# Foundations of Financial Economics
# Multi-period finance economies

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# Topics for today

- Arbitrage asset pricing
- Equilibrium asset pricing for a homogeneous household economy: the zero initial wealth case
- Equity premium puzzle again
- Non-zero initial wealth case

# Arbitrage asset pricing

# The structure of the asset market

There are two stochastic processes: $\{\mathbf{V}_t\}_{t=1}^{T}$ and $\{\mathbf{S}_t\}_{t=0}^{T-1}$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | $\mathbf{V}_1$ | | $\mathbf{V}_2$ | | | | $\mathbf{V}_{t+1}$ | |
| period | | 0 | | 1 | | | | $t$ | |
| time | 0 | | 1 | | 2 | | $t$ | | $t+1$ |
| | $\mathbf{S}_0$ | | $\mathbf{S}_1$ | | $\mathbf{S}_2$ | | $\mathbf{S}_t$ | | $\mathbf{S}_{t+1}$ |

- at every time $t = 0, \dots, T-1$ (not just at $t = 0$ as before) $K$ assets traded at the vector of price $\mathbf{S}_t$ is set
- asset deliver payoffs $\mathbf{V}_{t+1}$ in period $t = 0, \dots, T-1$, unknown at the time $t$ of price determination

# The price process

- The price process is $\{\mathbf{S}_t\}_{t=1}^{T} = \{\mathbf{S}_0, \mathbf{S}_1, \dots, \mathbf{S}_t, \dots, \mathbf{S}_{T-1}\}$, where

$$\mathbf{S}_t = (S_t^1, \dots, S_t^K), \text{ for } S_t^j = \begin{pmatrix} s_{t,1}^j \\ \dots \\ s_{t,s}^j \\ \dots \\ s_{t,N_t}^j \end{pmatrix}$$

  **conditional on the information at time** $t = 0$

- or, expanding, the possible realizations for the price at at time $t > 0$ are

$$\mathbf{S}_t = \begin{pmatrix} s_{t,1}^1 & \dots & s_{t,1}^j & \dots & s_{t,1}^K \\ \dots & \dots & \dots & \dots & \dots \\ s_{t,s}^1 & \dots & s_{t,s}^j & \dots & s_{t,s}^K \\ \dots & \dots & \dots & \dots & \dots \\ s_{t,N_t}^1 & \dots & s_{t,N_t}^j & \dots & s_{t,N_t}^K \end{pmatrix}$$

# The payoff process

- The payoff process $\{\mathbf{V}_t\}_{t=1}^{T} = \{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_t, \ldots, V_T\}$, where

$$\mathbf{V}_t = (V_t^1, \ldots, V_t^K), \text{ for } V_t^j = \begin{pmatrix} v_{t,1}^j \\ \cdots \\ v_{t,s}^j \\ \cdots \\ v_{t,N_t}^j \end{pmatrix}$$

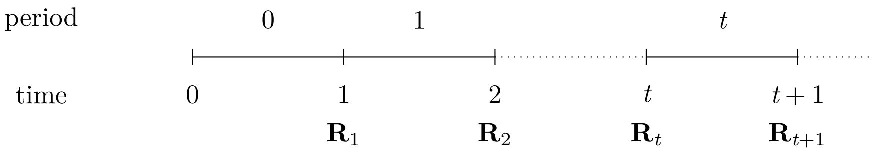
  **conditional on the information at time** $t = 0$

- or, expanding, the possible realizations for the payoff at time $t$ are

$$\mathbf{V}_t = \begin{pmatrix} v_{t,1}^1 & \cdots & v_{t,1}^j & \cdots & v_{t,1}^K \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{t,s}^1 & \cdots & v_{t,s}^j & \cdots & v_{t,s}^K \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{t,N_t}^1 & \cdots & v_{t,N_t}^j & \cdots & v_{t,N_t}^K \end{pmatrix}$$

# The structure of the asset market

The return process results: $\{\mathbf{R}_t\}_{t=1}^{T} = \{\mathbf{R}_1, \dots, \mathbf{R}_t, \dots, R_T\}$




- where the returns for every asset $\mathbf{R}_t = (R_t^1 \ \dots, R_t^K)$,
- where $R_t^j = (R_{t,1}^j, \dots, R_{t,N_t}^j)^\top$
- the return of asset $j$ at time $t$ is $R_t^j = \dfrac{V_t^j + S_t^j}{S_{t-1}^j}$
- is determined after the observation of price $S_t^j$, i.e., after being sold at the current market price.

