Foundations of Financial Economics
Deterministic GE asset pricing: two-period case

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February 26, 2021
(revised 5.3.2021)

Functions of finance

- Finance encompasses several functions, v.g:
  - intertemporal allocation of resources, which may or may not be time-dependent (consumption smoothing)
  - inter-state of nature allocation of resources which are uncertain (insurance)
  - financing investment (increase in the resource capacity)
  - matching timing profiles of expenditures and incomes of different agents
  - matching uncertainty profiles of different agents
  - information revelation and pooling
  - distribution of income and wealth
- in this course **we will be mainly concerned with the first two functions**
- in this lecture we deal only with the **first function** in the most simple framework: **two period perfect information models**

Topics covered

- Interest rates, asset prices and the intertemporal allocation problem under perfect information
- Under several economic environments, defined by
  - Fundamentals: preferences and technology
  - Market structures: Arrow-Debreu securities, financial assets

Syllabus

- 1. Intertemporal consumption preferences
- 2. General equilibrium in a representative agent Arrow-Debreu economy
- 3. General equilibrium in an heterogeneous agent Arrow-Debreu economy
- 4. General equilibrium in a frictionless finance economy
- 5. General equilibrium in a finance economy with frictions: heterogeneous market participation

1. Intertemporal consumption preferences

Intertemporal utility

- We **index** variables by time.
- In the simplest case, we have $\mathbb{T} = \{0, 1\}$
- Consider the sequences $\{c_0, c_1\}$, where c_t is consumption in **period** $t = 0, 1$
- The **value** of a sequence $\{c_0, c_1\}$ is measured by the **intertemporal utility functional** ,

$$U = U(\{c_0, c_1\}),$$

- The **optimum** is a sequence for which U **is maximum**

Intertemporal choice

- Behavioral assumptions are implicitly introduced through the mathematical properties of U
- In this case a intertemporal preference structure
- Indexing consumption by time t has two consequences:
  - introduces an **heterogeneity**:
    therefore, we can use **general** concepts and results of choice among **different goods** as in the slide basic utility theory
  - but it also introduces an **order relationship among consumption in different moments in time** c_0 and c_1:
    therefore, we need **particular** concepts and results related to **intertemporal arbitrage**

Intertemporal utility

As a generic utility function

- Dealing with **sequences** $\{c_0, c_1\}$ as a **vector of real non-negative numbers** $\mathbf{c} = (c_0, c_1) \in \mathbb{R}^2_+$
- Therefore the **intertemporal utility function** (IUF) can be seen as a mapping $U : \mathbb{R}^2_+ \to \mathbb{R}$,

$$U = U(c_0, c_1)$$

where U is a number allowing to rank vectors $\mathbf{c} = (c_0, c_1)$

Intertemporal utility

As a generic utility function

- First assumption: $U(\cdot)$ is **continuous** and **differentiable** in both argiuments
- Second assumption: **Positive marginal utility**:

$$U_0 \equiv \frac{\partial U(c_0, c_1)}{\partial c_0} > 0, \;\; U_1 \equiv \frac{\partial U(c_0, c_1)}{\partial c_1} > 0$$

  Intuition: increase in consumption in any period increases utility and there is no satiation
- Third assumption: the Allen-Uzawa elasticities exist

$$\varepsilon_{tt'}(c_0, c_1) = -\frac{U_{tt'}(c_0, c_1)\, c_{t'}}{U_t(c_0, c_1)} \text{ for } t, t' = 0, 1$$

  types of **intertemporal dependence**

$$\varepsilon_{t,t'} \begin{cases} > 0, & \text{intertemporal substitutability} \\ = 0, & \text{intertemporal independence} \\ < 0 & \text{intertemporal complementarity} \end{cases}$$

  these are gross or Edgeworth relationships

Intertemporal utility

As a generic utility function: cont.

- The **intertemporal marginal rate of substitution** is defined as

$$IMRS_{0,1}(c_0, c_1) = -\frac{dc_1}{dc_0}$$

- **Intuition**: how much we are willing to sacrifice consumption at $t = 1$ (tomorrow) in order to increase une unit of consumption at $t = 0$ (today)
- For a compensated change in c_0 and c_1 such that $dU = 0$, we have

$$U_0(c_0, c_1)dc_0 + U_1(c_0, c_1)dc_1 = 0$$

then the IMRS is equal to the ratio of the marginal utilities

$$\boxed{IMRS_{0,1}(c_0, c_1) = \frac{U_0(c_0, c_1)}{U_1(c_0, c_1)}\bigg|_{U=\text{constant}}}$$

Intertemporal utility

As a generic utility function: cont.

