Foundations of Financial Economics
Two period SGE: production economy

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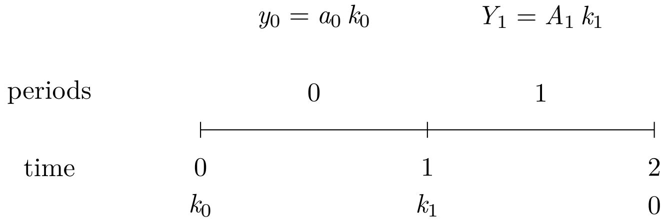
Topics

- Technology and income in a production economy
- AD equilibrium in a production economy
- Asset pricing consequences

Technology and income in a production economy

Timing of income



- productivity in period 1 $A_1 = (a_{1,1}, \ldots, a_{1,N})$
- income in period 1 $Y_1 = (y_{1,1}, \ldots, y_{1,N})$

Income in a production economy

- Assume a two-period economy with uncertainty for period 1: $s \in \{1, \ldots, N\}$;
- With production, the supply of goods is given by the sequence $\{y_0, Y_1\}$, where

$$\begin{aligned} y_0 &= a_0 k_0,\ t = 0 \\ y_{1,s} &= a_{1,s} k_1,\ t = 1,\ s = 1, \ldots, N \end{aligned}$$

  where k_t is the capital stock and A_t is a time-dependent productivity parameter.
- Therefore $Y_1 = A_1 k_1$, that is

$$Y_1 = \begin{pmatrix} y_{1,1} \\ \ldots \\ y_{1,s} \\ \ldots \\ y_{1,N} \end{pmatrix} = \begin{pmatrix} a_{1,1} \\ \ldots \\ a_{1,s} \\ \ldots \\ a_{1,N} \end{pmatrix} k_1$$

Endogeneity of Income in a production economy

- The capital stock at the beginning of period 1 is determined by savings in period 0

$$k_1 - k_0 = s_0 = y_0 - c_0$$

- Then the availability of the good in period 1 depends on the consumption at period 0

$$Y_1 = A_1(k_0 + s_0) = A_1((1 + a_0)k_0 - c_0)$$

- The **distribution** of income available at period 1 is

$$y_{1,s} = a_{1,s}\left((1 + a_0)k_0 - c_0\right), \; s = 1, \dots, N$$

The rate of growth

- The growth rate of the economy is endogenous

$$1 + g_s = \frac{y_{1,s}}{y_0} = (1 + \gamma_s)\left(1 + a_0 - \frac{c_0}{k_0}\right),$$

  where productivity grows at the rate γ_s

$$1 + \gamma_s = \frac{a_{1,s}}{a_0}$$

- If there is no stochastic productivity growth then the increase in income will be deterministic

$$1 + g = (1 + \gamma)\left(1 + a_0 - \frac{c_0}{k_0}\right),$$

- Even if $\gamma = 0$ we can have growth if $1 + a_0 > \dfrac{c_0}{k_0}$.

The stochastic discount factor with production

Questions

- How does the existence of a new motive for allocation of resources (investment in productive capital) changes the stochastic discount factor ?
- What are the consequences for asset prices (or asset rates of return ?)
- To answer those questions we need a general equilibrium model. We consider next the Arrow-Debreu economy (or the equivalent finance economy with complete markets)

AD equilibrium in a production economy

Arrow-Debreu economy with production

Endowment and production economies

- Continue to consider a two-period Arrow-Debreu economy with uncertainty for period 1;
- In an endowment economy the sequence of of the endowments of the good, $\{y_0, Y_1\}$ is independent of the household decision;
- In a production economy the household's endowment of goods in period 1 is dependent upon the savings decisions in period 0 and on a state-contingent productivity shock
- How do Arrow-Debreu prices (or the stochastic discount factor) change when comparing the two economies ?

The household's problem

- von Neumann-Morgenstern utility functional

$$\max_{c_0, C_1} u(c_0) + \beta \sum_{s=1}^{N} \pi_s u(c_{1,s})$$

- intertemporal constraint (where $Q = (q_s)_{s=1}^{N}$ are the Arrow-Debreu prices)

$$c_0 - y_0 + \sum_{s=1}^{N} q_s (c_{1,s} - y_{1,s}) \leq 0$$

- the constraint is equivalent to

$$c_0 - a_0 k_0 + \sum_{s=1}^{N} q_s \left[c_{1,s} - a_{1,s} \left((1 + a_0) k_0 - c_0 \right) \right] \leq 0$$