# Example

- Two-state binomial tree and $T = 3$ (information conditional at time $t = 0$)
- Two assets: $a$ and $b$
- Prices and payoffs processes
  - at time $t = 0$ only prices are observed $\mathbf{S}_0 = (S_0^a, S_0^b)$
  - at time $t = 1$, $\mathbf{V}_1 = \begin{pmatrix} V_{1,1}^a & V_{1,1}^b \\ V_{1,2}^a & V_{1,2}^b \end{pmatrix}$ and $\mathbf{S}_1 = \begin{pmatrix} S_{1,1}^a & S_{1,1}^b \\ S_{1,2}^a & S_{1,2}^b \end{pmatrix}$
  - at time $t = 2$

$$\mathbf{V}_2 = \begin{pmatrix} V_{2,1}^a & V_{2,1}^b \\ V_{2,2}^a & V_{2,2}^b \\ V_{2,3}^a & V_{2,3}^b \\ V_{2,4}^a & V_{2,4}^b \end{pmatrix}, \ \mathbf{S}_2 = \begin{pmatrix} S_{2,1}^a & S_{2,1}^b \\ S_{2,2}^a & S_{2,2}^b \\ S_{2,3}^a & S_{2,3}^b \\ S_{2,4}^a & S_{2,4}^b \end{pmatrix}$$

  - at terminal time $t = 3$

$$\mathbf{V}_3 = \begin{pmatrix} V_{3,1}^a & V_{3,1}^b \\ \dots & \dots \\ V_{3,8}^a & V_{3,8}^b \end{pmatrix}$$

# Arbitrage asset pricing

## Stochastic discount factor: intertemporal form

### Definition

*A* ***stochastic discount factor*** *(SDF) is a process* $\{M_t\}_{t=0}^{T-1}$*, such that,* ***for any asset*** $j = 1, \ldots, K$*:*

1. $M_t$ *is* $\mathcal{F}_t$*-measurable (v.g., has a tree structure) ,*
2. $M_0 = m_0 = 1$
3. *satisfies*

$$\boxed{M_t S_t^j = \mathbb{E}_t \left[ \sum_{\tau=t+1}^{T} M_\tau V_\tau^j \right], \text{ for } t = 0, \ldots, T-1}$$

*(i.e) the value of any asset j at time t is equal to the (conditional) mathematical expectation of the value of its future payoffs*

# Arbitrage asset pricing

## Stochastic discount factor: intertemporal form

Observations:

1. We say this is SDF definition is in the **intertemporal form**
2. the meaning of the conditional expectation $\mathbb{E}_t[.]$ is

$$M_t S^j_t = \mathbb{E}_t\left[\sum_{\tau=t+1}^{T} M_\tau V^j_\tau\right] = \mathbb{E}\left[\sum_{\tau=t+1}^{T} M_\tau V^{j,t}_\tau \,\middle|\, S^{j,t}, V^t\right], \text{ for any } j = 1, \dots$$

where $S^{j,t} = \{S^j_0, S^j_1, \dots S^j_t\}$ and $V^{j,t} = \{V^j_1, \dots V^j_t\}$ are the histories of the asset prices and payoffs of asset $t$ up until time $t$

# Arbitrage asset pricing

## Stochastic discount factor: recursive form

### Proposition

*The stochastic discount factor can be equivalently defined in the* ***recursive from***

$$\boxed{M_t S_t^j = \mathbb{E}_t \left[ M_{t+1} (S_{t+1}^j + V_{t+1}^j) \right], \textit{ for any } j = 1, \ldots, K}$$

- **Intuition**: the stochastic discount factor is a stochastic process $\{M_t\}$, that equalizes the **value** of the asset price in period $t$ with the conditional expected value of the **value** of the income in period $t + 1$ (the income is equal to the payoff plus the anticipated market price)

# Arbitrage asset pricing

## Stochastic discount factor: recursive form

Proof:

- using the definition of intertemporal form and expanding

$$M_t S_t^j = \mathbb{E}_t \left[ M_{t+1} V_{t+1}^j + \sum_{\tau=t+2}^{T} M_\tau V_\tau^j \right]$$

- by the law of iterated expectations

$$\mathbb{E}_t \left[ M_{t+1} V_{t+1}^j + \sum_{\tau=t+2}^{T} M_\tau V_\tau^j \right] = \mathbb{E}_t \left[ M_{t+1} V_{t+1}^j + \mathbb{E}_{t+1} \left[ \sum_{\tau=t+2}^{T} M_\tau V_\tau^j \right] \right]$$

- but

$$M_{t+1} S_{t+1}^j = \mathbb{E}_{t+1} \left[ \sum_{\tau=t+2}^{T} M_\tau V_\tau^j \right],$$

- then $M_t S_t^j = \mathbb{E}_t \left[ M_{t+1} V_{t+1}^j \right] + \mathbb{E}_t \left[ M_{t+1} S_{t+1}^j \right]$

