatives:
  \[
  (\text{own}) \quad F_{ii} \equiv \frac{\partial^2 F(x)}{\partial x_i^2} \quad (\text{crossed}) \quad F_{ij} \equiv \frac{\partial^2 F(x)}{\partial x_i \partial j}, \text{ for any } j \neq i \in I
  \]

- The **marginal contribution** of \( i \) for a variation in \( x_i \):
  \[
  \frac{\partial M_i}{\partial x_i} = F_{ii} = \begin{cases} > 0 & \text{increasing} \\ = 0 & \text{constant} \\ < 0 & \text{decreasing} \end{cases}
  \]

- **Pareto-Edgeworth** relationships: variation in \( M_i \) for a variation in any \( x_j \):
  \[
  \frac{\partial M_i}{\partial x_j} = F_{ij} = \begin{cases} > 0 & \text{complementarity} \\ = 0 & \text{independence} \\ < 0 & \text{substitutability} \end{cases}
  \]

- **Uzawa-Allen elasticities**: relative variation in \( M_i \) for a variation in any \( x_j \)
  \[
  (\text{own}) \quad \varepsilon_{ii} \equiv -\frac{F_{ii} x_i}{F_i} \quad (\text{crossed}) \quad \varepsilon_{ij} \equiv -\frac{F_{ij} x_j}{F_i}
  \]

- If \( i \) is a good and its quantity is positive then \( \varepsilon_{ii} > 0 \) and it is complementary with (substitutable by) \( j \) if \( \varepsilon_{ij} < 0 \) (\( \varepsilon_{ij} > 0 \)).
Marginalist concepts

Compensated variations

➤ The **marginal rate of substitution** of good \( i \) by good \( j \) is the variation in the quantity of good \( j \) by unit variation in good \( i \)

\[
\text{(definition)} \quad MRS_{ij} \equiv -\frac{dx_j}{dx_i}
\]

➤ Assume we want to know what would be \( dx_j \) if we change \( dx_i \) in such a way as to keep the value \( F \) constant, ie. if \( d\mathbf{x} = (0, \ldots, 0, dx_i, 0, \ldots, dx_j, 0, \ldots, 0) \) such that \( dF = 0 \). That is

\[
dF = \nabla F \cdot d\mathbf{x} = F_i \, dx_i + F_j \, dx_j = 0
\]

➤ Then

\[
\text{(Implication)} \quad MRS_{ij}(\mathbf{x}) = -\frac{F_i(\mathbf{x})}{F_j(\mathbf{x})} \quad \text{for} \quad F(\mathbf{x}) = \text{constant}
\]
Marginalist concepts

Elasticity of substitution

► A fundamental concept here is the **elasticity of substitution** of good $i$ by good $j$

\[
(\text{definition}) \quad ES_{ij}(x) \equiv \frac{d \ln(x_j/x_i)}{d \ln MRS_{ij}(x)}
\]

intuition: relative change in the $MRS_{ij}$ for a relative change in the ratio $x_j/x_i$.

► If $F$ is twice differentiable, then the $ES_{ij}$ is

\[
(\text{Implication}) \quad ES_{ij} = \frac{x_i F_i + x_j F_j}{x_j F_j \varepsilon_{ii} - 2 x_i F_i \varepsilon_{ij} + x_i F_i \varepsilon_{jj}}
\]

where $x_i F_i \varepsilon_{ij} = x_j F_j \varepsilon_{ji}$ and $F_{ij} = F_{ji}$ if $F$ is continuous.
Marginalist concepts

Elasticity of substitution: continuation

Sketch of the proof:

- remember we want to substitute $j$ by $i$ keeping the quantities of the other goods constant
- the numerator is

$$d \ln(x_j/x_i) = d \ln x_j - d \ln x_i = \frac{dx_j}{x_j} - \frac{dx_i}{x_i} =$$

$$= - \frac{dx_i}{x_i x_j F_j} (x_i F_i + x_j F_j) \text{ (because } F_i \, dx_i + F_j \, dx_j = 0)$$

- the denominator is

$$d \ln MRS_{ij} = d \ln \left( \frac{F_i(x_i, x_j)}{F_j(x_i, x_j)} \right) = d \ln F_i - d \ln F_j = \frac{dF_i}{F_i} - \frac{F_j}{F_j}$$

