Foundations of Financial Economics
Two period GE: limited participation

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Differences with the benchmark model

- The participation in the risky asset market is not proportional to household wealth: U.S data similar shape for different countries
- Potential explanations: differences in patience, risk aversion, information
- In this lecture: difference in beliefs together with a **friction** (agents cannot have short positions in assets)
- Wealth takes the form of financial wealth only
- There is positive net wealth (external finance: external money and a another risky asset in positive net supply)

Possible extensions and applications

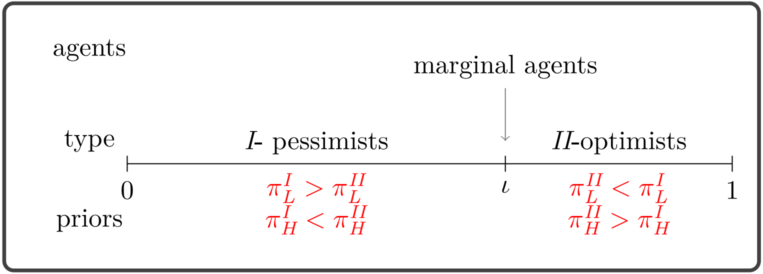
- The model is in the other extreme of the benchmark model we studied until this point
- A half-way model would consider that internal finance (short positions) is possible, but it is constrained by **collateral constraints**: short positions are limited by the existence of a long position in another asset that should be offered as collateral (for instance money)
- This partly explains
  - the demand for liquidity, for instance for firms,
  - the characteristics of the 2008 and the Euro crises (a crisis may be brewing without signs in the behavior of interest rates)
  - and the increasing consideration of the so-called balance effects in macro-finance models.

Topics

- Environemnt
- Types of agents: participation in the risky asset market
- Endogenous market participation (related to priors on the likelihood of the good and bad state: pessimists and optimists)
- The equilibrium interest rate depends on the participation in the risky asset market
- Interest rate response to news in an asymetric way
- **Welcome to financial economics post 2008 !**

Environment: fundamentals

- Information: two-period binomial tree with two states $s = L, H$
- **Heterogeneity in priors regarding the states of nature**.
  - there is a continuum of agents $i \in [0, 1]$
  - divided into two main groups: pessimists $i \in [0, \iota]$ giving more weight to $s = L$ and optimists $i \in [\iota, 1]$ giving more weight to $s = H$
  - There are marginal agents with $i = \iota$ (very small group).



- off course: $\pi_L^I + \pi_H^I = \pi_L^{II} + \pi_H^{II} = 1$

Markets

- I assume a finance economy in which there are two assets: **money** and a **risky asset**
- The asset prices and payoffs are

$$\mathbf{S} = (1, S_2), \ \mathbf{V} = \begin{pmatrix} 1 & v_{2L} \\ 1 & v_{2H} \end{pmatrix}$$

  L is a bad state and H is a good state: $v_{2L} < v_{2H}$

- Therefore, the return matrix is

$$\mathbf{R} = (R_1, R_2) = \begin{pmatrix} 1 & R_{2L} \\ 1 & R_{2h} \end{pmatrix}$$

  where $R_{2,s} = v_{2s}/S_2$ for $s = L, H$

- Assume there are no arbitrage opportunities:

$$R_{2L} < 1 < R_{2H}$$

Generic household problem

- The problem for household of type $i \in [0,1]$

$$\max_{c_0^i, C_1^i, \theta^i} u(c_0^i) + \beta \mathbb{E}^i[u(C_1^i)]$$

where $\mathbb{E}^i[u(C_1^i)] = \sum_{s \in \{H,L\}} \pi_s^i u(c_{1s}^i)$, subject to

$$\begin{aligned}
c_0^i + \theta_1^i + S_2\theta_2^i &= y_0^i + w^i \\
c_{1,s}^i &= \theta_1^i + v_{2s}\theta_2^i, \ s = L, H \\
\theta_1^i &\geq 0 \text{ (friction: no short position allowed)} \\
\theta_2^i &\geq 0 \text{ (friction: no short position allowed)}
\end{aligned}$$

- Assume that the initial wealth composition may contain the two types of assets for any type of agent

$$w^i = w_1^i + S_2 w_2^i$$

with $w_1^i > 0$ and $w_2^i > 0$ for any i given.

