Foundations of Financial Economics
Two period GE for a finance economy

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Topics

- Environment in a finance economy
- The household problem in a finance economy
- The household problem when financial markets are complete
- The household problem when financial markets are incomplete
- Generic optimality conditions for the household problem
- GE asset pricing: representative agent finance economy
- Equilibrium asset prices
- Equilibrium asset prices in the presence of a risk-free asset
- Equilibrium equity premium
- Equivalence between AD and finance economies

Environment in a finance economy

Markets in a finance economy

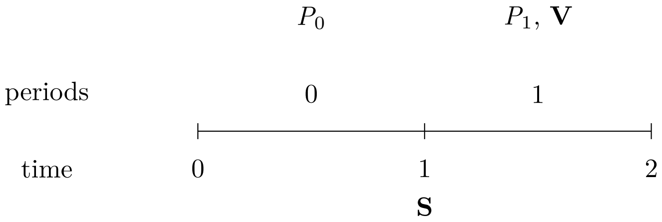
We assume an endowment finance economy, with the following **markets**:

1. **one real spot market for the good** opening in **every period** $t = 0$ **and** $t = 1$. We set output prices as $P_0 = 1$ and $P_1 = \mathbf{1} = (1, 1, \ldots, 1)^\top$;

   Note: differently from the AD economy, now the household cannot perform forward contracts in the good but can buy or sell it at time $t = 1$ in the spot market;

2. K **financial spot markets** opening at the **end of period** $t = 0$, where assets promising a contingent payoff V_j (at period $t = 1$) are traded at prices S_j, $j = 1, \ldots, K$.

Timing of asset prices and payoffs



- asset prices $\mathbf{S} = (S_1, \ldots, S_K)$
- asset payoffs: $\mathbf{V} = \begin{pmatrix} V_{1,1} & \ldots & V_{K,1} \\ \vdots & \ldots & \vdots \\ V_{1,N} & \ldots & V_{K,N} \end{pmatrix}$

The household problem in a finance economy

Timing of the household resources (exogeneous)

endowments $\quad y_0 \quad Y_1 = \begin{pmatrix} y_{1,1} \\ \dots \\ y_{1,s} \\ \dots \\ y_{1,N} \end{pmatrix}$

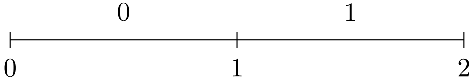
	0	1

$0 \qquad 1 \qquad 2$

financial wealth $\quad a_0$

Timing of the household decisions

consumption $\quad c_0 \quad C_1 = \begin{pmatrix} c_{1,1} \\ \dots \\ c_{1,s} \\ \dots \\ c_{1,N} \end{pmatrix}$



portfolio $\quad \theta = \begin{pmatrix} \theta_1 \\ \dots \\ \theta_j \\ \dots \\ \theta_K \end{pmatrix}$

Flow and stock accounting

budget constraints $c_0 = y_0 + z_0^\theta$ $\quad C_1 = Y_1 + Z_1^\theta$

period: 0 $\quad$ 1

time: 0 $\quad$ 1 $\quad$ 2

wealth $a_0 = 0$ $\quad a_1 = z_0^\theta$ $\quad A_2 = \mathbf{0}$

(net) income generated by portfolio θ

- in period $t = 0$: $z_0^\theta = -\mathbf{S}\,\theta = \sum_{j=1}^{K} S_j\,\theta_j$
- in period $t = 1$ $Z_1^\theta = \mathbf{V}\,\theta = \sum_{j=1}^{K} \mathbf{V}_j\,\theta_j$

NPG condition plus optimality $\Rightarrow A_2 = \mathbf{0}$

Constraints for the household

- The representative household receives a (contingent) stream of endowments $\{y_0, Y_1\}$ where $Y_1 = (y_{1,1}, \ldots, y_{1,s}, \ldots, y_{1,N})^\top$
- We say the **portfolio θ finances a (random) consumption sequence** $\{c_0, C_1\}$ if

$$\begin{aligned} c_0 &= y_0 + z_0^\theta \\ C_1 &= Y_1 + Z_1^\theta \end{aligned}$$

where

$$z_0^\theta = -\mathbf{S}\,\theta,\ Z_1^\theta = \mathbf{V}\,\theta$$

- we can write equivalently

$$\begin{aligned} c_0 + s &= y_0 \\ C_1 &= Y_1 + s\,\mathbf{R} \end{aligned}$$

where $\mathbf{R} = \mathbf{V}\,\mathrm{diag}(\mathbf{S})^{-1}$ and $s = \mathbf{S}\,\theta$ are savings