First order conditions of optimality

Assuming no satiation: i.e. $u^{'}(c) > 0$ for all $c > 0$,

- intertemporal arbitrage condition

$$q_s u^{'}(c_0^*) = \beta \pi_s u^{'}(c_{1,s}^*) \left(1 + \sum_{s=1}^{N} q_s a_{1,s} \right),\ s = 1, \ldots, N \qquad (1)$$

- intertemporal constraint

$$c_0^* - a_0 k_0 + \sum_{s=1}^{N} q_s \left[c_{1,s}^* - a_{1,s} \left((1 + a_0) k_0 - c_0^* \right) \right] = 0$$

First order conditions of optimality

Proof: The Lagrangean is

$$\mathcal{L} = u(c_0)+\beta\sum_{s=1}^{N}\pi_s u(c_{1,s})+\lambda\left(a_0k_0+\sum_{s=1}^{N}q_s\left[a_{1,s}\left((1+a_0)k_0-c_0\right)-c_{1,s}\right]-c_0\right)$$

- $\dfrac{\partial\mathcal{L}}{\partial c_0}=0 \iff u'(c_0)-\lambda\left(\sum_{s=1}^{N}q_s\,a_{1,s}+1\right)=0$
- $\dfrac{\partial\mathcal{L}}{\partial c_{1,s}}=0 \iff \beta\,\pi_s\,u'(c_{1,s})+\lambda\,q_s=0$
- $\dfrac{\partial\mathcal{L}}{\partial\lambda}=0 \iff a_0k_0+\sum_{s=1}^{N}q_s\left[a_{1,s}\left((1+a_0)k_0-c_0\right)-c_{1,s}\right]-c_0=0$
- Substituting λ in the two first foc we find equation (1)

Arrow-Debreu economy with production

first order conditions of optimality

Using $M = (m_s)_{s=1}^N$ is the stochastic discount factor where $q_s = \pi_s m_s$, the representative consumer optimal path $\{c_0^*, C_1^*\}$ is obtained from

$$m_s u^{'}(c_0^*) = \beta u^{'}(c_{1,s}^*) \left(1 + \mathbb{E}[MA_1]\right), \; s = 1, \ldots, N$$

$$c_0^*(1 + \mathbb{E}[MA_1]) + \mathbb{E}[M\,C_1] = \left(a_0(1 + \mathbb{E}[MA_1]) + \mathbb{E}[MA_1]\right) k_0$$

where the expected present value of C_1 is

$$\mathbb{E}[M\,C_1] = \sum_{s=1}^{N} \pi_s m_s c_{1,s}^*$$

and the expected discount value of the future productivity

$$\mathbb{E}[M\,A_1] = \sum_{s=1}^{N} \pi m_s a_{1,s}$$

General equilibrium

Is the sequence $\{c_0, C_1\}$ and k_1^* and M such that

- household's optimality conditions are satisfied

$$m_s u^{'}(c_0) = \beta u^{'}(c_{1,s})\left(1 + \mathbb{E}[MA_1]\right),\ s = 1, \ldots, N \qquad (2)$$

$$c_0^*(1 + \mathbb{E}[MA_1]) + \mathbb{E}[M\,C_1] = \left(a_0 + \mathbb{E}[MA_1](1 + a_0)\right)\right)k_0 \qquad (3)$$

- and market equilibrium conditions hold

$$c_0 + k_1 - k_0 = a_0 k_0 \qquad (4)$$

$$c_{1,s} = a_{1,s} k_1,\ s = 1, \ldots, N \qquad (5)$$

Arrow-Debreu economy with production

General equilibrium

- Solving (4) for $k_1 = (1 + a_0)k_0 - c_0$ and substituting in (5) we get

$$c_{1,s} = ((1 + a_0)k_0 - c_0)a_{1,s}$$

(then $\mathbb{E}[M\,C_1] = ((1 + a_0)k_0 - c_0)\mathbb{E}[M\,A_1]$)

- Substituting in (3) we obtain

$$c_0^* = a_0 k_0$$

- then

$$c_{1,s}^* = a_{1,s} k_0$$

- then the equilibrium stock of capital is stationary

$$k_1^* = (1 + a_0)k_0 - c_0^* = k_0$$

- we obtain m_s^* by substituting c_0 and $c_{1,s}$ in equation (2).

Arrow-Debreu economy with production

The equilibrium stochastic discount factor (SDF)

- The equilibrium stochastic discount factor $M = (m_s)_{s=1}^N$, where $m_s = \frac{q_s}{\pi_s}$ is implictly given by

$$m_s^* = \beta \frac{u^{'}(a_{1,s} k_0)}{u^{'}(a_0 k_0)} \left(1 + \mathbb{E}[M^* A_1]\right)$$

- where

$$\mathbb{E}[M A_1] = \sum_{s=1}^N \pi_s m_s^* a_{1,s}$$

- **Observation**: the SDF for a particular state of nature depends on the expected present value of future productivity, which depends on the distribution of M. In order to determine it we can assume a particular utility function.