Generic household problem

- **Simplifying assumptions**: $c_0^i = y_0^i$ and $Y_1^i = \mathbf{0}$: consumption in period 1 is only financed by financial returns)
- Then the constraint in period zero simplifies to:

$$\theta_1^i + S_2\theta_2^i = w^i$$

(meaning that is a change in the portofio such that $\theta_1^i - w_1^i = S_2\left(\theta_2^i - w_2^i\right)$)

Generic household problem

The problem for household of type $i \in [0, 1]$ is simplified to

$$
\begin{aligned}
\max_{\theta_1^i, \theta_2^i} & u(y_0^i) + \beta \mathbb{E}^i \big[u(\theta_1^i + V_2 \theta_2^i) \big] \\
& \text{subject to} \\
& \theta_1^i + S_2 \theta_2^i = w^i \\
& \theta_1^i \geq 0 \\
& \theta_2^i \geq 0
\end{aligned}
$$

Solving the generic agent's problem

- Lagrangean

$$\mathcal{L}^i = u(y_0^i) + \sum_{s\in\{H,L\}} \beta\pi_s^i u\big(\theta_1^i + v_{2s}\theta_2^i\big) + \lambda^i(w^i - \theta_1^i - S_2\theta_2^i) +$$
$$+ \mu_1^i\theta_1^i + \mu_2^i\theta_2^i$$

- Optimality conditions

$$\frac{\partial\mathcal{L}^i}{\partial\theta_1^i} = 0 \Leftrightarrow \lambda^i = \beta\left(\sum_{s\in\{H,L\}} \pi_s^i u'(c_{1s}^i)\right) + \mu_1^i$$
$$\frac{\partial\mathcal{L}^i}{\partial\theta_2^i} = 0 \Leftrightarrow S_2\lambda^i = \beta\left(\sum_{s\in\{H,L\}} \pi_s^i u'(c_{1s}^i) v_{2s}\right) + \mu_2^i$$

- Complementary slackness conditions

$$\mu_1^i\theta_1^i = 0,\ \mu_1^i \geq 0,\ \theta_1^i \geq 0$$
$$\mu_2^i\theta_2^i = 0,\ \mu_2^i \geq 0,\ \theta_2^i \geq 0$$

Behavior of agent of type I (pessimist):

- Agents of type I sell their initial stock of the risky asset and invests in money: $\theta_1^I > 0$ and $\theta_2^I = 0$
- Then

$$\begin{aligned} \theta_1^I &= w^I = w_1^I + S_2 w_2^I \\ c_{1s}^I &= w^I \end{aligned}$$

is state-independent

- From complementary slackness: $\mu_1^I = 0$ and $\mu_2^I > 0$. Then

$$\lambda^I = \beta \left(\sum_{s \in \{H,L\}} \pi_s^I u'(c_{1s}^I) \right) > \beta \left(\sum_{s \in \{H,L\}} \pi_s^I u'(c_{1s}^I) R_{2s} \right)$$

- Equivalently $\mathbb{E}^I[u'(C_1^I)] > \mathbb{E}^I[u'(C_1^I) R_2]$

Behavior of agent of type I (pessimist)

- Defining the utlity weighted prior:

$$\pi_s^{i_u} \equiv \frac{\pi_s^i \, u'(c_{1,s}^i}{\sum_{s\in\{H,L\}} \pi_s^I u'(c_{1s}^I)}, \text{for } s = L, H$$

- As $\pi_s^{i_u} > 0$ and $\pi_L^{i_u} + \pi_H^{i_u} = 1$
- Then $\mathbb{P}^{i_u} = \left(\pi_H^{i_u}, \pi_S^{i_u}\right)$ is a idiosyncratic probability distribution for agent i
- Given the return of asset j, $R_j = (R_{j,H}, R_{j,L}$ then

$$\mathbb{E}^i[u'(C_1^i)R_j] = \mathbb{E}^{i_u}[R_j], \text{ for every asset } j = 1, 2 \text{ for agent} i = I, II$$