- **Intertemporal elasticity of substitution**

$$\boxed{EIS_{0,1}(c_0, c_1) = \frac{d\ln(c_1/c_0)}{d\ln IMRS_{0,1}(c_0, c_1)} = \frac{c_0\,U_0 + c_1\,U_1}{c_1\,U_1\varepsilon_{00} - 2c_0\,U_0\varepsilon_{01} + c_0\,U_0\varepsilon_{11}}}$$

where $\varepsilon_{ij} = -\frac{U_{ij}c_j}{U_i}$ for $i = 0, 1$ and $U_{ij} = \frac{\partial^2 U}{\partial c_i c_j}$

- **Intuition**: how much does the rate of growth of the ratio c_1/c_0 changes for a one percent increase in the *IMRS*.
  This provides a measure of relative **intertemporal substitution/complementarity** of consumption
- In particular: $EIS_{0,1} > 0$ if there is intertemporal substitution, $EIS_{0,1} = 0$ intertemporal independence and $EIS_{0,1} < 0$ intertemporal complementarity (again in the Edgeworth sense)

Intertemporal choice

Introducing the time arrow

- Time can be introduced parametrically

$$U[\{c\}] = U[\{c_0, c_1\}; \mathbb{T}]$$

- **Discounting**: time is introduced via a time-weight: usually a **discount factor**

$$\{\beta^t\}_{t=0}^{T} = \{1, \beta, \beta^2, \dots, \beta^t \dots\}$$

  where

$$\beta \equiv \frac{1}{1+\rho}$$

  where ρ is the **rate of time preference**

- **Temporal utility functions** dependent on time: example the "temporal" preferences are different form different time periods

$$U[\{c\}] = U(u_0(c_0), u_1(c_0, x_1))$$

Intertemporal utility

Main assumptions regarding intertemporal preferences

We define:

- **Stationary preferences** if the temporal utility functions are independent of time (but there can be discounting)
- **Impatience**: if there is a preference for consumption today, at $t = 0$ rather than in the future $t = 1, 2, \ldots$
- Intertemporal allocation: let

$$U(c_0, c_1) = V(u(c_0), v(c_0, c_1))$$

  where V is called the Koopmans aggregator. There is
  - **intertemporal independence** if $v_{c_0} = 0$
  - **intertemporal substitution** if $v_{c_0} < 0$
  - **intertemporal complementarity** (addiction) if $v_{c_0} > 0$
- How can we identify those properties in particular utility functions ?

Intertemporal utility

Impatience and intertemporal complementarity

- Consider a stationary consumption process, i.e, $c_0 = c_1 = \bar{c}$ a constant
- **Impatience**: can be determined by using the *IMRS*. We say the **IUF displays impatience** if

$$IMRS_{0,1}(\bar{c}) = \frac{U_0(\bar{c})}{U_1(\bar{c})} > 1$$

  Intuition: to keep intertemporal utility constant, if we increase c_0 by one unit the reduction in consumption in period $t = 1$ be bigger then one unit. This means that **consumption at** $t = 0$ **has more value than consumption at** $t = 1$.
- **Intertemporal dependence** can be determined by the Allen-Uzawa elasticity ε_{01}.

$$\varepsilon_{0,1}(\bar{c}) = -\frac{U_{01}(\bar{c})\,\bar{c}}{U_0(\bar{c})} \begin{cases} > 0, & \text{intertemporal substitutability} \\ = 0, & \text{intertemporal independence} \\ < 0 & \text{intertemporal complementarity} \end{cases}$$

Intertemporal utility

Example 1: additive IUF

This is the simplest intertemporal utility function:

- **Assumption 1**: the IUF is intertemporally **additive**

$$U(c_0, c_1) = u(c_0) + \beta u(c_1), \text{ where } \beta \equiv \frac{1}{1+\rho}$$

  where $\beta \in (0,1)$ is the psychological discount factor and ρ is the rate of time preference, and $u(\cdot)$ is called the Bernoulli utility function

- **Assumption 2**: u is increasing and concave $u^{''}(c_t) < 0 < u^{'}(c_t)$, $t = 0, 1$

Intertemporal utility

Additive IUF

- Marginal utilities for c_t, $t = 0, 1$ are
$$U_0 = u^{'}(c_0),\;\; U_1 = \beta u^{'}(c_1)$$
- Derivatives of marginal utilities for c_t, $t = 0, 1$ are
$$U_{00} = u^{''}(c_0),\;\; U_{01} = 0,\, U_{11} = \beta u^{''}(c_1)$$
- The *IMRS* is
$$IMRS_{0,1} = \frac{U_0}{U_1} = \frac{u^{'}(c_0)}{\beta u^{'}(c_1)}$$
Therefore: marginal utility for period $t = 0$ is proportional to the discounted marginal utility for period $t = 1$ (from the perspective of period $t = 0$)
$$\boxed{u^{'}(c_0) = \beta u^{'}(c_1)\, IMRS_{0,1}}$$
we will see an analogous equation again and again translating the idea of intertemporal arbitrage.