Budget constraints

Therefore, the household/saver has a **random sequence of budget constraints**, conditional on the information he has at $t = 0$:

- for period $t = 0$

$$c_0 = y_0 - \sum_{j=1}^{K} S_j \theta_j$$

- for period $t = 1$

$$\begin{aligned} c_{1,1} &= y_{1,1} + \sum_{j=1}^{K} V_{j,1} \theta_j \\ &\dots \\ c_{1,s} &= y_{1,s} + \sum_{j=1}^{K} V_{j,s} \theta_j \\ &\dots \\ c_{1,N} &= y_{1,N} + \sum_{j=1}^{K} V_{j,N} \theta_j \end{aligned}$$

The household problem in a finance economy

- Assumptions: the agent has a von Neumann Morgenstern utility functional, and operates in a finance economy $(\mathbf{S}, \mathbf{V})$
- household problem:
  find an optimal sequence of consumption, $\{C_t\}_{t=0}^{1}$**, and portfolio composition,** θ, that maximizes

$$\max_{\{c_0, C_1\}, \theta} \mathbb{E}_0 \left[u(c_0) + \beta u(C_1) \right]$$

subject to the **sequence** of budget constraints

$$\begin{aligned} c_0 &\leq y_0 - S\theta \\ C_1 &\leq Y_1 + V\theta \end{aligned}$$

given $(\mathbf{S}, \mathbf{V})$ and process for the endowment $\{y_0, Y_1\}$.

The household problem in a finance economy

- Expanding the expressions, we have an equivalent representation :

$$\max_{\{c_0,(c_{1,1},\dots,c_{1,N})\},(\theta_1,\dots,\theta_K)} u(c_0) + \beta \sum_{s=1}^{N} \pi_s u(c_{1,s})$$

- subject to the $N+1$ restrictions

$$\begin{aligned} c_0 &\leq y_0 - \sum_{j=1}^{K} S_j \theta_j \\ c_{1,s} &\leq y_{1,s} + \sum_{j=1}^{K} V_{j,s} \theta_j, \; s = 1, \dots, N \end{aligned}$$

- Differences to the problem in an AD economy:
  (1) in the AD economy the household has a **single intertemporal** (stock) constraint;
  (2) in a finance economy he/she has a **sequence of intraperiod** (flow) constraints

Solving the household's problem

Lagrangean

- Assume there is no satiation, i.e., $u^{'}(c) > 0$ for all $c > 0$ this implies that the constraints are saturated
- The Lagrangean for this problem is

$$\mathcal{L} = u(c_0) + \beta \sum_{s=1}^{N} \pi_s u(c_{1,s}) +$$

$$+ \lambda_0 \left(y_0 - \sum_{j=1}^{K} S_j \theta_j - c_0 \right) + \sum_{s=1}^{N} \lambda_{1,s} \left(y_{1,s} + \sum_{j=1}^{K} V_{j,s} \theta_j - c_{1,s} \right)$$

- we have to maximize it for: $K + 2 \times (1 + N)$ variables:

  $1 + N$ consumption variables: c_0, $(c_{1,1}, \ldots, c_{1,N})$,
  K portfolio components: $(\theta_1, \ldots, \theta_K)$
  $1 + N$ Lagrange multipliers: λ_0, $(\lambda_{1,1}, \ldots, \lambda_{1,N})$

Solving the household's problem

First order conditions

- optimality conditions for consumption: $1 + N$ equations

$$\begin{cases} u^{'}(\hat{c}_0) = \lambda_0, & (t=0) \\ \beta \pi_s u^{'}(\hat{c}_{1s}) = \hat{\lambda}_{1s},\ s = 1, \dots, N & (t=1) \end{cases}$$

- the optimality conditions for portfolio expenditure: K equations

$$\lambda_0 S_j \quad = \quad \sum_{s=1}^{N} \hat{\lambda}_{1s} V_{js},\ j = 1, \dots, K,\ (t = 0 \text{ vs } t = 1)$$