Behavior of agent of type I (pessimist)

- Then $\mathbb{E}^I[u'(C_1^I)] > \mathbb{E}^I[u'(C_1^I)R_2]$ is equivalent to

$$\mathbb{E}^I[R_1] > \mathbb{E}^{I_u}[R_2],$$

- Then pessimists have a prior $(\mathbb{P}^{I_u})$, i.e., a risk- probability distribution, such that

$$\boxed{R_1 = 1 > \mathbb{E}^{I_u}[R_2]}$$

agent I invests in the risk-free asset because **he finds** its anticipated return on money (i.e., 1) higher than that of the risky asset.

Behavior of agent of type II (optimist)

- Agents of type II sell their initial stock of money and invest in risky asset: $\theta_1^{II} = 0$ and $\theta_2^{II} > 0$
- Then

$$\theta_2^{II} = \frac{w^{II}}{S_2} = \frac{w_1^{II} + S_2 w_2^{II}}{S_2}$$

$$c_{1s}^{II} = \frac{v_{2s}}{S_2} w^{II} = \frac{w^{II}}{R_{2s}}$$

is state-dependent (i.e., risky)

- From complementary slackness: $\mu_1^{II} > 0$ and $\mu_2^{II} = 0$. Then

$$\lambda_0^{II} = \beta \left(\sum_{s \in \{H,L\}} \pi_s^{II} u'(c_{1s}^{II}) R_{2s} \right) > \beta \left(\sum_{s \in \{H,L\}} \pi_s^{II} u'(c_{1s}^{II}) \right)$$

Then $\mathbb{E}^{II}[u'(C_1^{II}) R_1] = \mathbb{E}^{II}[u'(C_1^{II})] < \mathbb{E}^{II}[u'(C_1^{II}) R_2]$.

- Then optimists have a different prior ($\mathbb{P}^{II_u}$), i.e., an equivalent probability distribution such that

$$\boxed{\mathbb{E}^{II_u}[R_2] > 1}$$

Marginal agent

- Agents of type I prefer holding money to holding the risky asset because
$$\mathbb{E}^{I_u}[R_2] < 1$$

- Agents of type II prefer holding the risky asset rather than money because
$$\mathbb{E}^{II_u}[R_2] > 1$$

- By continuity, **there should exist a marginal agent (with wealth weight of zero)** having a probability distribution such that
$$\mathbb{E}^{\iota}[R_2] = 1 \Leftrightarrow \boxed{S_2 = \pi^{\iota} v_{2L} + (1 - \pi^{\iota}) v_{2H}} \tag{1}$$

Equilibrium in the asset markets

- Generic equilibrium conditions (total de mand = total supply)

$$\iota\theta_1^I + (1-\iota)\theta_1^{II} = w_1 = w_1^I + w_1^{II}$$
$$\iota\theta_2^I + (1-\iota)\theta_2^{II} = S_2 w_2 = S_2\big(w_2^I + w_2^{II}\big)$$

  where w_j is the aggregate stock of total of asset $j = 1, 2$ and ι is the proportion in the population (of size equal to 1) of agents of type I: **non-investors in the risky asset**

- Using the previous demand agent-level results

$$\iota\theta_1^I = \iota w^I = w_1$$
$$(1-\iota)\theta_2^{II} = (1-\iota)w^{II} = S_2 w_2$$

  (remember that $\theta_1^{II} = \theta_2^I = 0$ and $\theta_1^I = w^I$ and $\theta_2^{II} = w^{II}$)

- **The equilibrium values for S_2 and ι are jointly determined**: the asset price depends on the rate of participation.