Arrow-Debreu economy with production

Example: CRRA Bernoulli utility function

- Simplifying assumptions:

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \text{ and } a_{1,s} = (1+\gamma_s)a_0$$

- then

$$m_s^* = \beta(1+\gamma_s)^{-\theta}\left(1 + \mathbb{E}[MA_1]\right)$$

- but as $\mathbb{E}[MA_1] = \sum_{s=1}^{N} \pi_s m_s^* a_{1,s}$ we have (because $a_{1,s} = (1+\gamma_s)a_0$)

$$\begin{aligned}\mathbb{E}[MA_1] &= a_0 \sum_{s=1}^{N} \pi_s(1+\gamma_s)\beta(1+\gamma_s)^{-\theta}(1+\mathbb{E}[MA_1]) \\ &= \beta a_0(1+\mathbb{E}[MA_1])\mathbb{E}[(1+\gamma)^{1-\theta}]\end{aligned}$$

- Solving for $\mathbb{E}[MA_1]$ we obtain

$$\mathbb{E}[MA_1] = \frac{\beta a_0 \mathbb{E}[(1+\gamma)^{1-\theta}]}{1 - \beta a_0 \mathbb{E}[(1+\gamma)^{1-\theta}]}$$

Arrow-Debreu economy with production

- The equilibrium stochastic discount factor for a **production** economy is

$$m_s^{\text{prod}} = \beta(1+\gamma_s)^{-\theta}\Phi, \; s = 1, \ldots, N$$

where

$$\Phi \equiv \frac{1}{1 - \beta a_0 \mathbb{E}[(1+\gamma)^{1-\theta}]}$$

under the condition $\beta a_0 \mathbb{E}[(1+\gamma)^{1-\theta}] < 1$ then $\Phi > 1$

- If $\theta = 1$ (log utility) then

$$\Phi \equiv \frac{1}{1-\beta a_0} = \frac{1+\rho}{1+\rho-\beta a_0} > 1 \text{ for } 1+\rho > \beta a_0$$

where ρ is the rate of time preference

Arrow-Debreu economy with production

- We found the equilibrium stochastic discount factor for an **endowment** economy, with the same utility function was

$$m_s^{\text{end}} = \beta \frac{u'(y_{1,s})}{u'(y_0)} = \beta(1+\gamma_s)^{-\theta}$$

- Then: for every state of nature the stochastic discount factor is higher in a production economy than in a related endowment economy

$$m_s^{\text{prod}} > m_s^{\text{end}}$$

  because $\Phi > 1$.

Arrow-Debreu economy with production

- Φ is a growth factor which adds to the consumption smoothing effect which exists in an endowment economy
- Intuition: the decision on consumption at time 0,
  in addition to having into consideration the consumption smoothing between time and the states of natures (as in the endowment economy),
  also takes into account the changes in the level of intertemporal resources because production at time 1 now depends on consumption at time 0.

Asset pricing consequences

Finance economy with production

- In a related **finance economy** (with the same fundamentals) we found
$$\mathbb{E}[MR^j] = 1$$
where R^j is the return for any asset j.
- And for a risk free asset
$$R^f = \frac{1}{\mathbb{E}[M]}$$
- Then the risk free interest rate tends to be smaller in a production economy
$$R^{f,\text{prod}} < R^{f,\text{end}}$$
- this maybe explains the secular tendencies: the reduction in the interest rates and a positive growth rate of income

- As

$$\mathbb{E}[M] = \sum_{s=1}^{N} \pi_s \, m_s = \beta \, \mathbb{E}[(1+\gamma)^{-\theta}] \, \Phi$$

- Then the equilibrium risk-free return is

$$R^f = \frac{1 - \beta \, a_0 \, \mathbb{E}[(1+\gamma)^{1-\theta}]}{\beta \, \mathbb{E}[(1+\gamma)^{-\theta}]} = \frac{1 + \rho - \ a_0 \, \mathbb{E}[(1+\gamma)^{1-\theta}]}{\mathbb{E}[(1+\gamma)^{-\theta}]}$$

- For a log utility we have

$$R^f = \frac{1 + \rho - \ a_0}{\mathbb{E}[(1+\gamma)^{-1}]}$$

present increases in productivity decrease the interest rate, but expected future increases tend to increase it.