Intertemporal utility

Additive IUF

- The Allen-Uzawa elasticities are

$$\varepsilon_{00}(c_0) = -\frac{u^{''}(c_0)c_0}{u^{'}(c_0)},\ \varepsilon_{01} = 0,\ \varepsilon_{11}(c_1) = -\frac{u^{''}(c_1)c_1}{u^{'}(c_1)}$$

- The elasticity of intertemporal substitution between period 0 and 1 is

$$EIS_{0,1}(c_0, c_1) = \frac{c_0 u^{'}(c_0) + \beta c_1 u^{'}(c_1)}{\beta c_1 u^{'}(c_1)\varepsilon_{00}(c_0) + c_0 u^{'}(c_0)\varepsilon_{11}(c_1)}$$

Intertemporal utility

Additive IUF

For a stationary consumption path $\{\bar{c}, \bar{c}\}$ we find:

- The IMRS is independent from $\bar{c}$ and

$$IMRS_{0,1}(\bar{c}) = \frac{1}{\beta} = 1 + \rho > 1$$

  this means that the IUF displays **impatience**, and this effect is captured by time discounting

- It displays **intertemporal independence** because

$$\varepsilon_{0,1}(\bar{c}) = 0$$

  Intuition: the marginal valuation of consumption at time $t = 1$ is independent of the history of consumption

- The IES is

$$IES_{0,1}(\bar{c}) = -\frac{u^{'}(\bar{c})}{u^{''}(\bar{c})\bar{c}} > 0$$

  Intuition: this is interpreted as a measure of the preference for **consumption smoothing** through time

Intertemporal utility

Additive IUF

Particular case:

- Utility function (generalized logarithm)

$$u(c) = \frac{c^{1-\zeta} - 1}{1-\zeta}$$

- if $\zeta = 1$ we have $u(c) = \ln(c)$ (Prove this)
- Derivatives

$$U_0 = c_0^{-\zeta},\ U_1 = \beta\, c_1^{-\zeta},\ U_{00} = -\zeta\, c_0^{-\zeta-1},\ U_{01} = 0,\ U_{11} = -\zeta\, c_1^{-\zeta-1}$$

- The IMRS is

$$IMRS_{0,1} = \frac{1}{\beta}\left(\frac{c_1}{c_0}\right)^{\zeta}$$

- The UA elasticities are constant $\varepsilon_{00} = \varepsilon_{11} = \zeta$
- The IES is also constant

$$EIS_{0,1} = \frac{1}{\zeta}$$

this is why ζ is called the **inverse of the elasticity of intertemporal substitution**

Intertemporal utility

Non-additive IUF

- Case 1: **Uzawa and Epstein-Hynes** utility

$$U(c_0, c_1) = u(c_0) + b(c_0)v(c_1)$$

the discount factor is endogenous i.e. $\beta = b(c)$ with $b^{'}(.) < 0$ (rich people are more patient)

The crossed AU elasticity is for a stationary sequence is

$$\varepsilon_{0,1}(c) = -\frac{b^{'}(c)v^{'}(c)c}{u^{'}(c) + b^{'}(c)v(c)}$$

displays intertemporal dependence

- Case 2: **Habit formation**

$$U(c_0, c_1) = u(c_0) + \beta v(c_0, c_1).$$

where $v_{c_0}(c_0, c_1) < 0$.

The crossed AU elasticity is for a stationary sequence is

$$\varepsilon_{0,1}(c) = -\frac{\beta v_{c_0 c_1}(c) c}{u'(c) + \beta v_{c_0}(c)} c$$

can display intertemporal substitutability, independence or complementarity depending on the relationship between time discounting and the relative importance of habits, i.e., the magnitude of $v_{c_0}(c)$

Intertemporal utility

Case 3: habit formation example

- IUF
$$U(c_0, c_1) = \ln(c_0) + \beta \ln\left(\frac{c_1}{c_0}\right)^{\zeta},\ \zeta > 0$$

- Derivatives
$$U_0 = \frac{1-\beta\zeta}{c_0},\ U_1 = \frac{\beta\zeta}{c_1},\ U_{00} = -\frac{1-\beta}{c_0^2},\ U_{01} = 0,\ U_{11} = -\frac{\beta\zeta}{c_1^2}$$

- The IMRS is
$$IMRS_{0,1}(c_0, c_1) = \left(\frac{1-\beta\zeta}{\beta\zeta}\right)\frac{c_1}{c_0}$$

- The AU elasticities are constant
$$\varepsilon_{00} = \varepsilon_{11} = 1,\ \varepsilon_{01} = 0$$

- The IES is also constant
$$EIS_{0,1}(c_0, c_1) = 1$$

  for any (c_0, c_1)

Intertemporal utility

Case 2: habit formation example, cont

For a stationary sequence $c_0 = c_1 = c$

- The IMRS
$$IMRS_{0,1}(c) = \frac{1-\beta\zeta}{\beta\zeta}$$
  the utility displays impatience if $\zeta < \dfrac{1}{2\beta} = \dfrac{1+\rho}{2}$.
  Intuition: there is impatience (according to the above definition) if the weight of past consumption is not too strong
- As $\varepsilon_{01} = 0 = 0$ the model displays intertemporal independence (but this is special to this example).