- and the period budget constraints, evaluated at the optimum: $1 + N$ equations

$$\begin{cases} \hat{c}_0 + \sum_{j=1}^{K} S_j \theta_j = y_0, & (t=0) \\ \hat{c}_{1s} = y_{1s} + \sum_{j=1}^{K} V_{js} \theta_j,\ s = 1, \dots, N & (t=1) \end{cases}$$

Solving the household's problem

- If we define the **shadow price of the state of nature** s

$$\hat{q}_s \equiv \frac{\lambda_{1s}}{\lambda_0},\ s = 1, \ldots, N$$

then we get an arbitrage condition for consumption

$$\beta\pi_s u^{'}(\hat{c}_{1s}) = \hat{q}_s u^{'}(\hat{c}_0),\ s = 1, \ldots, N \tag{1}$$

the optimal portfolio conditions become

$$S_j = \sum_{s=1}^{N} \hat{q}_s V_{js},\ j = 1, \ldots, K \tag{2}$$

Solving the household's problem

and the budget constraints

$$\hat{c}_0 = y_0 - \sum_{j=1}^{K} S_j \theta_j, \tag{3}$$

$$\hat{c}_{1s} = y_{1s} + \sum_{j=1}^{K} V_{js} \theta_j, \ \ s = 1, \ldots, N \tag{4}$$

Solving the household's problem

In complete or incomplete financial markets

- Complete financial markets:
  - in this case $\det(\mathbf{V}) \neq 0$
  - then **real and financial decision separate** ($\hat{Q}$ can be uniquely determined from $(\mathbf{S}, \mathbf{V})$)
  - allowing for explicit solutions
- Incomplete financial markets:
  - in this case $\det(\mathbf{V}) = 0$ (or $K < N$)
  - then **real and financial decision do not separate** (Q cannot be uniquely determined from $(\mathbf{S}, \mathbf{V})$)
  - rarely allowing for explicit solutions

The household problem when financial markets are complete

Solving the household's problem

In a complete financial market

- The shadow price of the states of nature is (using the optimal portfolio conditions) equation (2)

$$\mathbf{S}^\top = \mathbf{V}^\top \hat{Q}^\top \Rightarrow \hat{Q}^\top = (\mathbf{V}^\top)^{-1}\,\mathbf{S}^\top$$

  where $\hat{Q} = (\hat{q}_1, \ldots, \hat{q}_N)$

- If $K = N$ and $\det(\mathbf{V}) \neq 0$ then

$$(\mathbf{V}^\top)^{-1}\,\mathbf{S}^\top = \hat{Q}^\top$$

  is equivalent to

$$\boxed{\hat{Q} = \mathbf{S}\,\mathbf{V}^{-1}}$$

  depends only on market (exogenous) data

Solving the household's problem

In a complete financial market

- Equation (4) can be written as

$$\sum_{j=1}^{K} \theta_j V_{js} = c_{1s} - y_{1s}, \; s = 1, \dots, N$$

- or

$$\begin{cases} \theta_1 V_{11} + \ldots + \theta_K V_{K1} = c_{11} - y_{11} \\ \ldots \\ \theta_1 V_{1N} + \ldots + \theta_K V_{KN} = c_{1N} - y_{1N} \end{cases}$$

or

$$\mathbf{V}\,\theta = C_1 - Y_1$$

- Again, if $K = N$ and $\det(\mathbf{V}) \neq 0$ then the optimal portfolio composition is a linear function of $C_1 - Y_1$

$$\boxed{\hat{\theta} = (\mathbf{V})^{-1}(\hat{C}_1 - Y_1)}$$

Solving the household's problem

In a complete financial market

- Substituting in equation (3)

$$\hat{c}_0 - y_0 = -\mathbf{S}\,\hat{\theta} = -\mathbf{S}\,(\mathbf{V})^{-1}(\hat{C}_1 - Y_1) = -\hat{Q}\,(\hat{C}_1 - Y_1)$$

- That is

$$\hat{c}_0 + \sum_{s=1}^{N} \hat{q}_s\,\hat{c}_{1,s} = \hat{y}_0 + \sum_{s=1}^{N} \hat{q}_s\,\hat{y}_{1,s}$$

is identical to the constraint for the household problem in an Arrow-Debreu economy