Equilibrium in the asset markets

- Assumption: homogeneity in the distribution of wealth among optimists and pessimists, that is $w^I = w^{II} = \bar{w}$
- Then the **equilibrium price for the risky asset** is

$$\boxed{S_2^* = S_2(\iota) = \left(\frac{1-\iota}{\iota}\right)\frac{w_1}{w_2}} \tag{2}$$

- As

$$\frac{\partial S_2}{\partial \iota} = -\frac{w_1}{\iota^2 w_2} < 0$$

  The asset price decreases with the proportion of non-participation ι (i.e., if there are more pessimists the asset price decreases)

- The asset price increases with the stock of money w_1

Equilibrium distribution of agents

- Assumption: the probability distribution of the marginal investor, π^ι, is a function of their weight in the total population ι. For simplicity let $\pi^\iota = \iota$.
- Then, from equations (1) and (2), the equilibrium value $\iota^* = \{\iota \in (0,1) : \mathcal{I}(\iota) = 0\}$ where

$$\mathcal{I}(\iota) \equiv (1-\iota) w_1 - (\iota v_{2L} + (1-\iota) v_{2H})\, \iota w_2$$

Proposition

There is one unique value $\iota^* \in (0,1)$,

$$\iota^* = \frac{v_{2H} w_2 + w_1}{2(v_{2H} - v_{2L}) w_2} - \left[\left(\frac{v_{2H} w_2 - w_1}{2(v_{2H} - v_{2L}) w_2} \right)^2 + \frac{4 v_{2L} w_1 w_2}{4(v_{2H} - v_{2L})^2 w_2^2} \right]^{\frac{1}{2}}$$

Equilibrium participation

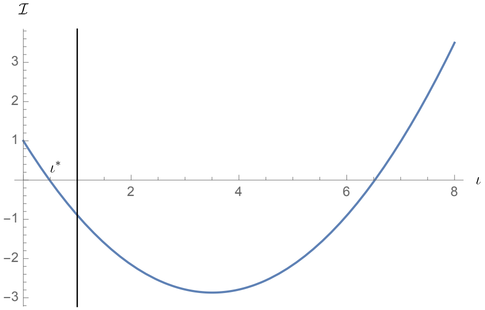


Figure: Proof of Proposition

Equilibrium distribution of agents

Proof

- **Proof that $\iota^* \in (0,1)$ exists and is unique.**
  Function $\mathcal{I}(\iota)$ is convex in ι (U-shaped) and therefore there can be zero, one or two values of ι such that $\mathcal{I}(\iota) = 0$ for $-\infty < \iota < \infty$. However, the domain of ι is $(0,1)$. It is easy to see that $\mathcal{I}(0) = w_1 > 0$, $\mathcal{I}^{'}(0) = -(w_1 + v_{2H}w_2) < 0$ and $\mathcal{I}(1) = -(w_1 + v_{2L}w_2) < 0$: therefore, in the interval $(0,1)$ there is one and only one value of ι, ι^* such that $\mathcal{I}(\iota) = 0$. Because the function is convex it has two points $0 < \iota_- < 1 < \iota_+$ such that $\mathcal{I}(\iota) = 0$ and the first one is the solution we are looking for.

Equilibrium distribution of agents

Properties

Proposition

The participation rate $(1-\iota^*)$ *increases with the payoff (for any state of nature) and the aggregate stock of the risky asset and reduces with the aggregate stock of money.*

- We showed that $\iota^* = \iota(v_{2H}, v_{2L}, w_1, w_2)$, and next we prove that

$$\boxed{\frac{\partial \iota^*}{\partial v_{2s}} < 0, \text{ for } s = H, L, \ \frac{\partial \iota^*}{\partial w_1} > 0, \ \frac{\partial \iota^*}{\partial w_2} < 0}$$

Equilibrium participation

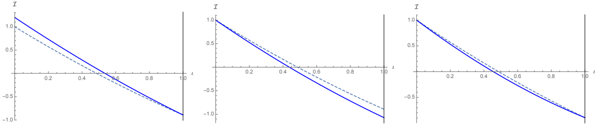


Figure: Change in participation

Equilibrium distribution of agents

Proof

- **Proof of the sign relationships for** $\frac{\partial \iota^*}{\partial v_{2s}}$