- But

$$dF_i = F_{ii} \, dx_i + F_{ij} \, dx_j = dx_i \left( F_{ii} + \frac{dx_j}{dx_i} F_{ij} \right) = dx_i \left( F_{ii} - \frac{F_i}{F_j} F_{ij} \right)$$

$$dF_j = F_{ji} \, dx_i + F_{jj} \, dx_j = dx_i \left( F_{ij} + \frac{dx_j}{dx_i} F_{jj} \right) = dx_i \left( F_{ij} - \frac{F_i}{F_j} F_{jj} \right)$$

- the rest of the proof is obtained by simplification and by using the definition of the Uzawa-Allen elasticities.
Example: Cobb-Douglas function

- The Coob-Douglas production function $F = \text{output}, \ x = (x_1, x_2) = \text{inputs}$

$$F = F(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}, \text{ for } 0 < \alpha < 1, \ x_1 > 0, \ x_2 > 0$$

- First derivatives: both inputs are productive (positive marginal productivities)

$$F_1 = \alpha \frac{F}{x_1} > 0, \ F_2 = (1 - \alpha) \frac{F}{x_2} > 0$$

- Second derivatives: they have decreasing marginal productivities and are Pareto-Edgeworth complements (but usually are substitutable in the Hicksian sense, i.e., when we consider their cost)

$$F_{11} = -\alpha(1 - \alpha) \frac{F}{(x_1)^2} < 0, \ F_{22} = -\alpha(1 - \alpha) \frac{F}{(x_2)^2} < 0,$$

$$F_{12} = F_{21} = \alpha(1 - \alpha) \frac{F}{x_1 x_2} > 0$$
Example: Cobb-Douglas function

- The Hicks-Allen elasticities are

\[ \varepsilon_{11} = 1 - \alpha > 0, \quad \varepsilon_{22} = \alpha > 0, \quad \varepsilon_{12} = -(1 - \alpha) < 0 \]

- The marginal rate of substitution is

\[ MRS_{12} = \frac{F_1}{F_2} = \frac{\alpha x_2}{(1 - \alpha) x_1} \]

- The elasticity of substitution is

\[ ES_{12} = \frac{x_1 F_1 + x_2 F_2}{x_2 F_2 \varepsilon_{11} - 2x_1 F_1 \varepsilon_{12} + x_1 F_1 \varepsilon_{22}} = \frac{F}{F} = 1 \]
Utility theory
The problem: optimal allocation

- The problem: consider an agent with a resource $W$ that wants to allocate it optimally among two goods, 1 and 2, having (given) costs $p_1$ and $p_2$.

- The optimality criterium is $U(c_1, c_2)$, where the quantities of the two goods are $c_1$ and $c_2$.

- Further assumptions:
  - The utility function $U(\cdot)$ is: continuous, differentiable, increasing and concave.
  - The endowment is positive: $W > 0$
  - Nominal expenditure $E \equiv E(c_1, c_2) = p_1 c_1 + p_2 c_2$
Optimal free allocation: definition

- Assume there are no other constraints with the exception of the resource constraint $E(c_1, c_2) = W$
- The problem is

$$V(W; p_1, p_2) = \max_{c_1, c_2} \left\{ U(c_1, c_2) : E(c_1, c_2) = W \right\}$$

- function $V(.)$ is called indirect utility or value function
- intuition: it gives the value of the endowment $W$ in utility terms
Optimal free allocation: solution

- The Lagrangean
\[ \mathcal{L} = u(c_1, c_2) + \lambda(W - E(c_1, c_2)) \]

- The solution (which always exists) \((c_1^*, c_2^*, \lambda^*)\) satisfies the conditions
\[
\begin{cases}
U_{c_j}(c_1, c_2) - \lambda p_j = 0, & j = 1, 2 \\
W - E(c_1, c_2) = 0
\end{cases}
\]