Solving the household's problem

In a complete financial market

- Therefore, we can obtain the optimal consumption path from equation (1) and this equation

$$\beta \pi_s u^{'}(c_{1,s}) = \hat{q}_s u^{'}(c_0), \; s = 1, \ldots, N$$

$$\hat{c}_0 + \sum_{s=1}^{N} \hat{q}_s \hat{c}_{1s} = h_0 = y_0 + \sum_{s=1}^{N} \hat{q}_s y_{1s}$$

where h_0 is total wealth (present value of endowments)

- using the definition of stochastic discount factor $\hat{m}_s = \hat{q}_s / \pi_s$ we have equivalently

$$\boxed{\begin{aligned} \beta u^{'}(c_{1,s}) &= \hat{m}_s u^{'}(c_0), \; s = 1, \ldots, N \\ \hat{c}_0 + \mathbb{E}[M \hat{C}_1] &= h_0 = y_0 + \mathbb{E}[M\, Y_1] \end{aligned}}$$

Solving the household's problem

In a complete financial market

- **Example**: for a log utility function, $u(c) = \ln(c)$ we have

$$\frac{\beta}{c_{1,s}} = \frac{\hat{m}_s}{c_0},\ s = 1, \dots, N$$

- then $c_0 + \mathbb{E}[M\, C_1] = c_0 + \sum_{s=1}^{N} \pi_s\, m_s\, \frac{\beta}{m_s} c_0 = (1+\beta) c_0 = h_0$
- Therefore

$$\hat{c}_0 = \frac{1}{1+\beta}\, h_0$$
$$\hat{c}_{1,s} = \frac{\beta}{(1+\beta) m_s}\, h_0$$

- Conclusion: consumption is a function of wealth defined as the present value of endowments (using a shadow discount factor, which in a complete financial market is equal to the market stochastic discount factor)

The household problem when financial markets are incomplete

Solving the household's problem

In an incomplete financial market

- if $\det(\mathbf{V}) = 0$ (or $K < N$) then the financial market is incomplete
- and we have to solve **jointly**, for c_0, C_1, Q, θ the equations

$$\begin{cases} \beta\pi_s u'(c_{1,s}) = \hat{q}_s u'(c_0), & s = 1, \dots, N \\ \sum_{s=1}^{N} \hat{q}_s V_{js} = S_j, & j = 1, \dots, K \\ \hat{c}_0 = y_0 - \sum_{j=1}^{K} S_j \theta_j \\ \hat{c}_{1s} = y_{1s} + \sum_{j=1}^{K} V_{js}\theta_j, & s = 1, \dots, N \end{cases}$$

- First, we solve $\hat{Q}$ (dimension N) and $\hat{\theta}$ (dimension K) from the $N + K$ equations

$$\begin{cases} \beta\pi_s u'\left(y_{1s} + \sum_{j=1}^{K} V_{js}\theta_j\right) = \hat{q}_s u'\left(y_0 - \sum_{j=1}^{K} S_j\theta_j\right), & s = 1, \dots, N \\ \sum_{s=1}^{N} \hat{q}_s V_{js} = S_j, & j = 1, \dots, K \end{cases}$$

and, second, substitute in the budget constraint expressions to get $\hat{c}_0$ and $\hat{C}_1$.

Solving the household's problem

In an incomplete financial market (cont.)

- Only for the case of quadratic preferences, because $u^{'}(x)$ is linear in x, we can obtain **explicit solutions** for θ and Q (in all other case usually we cannot get explicit solutions, or existence and uniqueness may not be guaranteed)
- Why ? The origin of the problem is related to the fact that we cannot get $\hat{q}_s$ directly from $\mathbf{S}$ and $\mathbf{V}$, i.e, we cannot have $\hat{Q} = Q = \mathbf{S}\,\mathbf{V}^{-1}$, that is, **we cannot equate market and shadow prices for the states of nature**

Solving the household's problem

In an incomplete financial market (cont.)