2. General equilibrium in a representative agent Arrow-Debreu economy

Two-period general equilibrium models

- Next we will address the (macro) **determination of the interest rate** in two-period general equilibrium models under perfect information (i.e., certainty)
- We consider two (equivalent) approaches and models
  - a micro-economic approach: Arrow-Debreu simultaneous equilibrium economy
  - a finance (or macro-finance) approach:a finance sequencial equilibrium economy
- For each model we proceed in two steps:
  - present and solve the consumer problem in each economy
  - we define and determine the general equilibrium

Arrow Debreu model

The consumer problem: set-up

- A consumer has an asset (resource, endowment) in positive amount ($w > 0$) which allows for a sequence of consumption in two periods, $\{c_0, c_1\}$, today c_0 and in the future c_1.
- There is a market for **forward contracts** allowing for contracting today for delivery in the future, at a price set today, $q > 0$. We take the price paid today as a *numéraire* and all the variables are denominated at todays' price
- The value of the consumption sequence is assessed by an intertemporal utility function: $U(c_0, c_1)$;
- The **budget constraint**, referring to payments made today, is

$$c_0 + q\, c_1 \le w$$

Arrow Debreu model

The consumer problem

- Formally, the intertemporal problem for the consumer is

$$v(w) = \max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + q\, c_1 \leq w \}$$

- The (interior) optimum (c_0^*, c_1^*) satisfies the conditions

$$\begin{cases} qU_0(c_0^*, c_1^*) = U_1(c_0^*, c_1^*) & \text{(optimality condition)} \\ c_0^* + pc_1^* = w & \text{(budget constraint)} \end{cases}$$

Arrow Debreu model

Optimality: interpretation

- At the optimum: the IMRS is equal to the relative price (internal = external valuation)

$$IMRS^*_{0,1} = IMRS_{0,1}(c_0^*, c_1^*) = \frac{U_0(c_0^*, c_1^*)}{U_1(c_0^*, c_1^*)} = \frac{1}{q}$$

- Intuition: at the optimum increasing one euro of consumption tomorrow should be matched by a reduction in $1/q$ euro of consumption today, ie $dc_0^* = -q dc_1^*$
- Therefore q is an **intertemporal relative price**: i.e., an opportunity cost for changing the consumption sequence across time.

Arrow Debreu model

Assumptions

- Now, we go from a microeconomic to a macroeconomic perspective

H1 Assume there is perfect information: **deterministic general equilibrium**

H2 Assume all agents are homogeneous (in behavior and in endowments): **representative agent economy**

H3 Assume that there is an exogenous sequence of resources sustaining consumption: **endowment economy**

H4 Assume that all trade is done at time $t = 0$: an **Arrow-Debreu economy**

- We want to determine (endogenously) the price q: the **Arrow-Debreu** price

Arrow Debreu model

The setup

- Assume that the resource of the economy takes the form of a **flow** of non-durable goods, that can be collected both at time $t = 0$ and $t = 1$, $\{y_0, y_1\}$.
- Again, assume that **trade can only take place at time** $t = 0$: this means that the price for contracts for delivery at time $t = 1$ has to be set at time $t = 0$. We call q the **Arrow-Debreu price**
- From this, wealth at time $t = 0$ is equal to the **present value of the flow of endowments**

$$w = y_0 + q\, y_1$$

Arrow Debreu model

The setup: continuation

- All the participants have **perfect information** (over prices and endowments referring to period $t = 1$) and solve their micro-economic problems;
- At every period, total consumption must be equal to total endowment (market equilibrium);
- **Representative agent economy**: we assume that all consumers solve the same problem (same utility function and same endowments);
- What is the equilibrium forward price q ?

Arrow Debreu model

General equilibrium for a representative agent economy

General equilibrium in this economy is defined by $(\{c_0^*, c_1^*\}, q^*)$ such that

- the consumer solves the problem

$$\max_{c_0, c_1} \{U(c_0, c_1) : c_0 + q\, c_1 \leq y_0 + q y_1\}$$

- markets clear

$$c_0 = y_0,$$
$$c_1 = y_1$$

Arrow Debreu model

General equilibrium for a representative agent economy

- General equilibrium conditions: (c_0, c_1, q) is determined from

$$\begin{cases} qU_0(c_0, c_1) = U_1(c_0, c_1) & \text{(micro: intertemporal optimality condition)} \\ c_0 + qc_1 = y_0 + qy_1 & \text{(micro: budget constraint)} \\ c_0 = y_0 & \text{(aggregate: market clearing for } t = 0\text{)} \\ c_1 = y_1 & \text{(aggregate: market clearing for } t = 1\text{)} \end{cases}$$

- There are only three independent conditions (Walras's law)

$$\begin{cases} qU_0(c_0, c_1) = U_1(c_0, c_1) \\ c_0^* = y_0 \\ c_1^* = y_1 \end{cases}$$

- In a representative agent economy there is **no trade** (consumption is equal to the endowment)