# Arbitrage asset pricing

## Stochastic discount factor and the rate of return

### Proposition

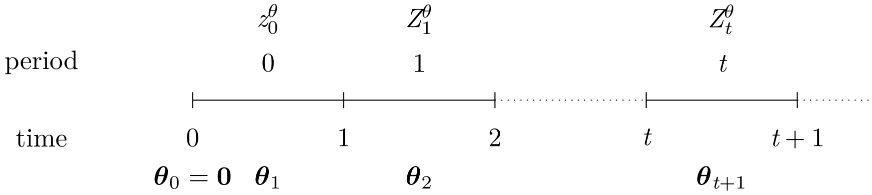
*The stochastic discount factor can be equivalently defined in the* ***recursive from*** *using the definition of the return of an asset*

$$\boxed{M_t = \mathbb{E}_t \left[ M_{t+1} R^j_{t+1} \right],\ \textit{for any}\ j = 1, \ldots, K}$$

- **Intuition**: the stochastic discount factor is a stochastic process $\{M_t\}$, that equalizes the **value** of the return of an investment (or loan) to be collected (or payed) at time $t+1$, to the price of money, conditional on the information at time $t$

# Portfolios and income

Stochastic processes for portfolios and income: $\{\boldsymbol{\theta}_t\}_{t=1}^{T-1}$ and $\{Z_t\}_{t=0}^{T}$




- at every time $t = 0, \dots, T-1$ (not just at $t = 0$ as before) a portfolio $\boldsymbol{\theta}_{t+1} = (\theta_{t+1}^1, \dots, \theta_{t+1}^K)$ can be detained
- it generates an income in period $t = 0, \dots, T$, $Z_t^\theta$

# Arbitrage asset pricing

## Transactions strategy

- The **income** stream $\{Z_t^\theta\}_{t=0}^T$ where (zero initial wealth) generated by a transactions strategy $\{\boldsymbol{\theta}_t\}_{t=1}^T$ is

$$
\begin{aligned}
z_0^\theta &= -\theta_1 S_0 = -\sum_{j=1}^K \theta_1^j S_0^j \\
&\dots \\
Z_t^\theta &= \theta_t(S_t + V_t) - \theta_{t+1} S_t = \sum_{j=1}^K \left(\theta_t^j (S_t^j + V_t^j) - \theta_{t+1}^j S_t^j\right), \\
Z_T^\theta &= \theta_T V_T = \sum_{j=1}^K \theta_T^j V_T^j
\end{aligned}
$$

- where $Z_t^\theta \in \mathbb{R}^{N_t}$ is $\mathcal{F}_t$-measurable, i.e.

$$Z_t^\theta = (z_{t,1}^\theta, \dots, z_{t,s}^\theta, \dots, z_{t,N_t}^\theta)$$

# Arbitrage asset pricing

## Transactions strategy

### Definition

*A* ***transactions strategy*** *is a sequence of portfolios* $\{\boldsymbol{\theta}_{t+1}\}_{t=0}^{T-1}$, *with* $\boldsymbol{\theta}_{t+1} = (\theta_{t+1}^{1} \dots \theta_{t+1}^{K})$, *where* $\theta_{t+1}^{j}$ *is* $\mathcal{F}_t$*-measurable, generating an* ***income*** *stream* $\{Z_t^{\theta}\}_{t=0}^{T} = \{z_0^{\theta}, Z_1^{\theta}, \dots, Z_T^{\theta}\}$.

### Definition

*If* $z_0^{\theta} = \dots = Z_t^{\theta} = \dots = Z_T^{\theta} = \mathbf{0}$ *we say the transactions strategy is* ***self-financed****.*

# Arbitrage asset pricing

## Absence of arbitrage opportunities

### Definition

*There is* **absence of arbitrage opportunities** *if there is a* ***positive process*** $\{M_t\}_{t=0}^{T-1}$ *such that the income stream* $\{Z_t^{\theta}\}_{t=0}^{T}$*, generated by the transaction strategy* $\{\boldsymbol{\theta}_{t+1}\}_{t=0}^{T-1}$*, satisfies*

$$\boxed{\mathbb{E}_0\left[\sum_{t=0}^{T} M_t Z_t^{\theta}\right] = 0}$$

**Intuition**: there are no arbitrage opportunities if, with a **zero initial investment**, the expected value of the present value of any transaction strategy is zero, if the discount factor is positive.