Example:

- Let $u(c) = ac - \frac{b}{2}c^2$, assume that $N = 2$ and $K = 1$ where the price is S and the payoff is $V = (v_{1,1}, v_{1,2})^\top$
- the first-order conditions are, if we substitute $\hat{q}$,

$$\begin{aligned} u^{'}(\hat{c}_0) &= \beta\left[\pi u^{'}(\hat{c}_{1,1})R_1 + (1-\pi)u^{'}(\hat{c}_{1,2})R_2\right] && (5) \\ \hat{c}_{1,1} &= y_{1,1} + (y_0 - \hat{c}_0)R_1 && (6) \\ \hat{c}_{1,2} &= y_{1,2} + (y_0 - \hat{c}_0)R_2 && (7) \\ \hat{\theta} &= \frac{y_0 - \hat{c}_0}{S} && (8) \end{aligned}$$

- as $u^{'}(c) = a - bc$ we can solve equation (5) for $\hat{c}_0$ to get

$$\hat{c}_0 = \frac{a(1 - \mathbb{E}(R)) - b\left[\mathbb{E}[RY_1] + y_0\mathbb{E}[R^2]\right]}{b\left(1 + \beta\mathbb{E}[R^2]\right)}$$

Generic optimality conditions for the household problem

Characterizing household's behavior

Intertemporal arbitrage condition

Proposition 1

The ***intertemporal arbitrage condition*** *for any asset* j

$$S_j \, u^{'}(\hat{c}_0) = \beta \, \mathbb{E}[u^{'}(\hat{C}_1) \, V_j], \;\; j = 1, \ldots, K$$

holds irrespective of the completeness of markets.
Equivalently, in terms of the rate of return for any asset j

$$u^{'}(\hat{c}_0) = \beta \mathbb{E}[u^{'}(\hat{C}_1) R_j], \;\; j = 1, \ldots, K$$

where $R_j = V_j / S_j$.

Characterizing household's behavior

Intertemporal arbitrage condition

- Proof: **Independently of the financial market structure**, we have two optimality conditions

$$\begin{aligned} \hat{q}_s u^{'}(\hat{c}_0) &= \beta \pi_s u^{'}(\hat{c}_{1,s}),\ s = 1, \ldots, N \\ S_j &= \sum_{s=1}^{N} q_s V_{js},\ j = 1, \ldots, K \end{aligned}$$

- multplying the second equation by $u^{'}(\hat{c}_0)$, we have

$$u^{'}(\hat{c}_0) S_j = \sum_{s=1}^{N} q_s u^{'}(\hat{c}_0) V_{js}$$

then, using the first equation,

$$S_j u^{'}(\hat{c}_0) = \beta \sum_{s=1}^{N} \pi_s u^{'}(\hat{c}_{1s}) V_{js} = \beta \mathbb{E}[u^{'}(\hat{c}_1) V_j],\ \ j = 1, \ldots, K$$

- and use the definition $R_j = V_j / S_j$ we get the intertemporal arbitrage condition for the household.

General equilibrium asset pricing: representative agent economy

Dynamic stochastic general equilibrium for a finance economy

Homogeneous agent economy

Definition: **General equilibrium with rational expectations for an homogeneous economy** : it is the (random) sequence of consumption and of optimal portfolio and asset prices $(\{c_0^*, C_1^*\}, \theta^*, S^*)$ where $C_1^* = (c_{11}^*, \ldots, c_{1N}^*)$, $\theta^* = (\theta_1^*, \ldots, \theta_K^*)$ and $S^* = (S_1^*, \ldots, S_K^*)$ such that, given $\{y_0, Y_1\}$ and V:

- (1) (c^*, θ^*) is the solution of the households' problem where households have common knowledge and rational expectations over $\{Y\}$ and V ;
- (2) asset markets clear (when assets are in zero net supply)

$$\theta^* = \mathbf{0},$$

- (3) the product market is in equilibrium (demand = supply)

$$\begin{aligned} c_0^* &= y_0, \\ c_{1s}^* &= y_{1s}, \quad s = 1, \ldots, N. \end{aligned}$$

DSGE for a representative agent finance economy

Characteristics

In equilibrium:

- There is **no trade**
- Agents do not change the asset position: they are **neither creditors nor borrowers**
- Uncertainty: there is **both idiosyncratic and aggregate uncertainty**
- Therefore: there is **no insurance**
- But we can determine, depending on the fundamentals (preferences and endowments):
  - the equilibrium stochastic discount factor, M^*
  - equilibrium asset prices, $\mathbf{S}^*$