Arrow Debreu model

Equilibrium AD price

- Then **the equilibrium AD price** is

$$\boxed{q^* = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)}}$$

- **We need more structure on preferences to get explicit results**

Arrow Debreu model

Example 1: additive utility

- For an intertemporally additive utility function

$$U(c_0, c_1) = u(c_0) + \beta\, u(c_1)$$

- General equilibrium determination

$$\begin{cases} q u'(c_0) = \beta\, u'(c_1) & \text{(micro: intertemporal optimality condition)} \\ c_0 + q c_1 = y_0 + q y_1 & \text{(micro: budget constraint)} \\ c_0 = y_0 & \text{(aggregate: market clearing for } t = 0\text{)} \\ c_1 = y_1 & \text{(aggregate: market clearing for } t = 1\text{)} \end{cases}$$

- From Walras's law: the independent equations are

$$\begin{cases} q u'(c_0) = \beta\, u'(c_1) \\ c_0 = y_0 \\ c_1 = y_1 \end{cases}$$

Arrow Debreu model

Example 1: additive utility

- Then, for an intertemporally additive utility the **equilibrium AD price** is

$$\boxed{q^*(\beta, y_0, y_1) = \beta \frac{u^{'}(y_1)}{u^{'}(y_0)}}$$

- concavity of $u(.)$, i.e., $u^{''}(c) < 0$, implies

$$\frac{\partial q^*(y_0, y_1)}{\partial y_0} = -\beta \frac{u^{'}(y_1) u^{''}(y_0)}{(u^{'}(y_0))^2} > 0$$

and

$$\frac{\partial q^*(y_0, y_1)}{\partial y_1} = \beta \frac{u^{''}(y_1)}{u^{'}(y_0)} < 0$$

- Then it increases (decreases) with an excess supply of present (future) **relative** to future (present) supply

Arrow Debreu model

Example 1: additive utility

- Particular case: if $u(c) = \ln(c)$ then

$$\boxed{q^*(y_0, y_1) = \beta \frac{y_0}{y_1} = \frac{\beta}{1+\gamma}}$$

- or, if we set $y_1 = (1+\gamma)\, y_0$ where γ is the anticipated rate of growth, and recall $\beta = (1+\rho)^{-1}$ then

$$\boxed{q^*(y_0, y_1) = \frac{1}{(1+\gamma)\,(1+\rho)}}$$

the AD price decreases with the rate of time preference and the anticipated rate of growth

Arrow Debreu model

Example 2: habit formation

- For the habit formation intertemporal utility

$$U(c_0, c_1) = u(c_0) + \beta\, v(c_0, c_1)$$

- General equilibrium determination

$$\begin{cases} q\big(u'(c_0) + \beta\, v_{c_0}(c_0, c_1)\big) = \beta\, v_{c_1}(c_0, c_1)\big) \\ c_0 + qc_1 = y_0 + qy_1 \\ c_0 = y_0 \\ c_1 = y_1 \end{cases}$$

- From Walras's law: the independent equations are

$$\begin{cases} q\big(u'(c_0) + \beta\, v_{c_0}(c_0, c_1)\big) = \beta\, v_{c_1}(c_0, c_1)\big) \\ c_0 = y_0 \\ c_1 = y_1 \end{cases}$$

Arrow Debreu model

Example 2: habit formation

- For the habit formation utility function

$$q^*(y_0, y_1) = \beta \frac{v_{c_1}(y_0, y_1)}{u'(y_0) + \beta v_{c_0}(y_0, y_1)}$$

- May have the same qualitative properties than with the additive model: recall

$$q^*_{y_0} > 0, \text{and } q^*_{y_1} < 0$$

Arrow Debreu model

Example 2: habit formation

- Example: setting $U(c_0, c_1) = \ln(c_0) + \beta \ln \left[\left(\frac{c_1}{c_0} \right)^{\zeta} \right]$ displaying intertemporal substitution

$$\boxed{q^* = \frac{\beta\zeta}{y_1} \left(\frac{1}{y_0} - \beta\zeta \frac{1}{y_0} \right)^{-1} = \frac{\beta\zeta y_0}{(1-\beta\zeta) y_1} = \frac{\beta\zeta}{(1-\beta\zeta)(1+\gamma)}}$$

- again $q^*_{y_0} > 0$
- however:
  $q^*_{y_1} < 0$ if $\zeta < \frac{1}{\beta} = 1 + \rho$ (low weight of habits)
  $q^*_{y_1} > 0$ if $\zeta > \frac{1}{\beta} = 1 + \rho$ (high weight of habits)