  We know that $\mathcal{I}(\iota, v_{2H}, v_{2L}) = 0$. Therefore, the response of ι to the payoffs is

$$\frac{\partial \iota^*}{\partial v_{2s}} = - \left. \frac{\mathcal{I}_{v_{2s}}}{\mathcal{I}_{\iota}} \right|_{\iota = \iota^*}, \ s = L, H$$

- Where $\mathcal{I}_{v_{2H}} = -\iota^*(1-\iota^*)w_2 < 0$ and $\mathcal{I}_{v_{2L}} = -(\iota^*)^2 w_2 < 0$ and

$$\mathcal{I}_{\iota} = 2(v_{2H} - v_{2L})w_2 \left(\iota^* - \frac{w_1 + v_{2H}w_2}{2(v_{2H} - v_{2L})w_2} \right) < 0$$

  because $0 < \iota^* < \frac{w_1 + v_{2H}w_2}{2(v_{2H} - v_{2L})w_2}$

Equilibrium rate of return for the risky asset

- Equilibrium rate of return of the risky asset is

$$\boxed{R^*_{2,s} = \frac{v_{2s}}{S_2(v_{2L}, v_{2H}, .)}, s = L, H} \tag{3}$$

- As

$$\frac{\partial S_2}{\partial \iota} < 0,\ \frac{\partial \iota}{\partial v_{2s}} < 0,\ s = H, L$$

  then

$$\frac{\partial S_2}{\partial v_{2s}} > 0 \text{ for any } s = H, L$$

- This means that **if there is an increase in** v_{2s} **generates two effects on** R_{2s}:
  - a direct positive effect (of the payoff in the "own" state)
  - a negative indirect effect, because the prices increases as a result of the change in the participation in the risky asset market
  - The **final effect is ambiguous**.

Equilibrium rate of return for the risky asset

- For the case in which there is **no change in participation** we have

$$\frac{d\bar{R}_{2s}}{dv_{2s}} = \frac{1}{\bar{S}_2} > 0, \frac{d\bar{R}_{2s'}}{dv_{2s}} = 0, \; s \neq s^{'} = H, L$$

- The rate of return outcome for a particular state of nature **only changes when the payoff outcome for the same state of nature changes**.

Equilibrium R distribution and news

Proposition

If there is a change in participation, then a change in any of the anticipated outcomes in the payoff distribution will change the rate of return, whatever the state of nature that occurs at time $t = 1$. However, the change will be state-dependent. In particular, we have

	ΔR_{2L}	ΔR_{2H}
Δv_{2L}	+ (+)	- (0)
Δv_{2H}	- (0)	+ (+)

Table: *In parenthesis no change in participation*

Equilibrium rate of return for the risky asset

- Proof: When there is a change in participation we have

$$\frac{\partial R_{2s}}{\partial v_{2s}} = \frac{1 - \epsilon_{\iota}^{S_2} \epsilon_{v_{2s}}^{\iota}}{S_2(\iota^*)}, \ \frac{\partial R_{2s'}}{\partial v_{2s}} = -\frac{v_{2s'}}{v_{2s}} \frac{\epsilon_{\iota}^{S_2} \epsilon_{v_{2s}}^{\iota}}{S_2(\iota^*)}, \ s \neq s^{'} = L, H$$

  where

  - the elasticity of S_2 to ι is

$$\epsilon_{\iota}^{S_2} = \frac{\partial S_2}{\partial \iota} \frac{\iota}{S_2} - \frac{1}{1 - \iota^*} < -1$$

  - the elasticity of ι to v_{2s} is

$$\epsilon_{v_{2s}}^{\iota} = \frac{\partial \iota^*}{\partial v_{2s}} \frac{v_{2s}}{\iota}, \ s = H, L$$

  - The rate of return outcome for a particular state of nature changes with **changes in the payoff of any state of nature** due to the change in participation.