Equilibrium asset prices

Equilibrium asset price

Homogeneous agent economy

Proposition 2

Consider the DGSE just defined and assume an utility function with no satiation. Then there is an equilibrium stochastic discount factor $M^* = (m_1^*, \ldots, m_N^*)$, *where*

$$m_s^* = \beta \frac{u^{'}(y_{1,s})}{u^{'}(y_0)}, \; s = 1, \ldots, N,$$

such that the equilibrium price for asset j *is equal to the expected payoff*

$$S_j^* = \mathbb{E}[M^* V_j], \; \textit{for each} \, j =, 1, \ldots, K.$$

Equilibrium asset price

Homogeneous agent economy

- Proof: in an **homogeneous agent** economy, the equilibrium conditions (i,e $(c_0^*, C_1^*, \theta^*, S^*)$) are

$$\begin{cases} S_j^* u^{'}(c_0^*) = \beta \mathbb{E}[u^{'}(c_1^*) V_j], & j = 1, \dots, K \\ \theta_j = 0, & j = 1, \dots, K \\ c_0^* = y_0, & \\ c_{1,s}^* = y_{1,s}^*, & s = 1, \dots, N \end{cases}$$

  we just need to substitute the equilibrium values of consumption into the household's arbitrage condition
- Therefore, the **equilibrium exists and is unique**.
- Observation: in **heterogenous** agents' finance economies the existence and uniqueness is not guaranteed: in particular if the financial markets are incomplete (remember that in these economies we do not have in general $\hat{q} = Q$ and Q is not unique).

Equilibrium asset price

Homogeneous agent economy

- Proof cont.: If we define the **equilibrium** stochastic discount factor as the random variable $M^* = (m_1^*, \ldots, m_N^*)$ such that

$$m_s^* = \beta \frac{u^{'}(y_{1,s})}{u^{'}(y_0)}, \; s = 1, \ldots, N$$

- Then equilibrium asset prices can be obtained explicitly from the **equilibrium arbitrage condition for asset market** j

$$S_j^* = \sum_{s=1}^{N} \pi_s^* m_s^* V_{j,s}, \; j =, 1, \ldots, K$$

Equilibrium returns

Homogeneous agent economy

Proposition 3

Consider the DGSE just defined and assume an utility function with no satiation. Then there is an equilibrium stochastic discount factor $M^* = (m_1^*, \ldots, m_N^*)$*, where*

$$m_s^* = \beta \frac{u^{'}(y_{1,s})}{u^{'}(y_0)},\; s = 1, \ldots, N,$$

such that the equilibrium return for asset j *is equal to one*

$$\mathbb{E}[M^* R_j] = 1,\; \textit{for every}\, j =, 1, \ldots, K.$$

Equilibrium returns

Homogeneous agent economy

- This implies, the following equilibrium arbitrage condition is satisfied

$$\mathbb{E}[M^* R_1^*] = \mathbb{E}[M^* R_2^*] \ldots = \mathbb{E}[M^* R_K^*] = 1$$

- In words: in equilibrium the mathematical expectation of the discounted returns for all assets is equalized

Example: CRRA utility function

- The utility function is CRRA

$$u(C) = \frac{C^{1-\eta} - 1}{1 - \eta}$$

and the endowment follows the process

$$Y_1 = (1 + \Gamma) y_0$$

where $\Gamma = (\gamma_1, \ldots, \gamma_N)$

- The equilibrium stochastic discount factor is a distribution

$$m_s^* = \beta(1 + \gamma_s)^{-\eta}, \; s = 1, \ldots, N$$

or in vector notation

$$M^* = \beta(1 + \Gamma)^{-\eta}$$

Example: CRRA utility function

- Then the equilibrium asset price is, for any asset j

$$S_j^* = \beta\,\mathbb{E}[(1+\Gamma)^{-\eta}\,V_j].$$

the price is equal to the expected payoff, using the equilibrium stochastic discount factor

- The equilibrium asset return for asset t is determined from

$$1 = \beta\,\mathbb{E}[(1+\Gamma)^{-\eta}\,R_j].$$

- or,

$$1+\rho = \mathbb{E}[(1+\Gamma)^{-\eta}\,(1+r_j)]$$

because the psychological discount factor is $\beta = \frac{1}{1+\rho}$ where ρ is the rate of time preference,

Equilibrium asset prices and returns in the presence of a risk-free asset

Equilibrium asset return

When there is a risk-free asset

- Assume there is a risk-free asset with return $R^f = 1 + i$
- At the equilibrium R^f is the inverse of $\mathbb{E}[M]$

$$\mathbb{E}[MR^f] = 1 \iff \mathbb{E}[M]\, R^f = 1 \Rightarrow R^f = \frac{1}{\mathbb{E}[M]}$$