# Arbitrage asset pricing

## Absence of arbitrage opportunities

### Proposition

*A necessary condition for the absence of arbitrage opportunities is that:*

1. *The terminal price satisfies* $M_T S_T = 0$ *if* $T$ *is finite;*
2. *ruling-out speculative bubbles condition holds:* $\lim_{t \to \infty} M_t S_t = 0$ *if* $T = \infty$

# Arbitrage asset pricing

## Absence of arbitrage opportunities

Proof (assuming $K = 1$):

- use the definition of stochastic discount factor (in the recursive form)

$$-M_0 Z_0^\theta = M_0 \theta_1 S_0 = \mathbb{E}_0 \left[ M_1 \theta_1 (S_1 + V_1) \right]$$

- use a little trick, introducing $\pm M_1 \theta_2 S_1$;

$$\begin{aligned} \mathbb{E}_0 \left[ M_1 \theta_1 (S_1 + V_1) \right] &= \mathbb{E}_0 \left[ M_1 \theta_1 (S_1 + V_1) \pm M_1 \theta_2 S_1 \right] = \\ &= \mathbb{E}_0 \left[ M_1 Z_1^\theta + M_1 \theta_2 S_1 \right] \end{aligned}$$

- use the definition of stochastic discount factor and the law of iterated expectations

$$\begin{aligned} \mathbb{E}_0 \left[ M_1 Z_1^\theta + M_1 \theta_2 S_1 \right] &= \mathbb{E}_0 \left[ M_1 Z_1^\theta + \mathbb{E}_1 \left[ M_2 \theta_2 (S_2 + V_2) \right] \right] \\ &= \mathbb{E}_0 \left[ M_1 Z_1^\theta + M_2 \theta_2 (S_2 + V_2) \right] \end{aligned}$$

# Arbitrage asset pricing

## Absence of arbitrage opportunities

Proof (assuming $K = 1$ continuation):

- by repeatedly using the previous steps we arrive at

$$-M_0 Z_0^\theta = \mathbb{E}_0 \left[ \sum_{t=1}^{T} M_t Z_t^\theta + M_T \theta_{t+1} S_T \right]$$

- then

$$\mathbb{E}_0 \left[ \sum_{t=0}^{T} M_t Z_t^\theta \right] = 0$$

  only if $M_T S_T = 0$

# Application 1: zero payoffs

## Absence of arbitrage opportunities

- **Zero payoffs (or no dividends case)**: Assume that there are no dividends, i.e., $V_t = \mathbf{0}$ for any $t = 1, \ldots, T$.
- If there are no arbitrage opportunities then

$$M_t S_t^j = \mathbb{E}_t \left[ M_{t+1} S_{t+1}^j \right]$$

- therefore:

### Proposition

*For a zero dividend process, the process* $\{M_t S_t\}_{t=0}^{T-1}$ ***is a martingale under measure*** $\mathbb{P}$,

## Proposition

*For a zero-dividend asset, absence of arbitrage opportunities is equivalent to the existence of an equivalent probability measure $\mathbb{Q}$ such that $\{S_t\}_{t=0}^{T-1}$ is a martingale under $\mathbb{Q}$, that is*

$$S_t^j = \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

# Arbitrage asset pricing

## Fundamental theorem

Sketch of proof

- Let us define the **conditional** stochastic discount factor

$$M_{t+1|t} \equiv \frac{M_{t+1}}{M_t}$$

- Then if there are no arbitrage opportunities (because $M_t$ is $\mathbb{F}_t$-measurable)

$$S_t^j = \mathbb{E}_t[M_{t+1|t} S_{t+1}^j]$$

- This is valid for the degenerate process $\{\mathbf{1}\}_{t=0}^T = \{\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}\}$, then

$$1 = \mathbb{E}_t[M_{t+1|t}]$$

$$S_t^j = \mathbb{E}_t[M_{t+1|t} S_{t+1}^j] = \frac{\mathbb{E}_t[M_{t+1|t} S_{t+1}^j]}{\mathbb{E}_t[M_{t+1|t}]} = \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

- from the Radon-Nikodym theorem $\mathbb{Q}$ is an equivalent martingale measure.

Observe that $\{M_t\}$ is also a martingale, under measure $\mathbb{Q}$ because

$$M_t = \mathbb{E}_t^{\mathbb{Q}}[M_{t+1}]$$

# Application 2: positive payoffs

- **Positive payoffs (or positive dividends case)**: if asset $j$ pays a positive dividend, that is $V_t^j \geq \mathbf{0}$ is a positive vector, then

$$\mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j + V_{t+1}^j] \geq \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

- Then $\{S_t\}$ is a **submartingale** under measure $\mathbb{Q}$

$$S_t^j \geq \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

## Proposition

*For a positive-dividend asset, absence of arbitrage opportunities is equivalent to the existence of an equivalent probability measure $\mathbb{Q}$ such that $\{S_t\}_{t=0}^{T-1}$ is a sub-martingale under $\mathbb{Q}$, that is*

$$S_t^j \geq \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

# Application 3: Existence of a risk-free asset

- Consider a bond, issued at every time $t = 0, \ldots, T-1$, with the maturity of one period and paying a (deterministic) payoff with unit face value
- Then

$$S_t^f = \frac{1}{1 + r_{t+1}}, \ V_{t+1}^f = \mathbf{1}, \ V_{t+2}^f = \mathbf{0}, \ldots V_T^f = \mathbf{0}$$