- We observe that, at the optimum that the \(MRS_{1,2}\) is equalized to the relative prices
\[
MRS_{1,2} = \frac{U_{c_1}(c_1^*, c_2^*)}{U_{c_2}(c_1^*, c_2^*)} = \frac{p_1}{p_2}
\]
and, in this case the resource is saturated
\[
E(c_1^*, c_2^*) = p_1 c_1^* + p_2 c_2^* = W
\]
Optimal free allocation: solution

- When there is free allocation, the solution is characterized by the equations,

\[ p_2 U_{c_1}(c_1^*, c_2^*) = p_1 U_{c_2}(c_1^*, c_2^*) \]  \quad (1)

\[ E(c_1^*, c_2^*) = W \]  \quad (2)

- Equation (1) is a first-order partial differential equation with solution (check this)

\[ U(c_1^*, c_2^*) = V \left( \frac{p_1 c_1^* + p_2 c_2^*}{p_1} \right) \]

- From equation (2), in the optimum we have

\[ U(c_1^*, c_2^*) = V(w), \quad w \equiv \frac{W}{p_1} \text{ (real resources deflated } p_1) \]

- If the utility function is strictly concave then with very weak conditions (differentiability) we have an unique interior optimum
Optimal free allocation: graphical representation

Figure: Interior optimum for a log utility function

\[ U(c_1, c_2) = \ln c_1 + b \ln c_2 \]
Utility theory
Optimal constrained allocation: definition

Let us assume that the agent is constrained in the allocation of resources to good 1. For instance, assume that $c_i \in [0, \bar{c}_1]$

The problem is now

$$V(W; p_1, p_2, \bar{c}_1) = \max_{c_1, c_2} \{ U(c_1, c_2) : E(c_1, c_2) = W, 0 \leq c_1 \leq \bar{c}_1 \}$$

Most models of financial frictions introduce constraints of this type

More generally we could assume there are restrictions in allocation resources to the two goods.

The problem would become

$$V(W; p_1, p_2, \bar{c}_1, \bar{c}_2) = \max_{c_1, c_2} \{ U(c_1, c_2) : E(c_1, c_2) = W, 0 \leq c_j \leq \bar{c}_j, j = 1, 2 \}$$
Utility theory
Optimal constrained allocation: optimality

The Lagrangean is now
\[
\mathcal{L} = u(c_1, c_2) + \lambda (W - E(c_1, c_2)) - \\
- \eta_1 c_1 - \eta_2 c_2 + \zeta_1 (\bar{c}_1 - c_1) + \zeta_2 (\bar{c}_2 - c_2)
\]

The solution (which always exists) \((c_1^*, c_2^*, \lambda^*, \eta_1^*, \eta_2^*, \zeta_1^*, \zeta_2^*)\) satisfies the Karush-Kuhn-Tucker conditions

\[
\begin{align*}
U_{c_j}(c_1, c_2) - \lambda p_j - \eta_j - \zeta_j &= 0, & j &= 1, 2 \\
\eta_j c_j &= 0, \quad \eta_j \geq 0, \quad c_j \geq 0, & j &= 1, 2 \\
\zeta_j (\bar{c}_j - c_j) &= 0, \quad \zeta_j \geq 0, \quad c_j \leq \bar{c}_j, & j &= 1, 2 \\
\lambda (W - E(c_1, c_2)) &= 0, \quad \lambda \geq 0, \quad E(c_1, c_2) \leq W
\end{align*}
\]
Optimal constrained allocation: solution

Corner solution: lower $c_1 = 0$

- Let $c_1^* = 0$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;

- FOC: $\eta_1^* > 0$ and $\eta_2^* = \zeta_1^* = \zeta_2^* = 0$, and

\[
p_2 U_{c_1}(c_1^*, c_2^*) = p_1 U_{c_2}(c_1^*, c_2^*) - p_2 \eta_1 \tag{3}
\]
\[
E(c_1^*, c_2^*) = W \tag{4}
\]

- Now, the MRS is smaller than the relative price

\[
MRS_{12} = \frac{U_{c_1}^*}{U_{c_2}^*} = \frac{p_1}{p_2} - \frac{\eta_1}{U_{c_2}^*} < \frac{p_1}{p_2}
\]

i.e., there is a ”wedge” between relative prices and the $MRS_{12}$

- Equation (3) is a first-order partial differential equation with solution

\[
U(c_1^*, c_2^*) = \frac{\eta_1 c_2^*}{p_1} + V \left( \frac{p_1 c_1^* + p_2 c_2^*}{p_1} \right)
\]