3. General equilibrium in an heterogeneous agent Arrow-Debreu economy

Arrow Debreu model

Assumptions

- The previous model is more general than it looks

H1 idem

H2 Assume heterogeneity in endowments

H3 idem

H4 idem

- What are the consequences for the equilibrium q

Arrow-Debreu economy

Beyond the representative agent case

- Assumptions: there are two groups of agents
  - **with the same preferences**
  - but **endowments are different** (y_t^i is the endowment of agent i at time t)
$$y^1 = \{y_0^1, y_1^1\}, y^2 = \{y_0^2, y_1^2\}$$
    and we assume $y^1 \neq y^2$
- The aggregate **flow of total endowments** are
$$y_0 = y_0^1 + y_0^2$$
$$y_1 = y_1^1 + y_1^2$$
- The general equilibrium is now

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

General equilibrium in this economy is defined by the allocations $(\{c_0^{1*}, c_1^{1*}\}, \{c_0^{2*}, c_1^{2*})\}$ and the price q^* such that

- consumer $i \in \{1, 2\}$ solves the problem

$$\max_{c_0^i, c_1^i} \{ U(c_0^i, c_1^i) : c_0^i + q\, c_1^i \leq y_0^i + q\, y_1^i \}, \text{ for } i = 1, 2$$

- market clearing for $t = 0, 1$,

$$c_t = y_t, \text{ for } t = 0, 1$$

- aggregation

$$c_t = c_t^1 + c_t^2, \text{ for } t = 0, 1$$
$$y_t = y_t^1 + y_t^2, \text{ for } t = 0, 1$$

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- General equilibrium conditions (considering that the Walras' law holds)

$$\begin{cases} qU_0(c_0^1, c_1^1) = U_1(c_0^1, c_1^1) & \text{(optimality condition for agent 1)} \\ qU_0(c_0^2, c_1^2) = U_1(c_0^2, c_1^2) & \text{(optimality condition for agent 2)} \\ c_t = y_t & \text{(market clearing for period } t = 0, 1\text{)} \\ c_t = c_t^1 + c_t^2 & \text{(aggregation of consumption for } t\text{)} \\ y_t = y_t^1 + y_t^2 & \text{(aggregation of endowment for } t\text{)} \end{cases}$$

- In this case there can be trade, because $c_t^1 - y_t^1 = y_t^2 - c_t^2$ can be different from zero, but the budget constraint should hold for every agent. (check this !)

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- Because we assumed homogeneity in preferences, $U(.,.)$ is the same for both consumers.
- Therefore, it also holds for the aggregate consumption

$$qU_0(c_0, c_1) = U_1(c_0, c_1)$$

that is

$$qU_0\big(c_0^1 + c_0^2, c_1^1 + c_2^1\big) = U_1(c_0, c_1)$$

- Using the market clearing conditions we have again

$$\boxed{q^* = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)} = \frac{U_1(y_0^1 + y_0^2, y_1^1 + y_1^2)}{U_0(y_0^1 + y_0^2, y_1^1 + y_1^2)}}$$

- Conclusion: if agents are **homogeneous as regards preferences** but are **heterogeneous as regards endowments** the distribution of income between agents **has no influence** the AD price. It is only determined by the aggregate endowment
- If there is heterogeneity in preferences, this result **will not hold** in general.

4. General equilibrium in a frictionless finance economy

Finance economy model

Assumptions

- Now we change the market structure

H1 idem

H2 Assume a representative agent economy

H3 idem

H4 Assume a **sequence of spot asset markets** (for the good opening at $t = 0$ and $t = 1$, for an asset opening at $t = 0$)

- What is the equilibrium asset price

Finance economy model

The economy

- Two periods and full information
- We still assume that the agent receives a flow of endowments $y = \{y_0, y_1\}$ The agent can consume the totality of the income, or not, at the end of period 1
- Agents can reallocate the endowments through a **spot financial** contract
- There is an asset (that can be seen as a durable good) that agents can lend and borrow at period $t = 0$ paying or receiving an interest income at period $t = 1$. The asset is in non-negative net supply at the beginning to period $t = 0$ and there is a market for the asset at time $t = 0$.
- Every agent has now a **sequence of budget constraints** (because trade in the good market can take place at period 1)

Finance economy model

Micro-economic problem in the finance economy

- The problem (assuming an additive intertemporal utility)

$$\max_{c_0, c_1, a_1, a_2} U(c_0, c_1) = u(c_0) + \beta u(c_1) :$$

- subject to

$$\begin{cases} c_0 + a_1 = y_0 + a_0 \\ c_1 = y_1 + (1+r)a_1 - a_2 \end{cases}$$

  where a_0 is the level of the asset at beginning of period 0 and a_1 and a_2 are the levels at the end of period 0 and 1, and r is the real interest rate.