- If there are no arbitrage opportunities then

$$\frac{1}{1 + r_{t+1}} = \mathbb{E}_t \left[ M_{t+1|t} \right].$$

# Application 3: Existence of a risk-free asset

## Proposition

*Assume there are no arbitrage opportunities and there is a risk-free asset with the (deterministic) return process* $\{R_t^f\}_{t=1}^T$. *Then there is a probability process* $\mathbb{Q}$ *such that the return for asset* $j$ *satisfies*

$$\boxed{R_{t+1}^f = \mathbb{E}_t^{\mathbb{Q}}\left[R_{t+1}^j\right], \textit{ for any } j = 1, \ldots K, \textit{for any } t = 0, \ldots T}$$

# Application 3: Existence of a risk-free asset

- Proof: for any other risky asset, $j$, we can write

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left[S_{t+1}^j + V_{t+1}^j\right] &= \frac{\mathbb{E}_t\left[M_{t+1|t}\left(S_{t+1}^j + V_{t+1}^j\right)\right]}{\mathbb{E}_t\left[M_{t+1|t}\right]} = \\ &= (1 + r_{t+1})\,\mathbb{E}_t\left[M_{t+1|t}\left(S_{t+1}^j + V_{t+1}^j\right)\right] = \\ &= (1 + r_{t+1})\,S_t^j \end{aligned}$$

- Then

$$S_t^j = \frac{1}{1 + r_{t+1}}\mathbb{E}_t^{\mathbb{Q}}\left[S_{t+1}^j + V_{t+1}^j\right]$$

- Dividing by $S_t^j$ we find

$$R_{t+1}^f = \mathbb{E}_t^{\mathbb{Q}}\left[R_{t+1}^j\right]$$

- It can also be proved that

$$S_t^j = \mathbb{E}_t^{\mathbb{Q}} \left[ \sum_{\tau=t+1}^{T} D_{t+1,\tau} \, V_\tau^j \right]$$

the asset price at time $t$ is the conditional expected value of the present value of the future payoffs;

- where the discount factor is

$$D_{t+1,\tau} = \prod_{h=t+1}^{\tau} \frac{1}{1+r_h}, \ \tau \geq t+1.$$

- Exercise: prove this.

# Equilibrium asset pricing

# Equilibrium asset pricing

## Real part of the economy: resources

- There is a **given** sequence of endowments

$$Y^T \equiv \{ Y_t \}_{t=0}^{T} = \{ Y_0, Y_1, \ldots, Y_t, \ldots, Y_T \}$$

- where $Y_t$ is $\mathcal{F}_t$- mensurable, such that

$$Y_t = \begin{pmatrix} y_{t,1} \\ \ldots \\ y_{t,N_t} \end{pmatrix}$$

# Equilibrium asset pricing

## Real part of the economy: preferences and distribution

- households choose a contingent-consumption sequence belonging to the set

$$C^T \equiv \{C_t\}_{t=0}^T = \{C_0, C_1, \dots, C_t, \dots, C_T\}$$

where $C_t$ is $\mathcal{F}_t$- mensurable,

- through an intertemporal von-Neumman-Morgenstern functional

$$\mathbb{E}_0 \left[ \sum_{t=0}^T \beta^t u(C_t) \right]$$

- expansion of the utility functional

$$\begin{aligned} \mathbb{E}_0 \left[ \sum_{t=0}^T \beta^t u(C_t) \right] &= \sum_{t=0}^T \beta^t \mathsf{P}_t u(C_t) = \\ &= u(C_0) + \beta \mathsf{P}_1 u(C_1) + \dots + \beta^t \mathsf{P}_t u(C_t) + \dots + \beta^T \mathsf{P} \end{aligned}$$

where

$$\mathsf{P}_t u(C_t) = \sum_{s=1}^{N_t} \pi_{t,s} u(c_{t,s})$$

# Equilibrium asset pricing

## Market structure

There are assets markets with the structure we have just presented, opening at every time $t \in \{0, \ldots, T\}$

# Equilibrium asset pricing: zero initial wealth

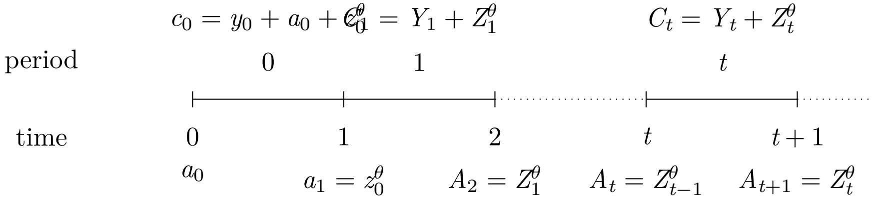
**Assumption**: the level of initial net wealth is zero $a_0 = 0$