Equilibrium R distribution and news

- Proof (cont): For a change in v_{2H} we have a change in the distribution of R_2
  - if the good state occurs
$$\frac{\partial \bar{R}_{2H}}{\partial v_{2H}} = -\frac{1}{S_2(\iota^*)}\left(\frac{2(v_{2H} - v_{2L})w_2\iota^*(1-\iota_+)}{\mathcal{I}_\iota}\right) > 0$$
  - if the bad state state occurs
$$\frac{\partial \bar{R}_{2L}}{\partial v_{2H}} = -\frac{1}{S_2(\iota^*)}\frac{v_{2L}}{v_{2H}}\epsilon_\iota^{S_2}\epsilon_{v_{2H}}^\iota < 0$$
- For a change in v_{2L} we have a change in the distribution of R_2
  - if the good state occurs
$$\frac{\partial \bar{R}_{2L}}{\partial v_{2L}} = -\frac{w_2}{\mathcal{I}_\iota} > 0$$
  - if the bad state state occurs
$$\frac{\partial \bar{R}_{2H}}{\partial v_{2L}} = -\frac{1}{S_2(\iota^*)}\frac{v_{2H}}{v_{2L}}\epsilon_\iota^{S_2}\epsilon_{v_{2L}}^\iota < 0$$

Equilibrium R distribution and news

- A **positive news regarding the good state** v_{2H}, $\Delta v_{2H} > 0$, generates an increase in the rate of return if the good state occurs and a decrease in the rate of return if the bad state occurs:

$$\Delta v_{2H} > 0 \ \Rightarrow \Delta R_{2L} < 0 < \Delta R_{2H}$$

This is because

$$v_{2H} \uparrow \rightarrow \iota \downarrow \rightarrow S_2 \uparrow \rightarrow \begin{cases} R_{2L} = v_{2L}/S_2 & \downarrow \\ R_{2H} = v_{2H}/S_2 & \uparrow \end{cases}$$

- a **negative news regarding the bad state**, v.g., $\Delta v_{2L} < 0$, there is an increase in the rate of return if the good state occurs and a reduction if the bad state occurs

$$\Delta v_{2L} < 0 \ \Rightarrow \Delta R_{2L} < 0 < \Delta R_{2H}$$

this is because

$$v_{2L} \downarrow \rightarrow \iota \uparrow \rightarrow S_2 \downarrow \rightarrow \begin{cases} R_{2L} = v_{2L}/S_2 & \downarrow \\ R_{2H} = v_{2H}/S_2 & \uparrow \end{cases}$$

Equilibrium participation

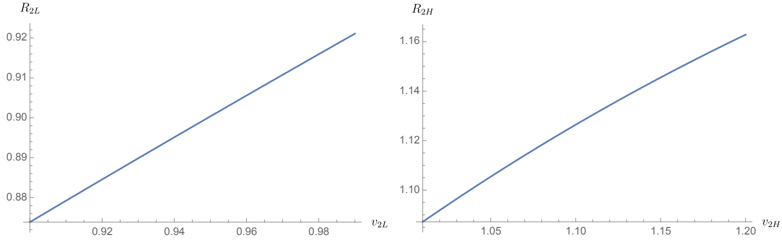


Figure: Reaction to news: R_{2L} to v_{2L} and R_{2H} to v_{2H}

Conclusions

- We showed that when priors differ, and there are participation frictions in the asset market, **asymmetric expected changes in payoffs have an effect on the whole distribution of the rate of return** for risky assets
- Good news regarding the good state or bad news regarding the bad state lead to a kind of an **amplification** response of the rate of return: a higher realized rate of return if the good state realizes and a lower rate of return if the bad state realizes.
- Other results: an expansion in the money supply $M = w_1$ will increase the rate of return for all states of nature

$$M \uparrow \rightarrow \iota \uparrow \rightarrow S_2 \downarrow \rightarrow \begin{cases} R_{2L} = v_{2L}/S_2 & \uparrow \\ R_{2H} = v_{2H}/S_2 & \uparrow \end{cases}$$

References

This lecture is adapted from Geanakoplos (2010) and Fostel and Geanakoplos (2014).

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