- Introduce a **market probability measure** (recall our definition in the last lecture)

$$\pi_s^M = \frac{\pi_s m_s}{\mathbb{E}[M]} = \frac{\pi_s m_s}{\sum_{s=1}^{N} \pi_s m_s}$$

  we get a **market probability distribution** $\mathbb{P}^M = (\pi_1^M, \dots, \pi_N^M)$

- Then from $\mathbb{E}[M^* R_j^*] = 1$ we have

$$\boxed{\mathbb{E}^M[R_1^*] = \mathbb{E}^M[R_2^*] \ldots = \mathbb{E}^M[R_K^*] = R^f}$$

  the expected return for all assets are equal and equal to the risk free rate.

Equilibrium equity premium

Equity premium (in equilibrium)

- Assume there are two assets:
  - one equity or any risky asset with return $R = (1 + r) = (1 + r_1, \dots 1 + r_N)^\top$
  - one risk-free asset with return $R^f = 1 + i$
- The **equity premium** is the differences in the rates of return:

$$\text{EP} \equiv R - R^f = r - i = \begin{pmatrix} r_1 - i \\ \dots \\ r_N - i \end{pmatrix}$$

- and the **Sharpe index** is the rate

$$\frac{\mathbb{E}[EP]}{\sigma[EP]} = \frac{\mathbb{E}[R - R^f]}{\sigma[R - R^f]} = \frac{\mathbb{E}[r - i]}{\sigma[r]}$$

where $\mathbb{E}[r - i] = \sum_{s=1}^{N} \pi_s (r_s - i)$ and $\sigma[r] = \sum_{s=1}^{N} \pi_s r_s^2$ are the expected value and the standard deviation for the risk premium

Equilibrium equity premium

- From the previous model the equilibrium equity premium is

$$\mathbb{E}[M(r-i)] = 0$$

- Assuming a CRRA utility function and a growing endowment economy we have

$$\mathbb{E}[(1+\Gamma)^{-\eta}(r-i)] = 0$$

- It can be proved that the Sharpe index should verify, if this theory is correct

$$\boxed{\frac{|\mathbb{E}[r-i]|}{\sigma[r]} \leq \sigma[(1+\Gamma)^{-\eta}]}$$

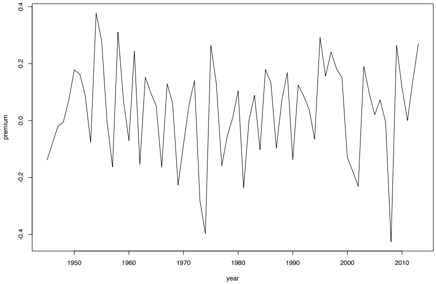
(prove this)

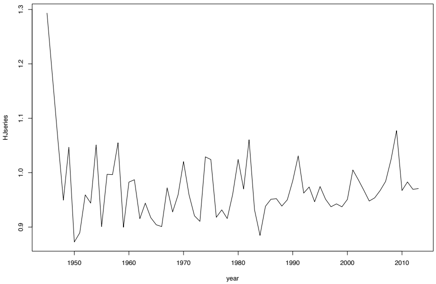
- The term, $\sigma[(1+\Gamma)^{-\eta}]$ is called the **Hansen-Jagannathan bound**.

Equity premium puzzle

- Equity premium puzzle: Mehra and Prescott (1985) provide a simple test to the theory
- In our framework this can be seen simply by comparing data with the theory:
  - **Mehra and Prescott (US)** the Sharpe ratio is roughly around 0.5
$$\frac{\mathbb{E}[r-i]}{\sigma[r]} \approx \frac{0.07}{0.16} \approx 0.45$$
  - **Theory**: for a log utility ($\eta = 1$) the standard deviation of the stochastic discount factor (taking real per capita growth) is
$$\sigma[(1+\Gamma)^{-1}] \approx 0.01$$
  - but the Hansen-Jaganathan bounds will only hold for $\eta > 20$, which is considered irrealistic