**Consequence**: we can transform the household problem in a finance economy into the household problem in an equivalent AD economy

**Non-zero initial wealth**: we have to apply other methods for solving the household-investor problem (v.g, dynamic programming or optimal control)

# Flow and stock accounting

Adaptoing the timing



# Equilibrium asset pricing: Zero initial wealth

Radner or sequential general equilibrium

### Definition

*The* **Radner or sequential general equilibrium** *is defined by the processes* $\{C_t\}_{t=0}^{T}$, $\{\boldsymbol{\theta}_t\}_{t=1}^{T}$ *and* $\{\mathbf{S}_t\}_{t=0}^{T-1}$ *such that,* ***given*** *the processes of endowments* $\{Y_t\}_{t=0}^{T}$ *and payoffs* $\{\mathbf{V}_t\}_{t=1}^{T}$*:*
*(1) the household solves his* ***consumption-portfolio problem****, with rational expectations regarding future asset prices, and*
*(2) the* ***markets clear****,*

$$\begin{aligned} C_t &= Y_t,\ t = 0, \dots, T \\ \boldsymbol{\theta}_t &= 0,\ t = 1, \dots, T. \end{aligned}$$

# Equilibrium asset pricing: Zero initial wealth

## The (sequential) household-investor problem

Find the process for consumption $\{C_t\}_{t=0}^T$ and a transactions' strategy $\{\theta_t\}_{t=1}^T$

- that maximizes the value functional

$$V_0(\{C_t\},\{\theta_t\}) \equiv \mathbb{E}_0\left[\sum_{t=0}^{T}\beta^t u(C_t)\right]$$

- subject to the sequential budget constraints

$$\begin{aligned}
c_0 + \sum_{j=1}^{K}\theta_1^j S_0^j &\leq y_0 \\
&\dots \\
C_t + \sum_{j=1}^{K}\theta_{t+1}^j S_t^j &\leq Y_t + \sum_{j=1}^{K}\theta_t^j(S_t^j + V_t^j),\ t = 1,\dots,T-1\ (\mathcal{F}_t - \text{adapt} \\
&\dots \\
C_T &\leq Y_T + \sum_{j=1}^{K}\theta_T^j V_T^j\ (\mathcal{F}_T - \text{adapted})
\end{aligned}$$

# Equilibrium asset pricing: Zero initial wealth

## The (sequential) household problem

- We can write the sequence of budget constraints equivalently as

$$\begin{aligned} C_0 &\leq Y_0 + Z_0^\theta \\ &\cdots \\ C_t &\leq Y_t + Z_t^\theta,\ t = 1, \ldots, T-1\ (\mathcal{F}_t - \text{adapted}) \\ &\cdots \\ C_T &\leq Y_T + Z_T^\theta\ (\mathcal{F}_T - \text{adapted}) \end{aligned}$$

  where $Z_t^\theta$ is the income generated at time $t$ by the transaction strategy $\{\theta_t\}_{t=1}^T$.
- If the utility function $u(.)$ displays no-satiation the constraints hold with equality in the optimum.

# Equilibrium asset pricing: Zero initial wealth

## Equivalent simultaneous household problem

- **If there are no arbitrage opportunities**, then there is stochastic discount factor process $\{M_t\}_{t=0}^{T-1}$, such that

$$-\mathbb{E}_0\left[\sum_{t=0}^{T} M_t Z_t^{\theta}\right] = \mathbb{E}_0\left[\sum_{t=0}^{T} M_t (Y_t - C_t)\right] = 0.$$

- Then, the household's problem is (the same as in the AD economy)

$$\max_{\{C_t\}_{t=0}^{T}} \mathbb{E}_0\left[\sum_{t=0}^{T} \beta^t u(C_t)\right] \text{ s.t } \mathbb{E}_0\left[\sum_{t=0}^{T} M_t (Y_t - C_t)\right] = 0$$

- We already found the f.o.c

$$u'(c_0) M_t = \beta^t u'(C_t), \ (\mathcal{F}_t - \text{adapted})$$

# Equilibrium asset pricing: Zero initial wealth

## Equilibrium stochastic discount factor

- The household arbitrage condition and the market equilibrium conditions

$$\begin{cases} u'(c_0)M_t = \beta^t u'(C_t) & t = 1, \dots, T \\ C_t = Y_t & t = 0, \dots, T \end{cases}$$