- if we use equation (6) in the optimum we have

\[
U(c_1^*, c_2^*) = -\eta_1^* w + V(w) < V(w)
\]
Figure: Corner solution: the indirect utility level is smaller than for the unconstrained case
Optimal constrained allocation: solution

Corner solution: upper constraint $c_1 = \bar{c}_1$

- Let $c_1^* = \bar{c}_1$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;
- then $\zeta_1^* > 0$ and $\eta_1^* = \eta_2^* = \zeta_1^* = \zeta_2^* = 0$
- In addition
  \[
  p_2 U_{c_1}(c_1^*, c_2^*) = p_1 U_{c_2}(c_1^*, c_2^*) + p_2 \zeta_1 \tag{5}
  
  E(c_1^*, c_2^*) = W \tag{6}
  \]

- There is again a ”wedge” between the $MRS_{12}$ and the relative price, but now
  \[
  MRS_{12} = \frac{U_{c_1}^*}{U_{c_2}^*} = \frac{p_1}{p_2} + \frac{\zeta_1}{U_{c_2}^*} > \frac{p_1}{p_2}
  \]

- Equation (5) is a first-order partial differential equation with solution
  \[
  U(c_1^*, c_2^*) = -\frac{\zeta_1 c_2^*}{p_1} + V\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)
  \]

- if we use equation (6) in the optimum we have
  \[
  U(c_1^*, c_2^*) = -\frac{\zeta_1 p_1 (w - \bar{c}_1)}{p_2} + V(w) < V(w)
  \]
Consumer problem
Corner solution 2

Figure: Corner solution: the indirect utility level is smaller than for the unconstrained case
Equivalent interpretation

Let the value function in which there are constraints on the consumer be denoted by $\tilde{V}(w)$.

Looking at the previous cases we can write

$$\tilde{v}(w) = V(w) - \delta(w)$$

where $\delta(w) \geq 0$ measures the welfare loss introduced by the constraint $c_1 \in [0, \bar{c}_1]$.

We could obtain a similar solution for the consumer problem is instead of considering the endowment level $w$ we consider the resource level

$$\tilde{w} = \{x : (\tilde{v}^{-1})(x) = 0\} < w$$

that is a smaller level for the endowment.
Conclusion

Constraints on the free allocation of resources between the two consumption goods

1. create a (algebraic) wedge between the the $MRS$ and the relative prices
2. generate welfare losses
3. this gives a rough idea on the effects of constraints in the intertemporal or intra-state of nature allocation of resources (at least for a benchmark model)
Example

1. Assume the utility function is of Cobb-Douglas type

\[ U = U(c_1, c_2) = c_1^\alpha c_2^{1-\alpha}, \text{ for } 0 < \alpha < 1 \]

2. Case 1: Assume that \((c_1, c_2)\) are only constrained by the budget constraint

\[ p_1 c_1 + p_2 c_2 = W \]

3. Case 2: in addition to the budget constraint impose the constraint \(c_1 > 0\)

4. Case 3: in addition to the budget constraint impose the constraint

\[ c_1 \leq \beta \frac{W}{p_1} \text{ with } 0 < \beta < \alpha \]

5. Observe that

\[ U_1 = \frac{\partial U}{\partial c_1} = \alpha \frac{U}{c_1} > 0, \text{ and } U_2 = \frac{\partial U}{\partial c_2} = (1 - \alpha) \frac{U}{c_2} > 0 \]

which means that the objects indexed by 1 and 2 are both goods
Example

Case 1: free allocations

- the first order conditions are

\[
\begin{align*}
    p_2 U_1 &= p_1 U_2 \\
    p_1 c_1 + p_2 c_2 &= W
\end{align*}
\]

\(\Leftrightarrow\)

\[
\begin{align*}
    (1 - \alpha) p_1 c_1 - \alpha p_2 c_2 &= 0 \\
    p_1 c_1 + p_2 c_2 &= W
\end{align*}
\]

then the optimal consumption allocation is, therefore

\[
\begin{align*}
    c_1^* &= \alpha \frac{W}{p_1} \\
    c_2^* &= (1 - \alpha) \frac{W}{p_2}
\end{align*}
\]