- other constraints:

$$c_0 \geq 0, c_1 \geq 0, a_1 \text{free}, a_2 \geq 0$$

Finance economy model

Micro-economic problem in the finance economy

Meaning of the other constraints:

- consumption cannot be negative

$$c_0 \geq 0, c_1 \geq 0$$

- a_1 **free** means the consumer can be in one of the three positions
  - can be a net debtor if $a_1 < 0$
  - can be a net creditor if $a_1 > 0$
  - neither a debtor nor a creditor $a_1 = 0$
- the **non-Ponzi game condition**: cannot be a debtor at the end of the last period

$$a_2 \geq 0$$

- Next, we prove that, it will never be optimal to have $a_2 > 0$

Finance economy model

Optimality of $a_2 = 0$

- Substitute c_0 and c_1 in the utility function, assume that $\beta > 0$ and r is finite, and consider the constraint for a_2

$$\max_{a_1, a_2} \{u(y_0 + a_0 - a_1) + \beta u(y_1 + (1+r)a_1 - a_2) : a_2 \geq 0\}$$

- The first order conditions are

$$u^{'}(c_0) = \beta(1+r)u^{'}(c_1)$$
$$\beta u^{'}(c_1) = \lambda$$
$$\lambda a_2 = 0, \ \lambda \geq 0, \ a_2 \geq 0$$

- We have $a_2 > 0$ if and only if $\lambda = 0$, but in this case either there is satiation or $c_1 \to \infty$ and $c_0 \to \infty$. But this is only possible if $a_0 - \to \infty$. Therefore we should have $a_2 = 0$ and $\lambda > 0$.

- Taking $a_2 = 0$ and assuming a_1 is free (i.e., the consumer can borrow or lend freely) we can eliminate a_1 in the sequence of budget constraints, to get

$$c_0 + mc_1 = a_0 + y_0 + my_1$$

  where m is the **market discount factor**

$$m \equiv \frac{1}{1+r} \equiv \frac{1}{R}$$

- This implies that if the initial wealth is zero, $a_0 = 0$ and the non-Ponzi game condition holds, such that the consumer chooses optimally the last time financial wealth $a_2 = 0$
- the **sequence of budget constraints** is equivalent to an **intertemporal budget constraint** formally similar to the constraint in the Arrow-Debreu economy.

$$c_0 + mc_1 = y_0 + mc_1$$

- where the **AD-price** q **is formally equal to the discount factor** m

Finance economy without frictions

General equilibrium for a representative agent finance economy

General equilibrium in this economy is defined by $\left(\{c_0^*, c_1^*\}, m*\right)$ such that

- the consumer solves the problem

$$\max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + mc_1 = a_0 + y_0 + mc_1 \}$$

- market clearing hold

$$c_0 = y_0, \; c_1 = y_1$$

Finance economy without frictions

General equilibrium for a representative agent finance economy

- The equilibrium equations are (from Walras's law)

$$
\begin{aligned}
m u^{'}(c_0) &= \beta u^{'}(c_1) \\
c_0 &= a_0 + y_0 \\
c_1 &= y_1
\end{aligned}
$$

- The **equilibrium discount factor** is

$$
m^* = m(a_0, y_0, y_1) = \beta \frac{u^{'}(y_1)}{u^{'}(a_0 + y_0)}
$$

- Because $R = \dfrac{1}{m}$ and $\beta = \dfrac{1}{1+\rho}$ where ρ is the psychological discount factor

Finance economy without frictions

General equilibrium for a representative agent finance economy

- The **equilibrium discount factor** is

$$\boxed{m^* = m^*(\beta, y_0, y_1, a_0) = \frac{\beta\, u^{'}(y_1)}{u^{'}(y_0 + a_0)}}$$

- concavity of $u(.)$, i.e., $u^{''}(c) < 0$, implies

$$\frac{\partial m^*}{\partial y_0} = \frac{\partial m^*}{\partial a_0} = -\beta \frac{u^{'}(y_1) u^{''}(y_0 + a_0)}{(u^{'}(y_0 + a_0))^2} > 0$$

$$\frac{\partial m^*}{\partial y_1} = \beta \frac{u^{''}(y_0)}{u^{'}(y_1)} < 0$$

- There are two main effects:
  - a direct effect: high y_0 or a_0 **increase** the DF
  - an anticipation effect: high y_1 **decreases** the DF

Finance economy without frictions

Asset return in a frictionless economy

- The **equilibrium asset return** (recall)

$$\boxed{R^* = 1 + r^* = (1+\rho)\frac{u^{'}(a_0 + y_0)}{u^{'}(y_1)}}$$

- But $R^* = R(a_0, y_0, y_1)$, with partial derivatives

$$\frac{\partial R}{\partial a_0} = \frac{\partial R}{\partial y_0} = (1+\rho)\frac{u^{''}(a_0 + y_0)}{u^{'}(y_1)} < 0$$

$$\frac{\partial R}{\partial y_1} = -(1+\rho)\frac{u^{''}(y_1)u^{'}(a_0 + y_0)}{(u^{'}(y_1))^2} > 0$$

- There are two main effects:
  - a direct effect: high y_0 or a_0 **reduce** the interest rate
  - an anticipation effect: high y_1 **increases** the interest rate