- imply that, at equilibrium, as in the AD economy

$$\boxed{M_t = \beta^t \frac{u'(Y_t)}{u'(Y_0)} \; (\mathcal{F}_t - \text{adapted})}$$

- In terms of the possible realizations

$$M_t = \begin{pmatrix} m_{t,1} \\ \dots \\ m_{t,N_t} \end{pmatrix}, \; t = 0, \dots, T-1$$

where

$$m_{ts} = \beta^t \frac{u'(y_{ts})}{u'(y_0)}, \text{ for } s = 1, \dots, N_t, \text{and}, \; t = 0, \dots T-1.$$

# Equilibrium asset pricing: Zero initial wealth



- If there are no arbitrage opportunities, we proved that, for any asset $j$

$$M_t S_t^j = \mathbb{E}_t \left[ M_{t+1} ( V_{t+1}^j + S_{t+1}^j ) \right], \;\; j = 1, \ldots, K$$

- Then the **GE equilibrium** asset pricing is

$$\boxed{u'(Y_t) S_t^j = \beta \mathbb{E}_t \left[ u'(Y_{t+1}) ( V_{t+1}^j + S_{t+1}^j ) \right], \;\; j = 1, \ldots, K}$$

- determines asset price process $\{S_t^j\}$ given the processes $\{V_t^j\}$ and $\{Y_t\}$.

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Equivalent representations:

1. The **equilibrium rate of return** for asset $j$ is determined from

$$\boxed{\mathbb{E}_t\left[M_{t+1|t}R^j_{t+1}\right] = 1}$$

   where the **equilibrium recursive stochastic discount factor** is

$$M_{t+1|t} \equiv \beta \frac{u'(Y_{t+1})}{u'(Y_t)}$$

   and the return is

$$R^j_{t+1} = \frac{V^j_{t+1} + S^j_{t+1}}{S^j_t}$$

2. or, equivalently

$$\boxed{u'(Y_0)S^j_0 = \mathbb{E}_0\left[\sum_{t=1}^{T}\beta^t u'(Y_t)\,V^j_t\right],\ \ j = 1, \ldots, K.}$$

# Infinite horizon case, $T = \infty$

- The arbitrage condition is, off course, still valid.

## Proposition

*Fundamental equilibrium arbitrage condition: if we rule out speculative bubbles, then the price for asset j satisfies*

$$S_t^j = \mathbb{E}_t \left[ \sum_{\tau=1}^{\infty} \beta^\tau \frac{u'(C_{t+\tau})}{u'(C_t)} V_{t+\tau}^j \right], \quad j = 1, \ldots, K, \quad t \in [0, \infty) \qquad (1)$$

-

# Infinite horizon case, $T = \infty$

Proof:

$$\begin{aligned} u'(C_t)S^j_t &= \beta \mathbb{E}_t \left[ u'(C_{t+1})(S^j_{t+1} + V^j_{t+1}) \right] = \\ &= \lim_{k\to\infty} \beta^k \mathbb{E}_t \left[ u'(C_{t+k}) S^j(t+k) \right] + \\ & \quad + \mathbb{E}_t \left[ \sum_{\tau=1}^{\infty} \beta^\tau u'(C_{t+\tau}) V^j(t+\tau) \right] \end{aligned}$$

If we rule out speculative bubbles, that is

$$\lim_{k\to\infty} \beta^k \mathbb{E}_t \left[ u'(C_{t+k}) S^j_{t+k} \right] = 0$$

we get equation (**??**)

# Risky and risk-free assets

- For a risky asset

$$\mathbb{E}_t \left[ M_{t+1|t} R^j_{t+1} \right] = 1$$

- For a riskless asset with return $R^f_t = 1 + r^f_t$ we have

$$\mathbb{E}_t \left[ M_{t+1|t} \right] R^f_{t+1} = 1$$

- Then, for any asset $j$

$$\mathbb{E}_t \left[ M_{t+1|t} R^j_{t+1} \right] = \mathbb{E}_t \left[ M_{t+1|t} \right] R^f_{t+1}$$

# Equilibrium equity premium: example

## Equilibrium risk premium for a Markovian case

- **Assumptions:**
  1. Homogeneous agent finance economy
  2. CRRA Bernoulli utility function
  3. growth factor for the return is Markovian following an iid log-normal distribution
  4. there is one riskless and one risky asset such that the return is Markovian following an iid log-normal distribution
- Problem: **Derive the distribution for the multiplicative risk premium for the risky asset $R^j/R^f$**
- Solution: the risk premium for asset $j$, satisfies

$$\boxed{\ln \mathbb{E}_t[R^j_{t+1}] = \ln R^f_{t+1} + \zeta \text{Cov}_t \left[\ln(1+\gamma_{t+1}), \ln R^j_{t+1}\right],\ j = 1, \ldots, K}$$