- Properties: as

\[
c_1^* = c_1^*(p_1, W), \quad c_2^* = c_2^*(p_2, W)
\]

1. Each type of consumption is proportional to nominal wealth deflated by its price
2. there is no complementarity or substitutability in the Hicksian sense, i.e. their cross-derivatives relative to the price of the other good are zero

\[
\frac{\partial c_1^*}{\partial p_2} = \frac{\partial c_2^*}{\partial p_1} = 0.
\]
Example
Case 1: free allocations

1. Substituting in the utility function we get the value of the resource $W$

$$V(W) = \left( \frac{\alpha}{p_1} \right)^\alpha \left( \frac{1-\alpha}{p_2} \right)^{1-\alpha} W =$$

$$= \chi(\alpha) \frac{W}{P}$$

where $P \equiv p_1^\alpha p_2^{1-\alpha}$ is the consumers price index

2. The value of the resource $W$, assuming there is an optimal free allocation among the two goods, is proportional to the real value of the resource deflated by the consumer’s own price index (which is a geometrical mean whose weights are given by those of the utility function.)
Example
Case 2: positive allocations to good 1

- In this case we require that $c_1 \geq 0$.
- As we saw in the free allocation case that $c^* = \alpha \frac{W}{p_1} > 0$ then the optimum will be interior
- This means that the constrains is not binding.
- Therefore the solution is the same as in case 1

\[
\begin{align*}
  c_1^* &= \alpha \frac{W}{p_1} \\
  c_2^* &= (1 - \alpha) \frac{W}{p_2}
\end{align*}
\]
Example

Case 3: upper bound on the allocations to good 1

▶ In this case we require that $c_1 \leq \bar{c}_1$ and $\bar{c}_1 = \beta \frac{W}{p_1}$, for $\beta < \alpha$

▶ As we saw in the free allocation case that $c^* = \alpha \frac{W}{p_1} > \bar{c}_1$ which means that this solution is not admissible.

▶ The first order conditions are now (5) and (6) with $c_1 = \bar{c}_1$

\[
\begin{align*}
\alpha p_2 c_2 &= (1 - \alpha) p_1 \bar{c}_1 + p_2 \bar{c}_1 c_2 \zeta_1 \\
p_1 \bar{c}_1 + p_2 c_2 &= W
\end{align*}
\]

that we need to solve for $c_2$ and $\zeta_1$.

▶ The solution is

\[
c_1^* = \bar{c}_1 = \beta \frac{W}{p_1} < \alpha \frac{W}{p_1}
\]

\[
c_2^* = (1 - \beta) \frac{W}{p_2} > (1 - \alpha) \frac{W}{p_2}
\]

\[
\zeta_1 = \frac{(\alpha - \beta) p_1}{\beta(1 - \beta) W} > 0
\]

Therefore: the consumption of good 1 (2) will smaller (larger) than in the free allocation case
Example

Case 3: upper bound on the allocations to good 1

- However, **there will be a loss in value.**
- To see this observe that the value of the resource is now

\[
V(W) = \left( \frac{\beta}{p_1} \right)^\alpha \left( \frac{1 - \beta}{p_2} \right)^{1-\alpha} W = \\
= \beta^\alpha (1 - \beta)^\alpha \frac{W}{P} = \\
X(\beta) \chi(\alpha) \frac{W}{P} < \chi(\alpha) \frac{W}{P}
\]

which is smaller than for the free allocation case.

- To prove this let

\[
X(\beta) \equiv \left( \frac{\beta}{\alpha} \right)^\alpha \left( \frac{1 - \beta}{1 - \alpha} \right)^{1-\alpha} > 0
\]

and remember that we assume that $\beta < \alpha$

- and show that $X(\alpha) = 1$ and that

\[
\frac{\partial X}{\partial \beta} = \left( \frac{\alpha - \beta}{\beta(1 - \beta)} \right) X > 0
\]

Then $X(\beta) < 1$ for $\beta < \alpha$. 