5. General equilibrium in a finance economy with frictions: heterogeneous market participation

A simple finance economy with frictions

Assumptions

- Now we introduce heterogeneity

H1 idem

H2 Assume agents face financing constraints

H3 idem

H4 idem

- What is the equilibrium asset price

Finance economy with heterogeneity

Heterogenous participation

- Assume there are two agents in the economy: agent b is a borrower and agent l is a lender, the only one that has positive assets at time 0 ($a_0^l > 0$, $a_0^b = 0$)
- To simplify, assume agent b is the only one that receives the flow of endowments $\{y_0, y_1\}$ and agent b can only earn interest income
- Assume there are no constraints in the credit market
- Assume that agents have homogeneous preferences

Finance economy with heterogeneity

Agents' problems

- The **lender**'s problem is

$$\max_{c_0^l, c_1^l} \{u(c_0^l) + \beta u(c_1^l) : \; c_0^l + l^l = a_0, \; c_1^l = (1+r)l^l\}$$

- Because l^l is free it can be simplified to

$$\max_{l^l} \{u(a_0 - l^l) + \beta u((1+r)l^l)\}$$

- The optimality condition is

$$u^{'}(a_0 - l^l) = \beta(1+r)u^{'}((1+r)l^l)$$

or equivalently

$$u^{'}(c_0^l) = \beta(1+r)u^{'}(c_1^l)$$

Finance economy with heterogeneity

Agents' problems

- The **borrower** problem is

$$\max_{c_0^b, c_1^b} \{u(c_0^b) + \beta u(c_1^b) : \; c_0^b = y_0 + l^b, \; c_1^l + (1+r)l^b = y_1\}$$

- Because l^b is free it can be simplified to

$$\max_{l^b} \{u(y_0 + l^b) + \beta u(y_1 - (1+r)l^b)\}$$

- The optimality condition is

$$u^{'}(y_0 + l^b) = \beta(1+r)u^{'}(y_1 - (1+r)l^b)$$

or equivalently

$$u^{'}(c_0^b) = \beta(1+r)u^{'}(c_1^b)$$

Finance economy with heterogeneity

Equilibrium equations

- The equilibrium equations are

$$\begin{aligned} u^{'}(c_0^l) &= \beta(1+r)u^{'}(c_1^l) \\ u^{'}(c_0^b) &= \beta(1+r)u^{'}(c_1^b) \\ c_0^l + c_0^b &= y_0 + a_0 \\ c_1^l + c_1^b &= y_1 \end{aligned}$$

- Because preferences are homogeneous we can use the same argument as before, to get

$$u^{'}(y_0 + a_0) = \beta(1+r)u^{'}(y_1)$$

Finance economy with heterogeneity

Equilibrium interest rate

- The **equilibrium return** is again

$$\boxed{R^* = 1 + r^* = (1+\rho)\frac{u'(y_0 + a_0)}{u'(y_1)}}$$

- Is formally similar to the representative agent economy case;
- Again, we have
  - negative liquidity effect $R^*_{a_0} < 0$;
  - a negative income effect, $R^*_{y_0} < 0$
  - a positive anticipation effect, $R^*_{y_1} > 0$
- Conclusion: without other sources of heterogeneity, limited participation has no effect on the market interest rate.

Taking the model to data

- In the long run we have (see introduction):
  - an upward trend of the growth rates
  - a downward trend of the interest rates
- then: the simple model has the **wrong** correlation (why ?)
- In the shorter run we have

Taking the model to data

- data from http://www.nber.org/papers/w24112.pdf
  $R_{safe} = 1.0188$ (average safe return) $R_{wealth} = 1.0678$ (average wealth return) $\gamma = 0.0287$ (average rate of growth)
- calibrated parameters: $\rho = 0.02$
- Utility functions
  - isoelastic utility function

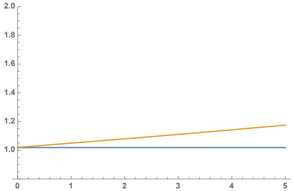
$$U(c_0, c_1) = \frac{c_0^{1-\zeta} - 1}{1-\zeta} + \beta \frac{c_1^{1-\zeta} - 1}{1-\zeta}$$

  - habit formation:

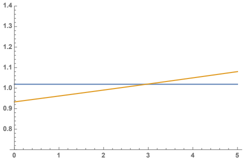
$$U(c_0, c_1) = \ln c_0 + \beta \left[\frac{\left(\frac{c_1}{c_0}\right)^{\phi(1-\zeta)} - 1}{1-\zeta} \right]$$

Taking the model to data

- Additive utility: interest rate puzzle (the model over predicts the observed risk-free interest rate, for any value of the *EIS*)



- Habit formation: it is possible to find values for the parameter ϕ, in the case $\phi \approx 0.5$ such that the model matches the observed R for "acceptable" values fo ζ



Questions

- The previous results hold for cases in which there is
  - full information (deterministic general equilibrium)
  - agents have homogeneous preferences (with or without homogeneous resources)
  - frictionless economy (for the case of a finance economy)
- Do those results hold under:
  - imperfect information (uncertainty) ?
  - heterogeneity in agents' preferences ?
  - frictions in a finance economy (ex: credit constraints) ?