Equity premium puzzle: data





Equity premium puzzle

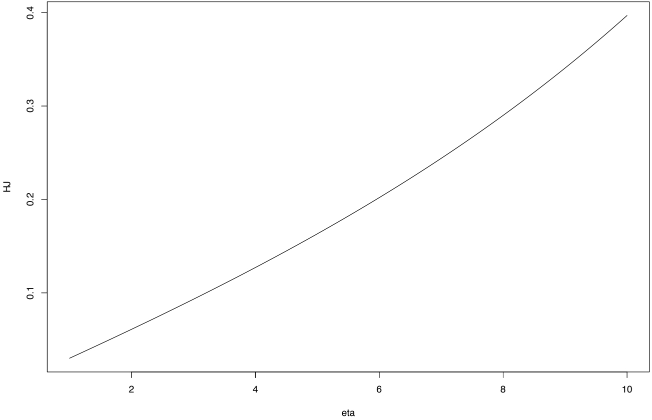


Figure: HJ bounds for different values of η

Equity premium puzzle

- The Puzzle: the risk premium is much larger in the data as compared to what is implied by the model.
- Intuition: the response of consumption to (future) income shocks is much stronger in the model than in the data
- Solution: hundreds of papers, books, Phd theses with different solutions: how to account for the smoother dynamics of consumption (or less responding of asset holdings regarding changes in the rates of return) in the data in comparison with the model.

Equivalence between finance and Arrow-Debreu economies

Equivalence between Radner and Arrow-Debreu equilibria

Proposition 3

Consider a finance economy in which there is absence of arbitrage opportunities and the markets are complete. Then the general equilibrium in this finance economy and in a Arrow-Debreu economy with the same fundamentals (preferences and endowments) are equivalent

Equivalence between Radner and Arrow-Debreu equilibria

Proof

The households' problems' in the two economies are

- the intertemporal utility functional is the same

$$U(\{C\}) = u(c_0) + \beta \mathbb{E}_0\big[u(C_1)\big]$$

  where $\{C\} = \{c_0, C_1\}$

- in the **AD** economy it has a single constraint

$$c_0 - y_0 + q(C_1 - Y_1) = 0$$

- but in the **finance** economy there is a sequence of period budget constraints

$$c_0 - y_0 + \mathbf{S}\,\theta = 0$$
$$C_1 - Y_1 - \mathbf{V}\,\theta = 0$$

Equivalence between Radner and Arrow-Debreu equilibria

Proof (cont.)

If **markets are complete** :

- then we can **transform the sequence of period budget constraints** for a finance economy in an **unique intertemporal budget constraint**:
  1. because $\det(\mathbf{V}) \neq 0$ then from the $t = 1$ constraints we get uniquely
  $$\theta = \mathbf{V}^{-1}(C_1 - Y_1)$$
  2. then substituting at the $t = 0$ constraint
  $$c_0 - y_0 + \mathbf{S}\,\mathbf{V}^{-1}(C_1 - Y_1) = 0$$
  where $Q = SV^{-1} >> 0$ is the vector of AD prices **if there are no arbitrage opportunities**.

Equivalence between Radner and Arrow-Debreu equilibria

- When financial markets are **complete** : we determine the prices of AD contracts from asset prices and payoffs as $Q^{\top} = (\mathbf{R}^{\top})^{-1}\mathbf{1}$.

  The general equilibria in the two economies are equivalent because:

  - market equilibrium conditions are the same $c_0^* = y_0$ and $C_1^* = Y_1$
  - the state prices of the states of nature are equal $q_s = \beta\pi_s(u^{'}(y_{1,s})/u^{'}(y_0))$

- In a **homogeneous agent economy** the equivalence always exists irrespective of the finance market structure

- In **heterogeneous agent economies** the equivalence always exists if the financial markets are complete, but equivalence may not hold if financial markets are incomplete.

Take away

- When markets are complete the own state price of the household is equal to the market state price and the household problem is formally similar to the problem for an agent in an Arrow-Debreu economy
- When markets are incomplete, although we may solve the household problems, it may not have closed form solutions
- In a representative agent economy the arbitrage condition holds irrespective of the market being complete or incomplete

$$u^{'}(c_0) = \beta\, \mathbb{E}[u^{'}(C_1)\, R_j] \text{ for every asset j}$$

- In a simple representative agent economy the equilibrium stochastic discount factor is negatively correlated to the growth rate of the endowments, if agents are risk averse
- The simple version of the model needs to be changed in order to match empirical data (equity premium puzzle)
- Associated problem set: Problem set 6