# Auxiliary: log-normal distributions

## Some properties

Assume two random variables $X$ and $Y$ following log-normal distributions. Then $\ln X$ and $\ln Y$ are normally distrbuted. Then:

$$\ln \mathbb{E}[X] = \mathbb{E}[\ln X] + \frac{1}{2}\mathbb{V}[\ln X]$$

$$\ln \mathbb{E}[\alpha X] = \ln \alpha + \mathbb{E}[\ln X] + \frac{1}{2}\mathbb{V}[\ln X],\ \alpha \text{ constant}$$

$$\ln \mathbb{E}[\alpha X^{\beta}] = \ln \alpha + \beta\mathbb{E}[\ln X] + \frac{\beta^2}{2}\mathbb{V}[\ln X],\ \alpha, \beta,\ \text{constants}$$

$$\ln \mathbb{E}[XY] = \mathbb{E}[\ln X] + \mathbb{E}[\ln Y] + \frac{1}{2}\left\{\mathbb{V}[\ln X] + \mathbb{V}[\ln Y] - 2\,Cov(\ln X, \ln Y)\right\}$$

$$\ln \mathbb{E}[X^{\beta} Y] = \beta\mathbb{E}[\ln X] + \mathbb{E}[\ln Y] + \\ + \frac{1}{2}\left\{\beta^2\mathbb{V}[\ln X] + \mathbb{V}[\ln Y] - 2\beta\,Cov(\ln X, \ln Y)\right\}$$

because $Cov[\beta X, Y] = \beta\,Cov[XY]$.

# Equilibrium equity premium example: proof

solution

- The risky asset $j$ follows a iid log-normal distribution: then

$$\ln \mathbb{E}_t[R^j_{t+1}] = \mathbb{E}_t[\ln R^j_{t+1}] + \frac{1}{2}\mathbb{V}_t[\ln R^j_{t+1}]$$

- the endowment process satisfies $Y_{t+1} = (1+\gamma_{t+1})\,Y_t$, where the growth factor follows also a iid log-normal distribution: then

$$\ln \mathbb{E}_t[1+\gamma_{t+1}] = \mathbb{E}_t[\ln(1+\gamma_{t+1})] + \frac{1}{2}\mathbb{V}_t[\ln(1+\gamma_{t+1})]$$

- the utility function is CRRA $u(C) = \frac{C^{1-\zeta}-1}{1-\zeta}$ then the stochastic discount factor is

$$M_{t+1|t} = \beta(1+\gamma_{t+1})^{-\zeta}$$

- then, for any asset, the arbitrage condition holds as

$$1 = \beta \mathbb{E}_t\left[(1+\gamma_{t+1})^{-\zeta} R^j_{t+1}\right]$$

# Equilibrium equity premium example: proof

solution (cont.)

- for the riskless asset, after taking logs to the arbitrage condition, we have

$$
\begin{aligned}
0 &= \ln\beta + \ln\mathbb{E}_t\left[(1+\gamma_{t+1})^{-\zeta}R^f_{t+1}\right] \\
&= \ln\beta + \mathbb{E}_t\left[\ln\left((1+\gamma_{t+1})^{-\zeta}R^f_{t+1}\right)\right] + \\
&\quad + \frac{1}{2}\mathbb{V}_t\left[\ln\left((1+\gamma_{t+1})^{-\zeta}R^f_{t+1}\right)\right] \\
&= \ln\beta - \zeta\mathbb{E}_t[\ln(1+\gamma_{t+1})] + \ln R^f_{t+1} + \frac{\zeta^2}{2}\mathbb{V}_t[\ln(1+\gamma_{t+1})]
\end{aligned}
$$

# Equilibrium equity premium example: proof

## solution (cont.)

- for the risky asset $j$, after taking logs to the arbitrage condition, we have

$$\begin{aligned}
0 &= \ln\beta + \ln\mathbb{E}_t\left[(1+\gamma_{t+1})^{-\zeta}R^j_{t+1}\right] = \\
&= \ln\beta - \zeta\mathbb{E}_t[\ln(1+\gamma_{t+1})] + \mathbb{E}_t[\ln R^j_{t+1}] + \\
&\quad + \frac{1}{2}\left\{\zeta^2\mathbb{V}_t[\ln(1+\gamma_{t+1})] + \mathbb{V}_t[\ln R^j_{t+1}] - \right. \\
&\quad \left. -2\zeta\mathrm{Cov}_t\left[\ln(1+\gamma_{t+1}), \ln R^j_{t+1}\right]\right\} = \\
&= -\ln R^f_{t+1} + \mathbb{E}_t[\ln R^j_{t+1}] + \frac{1}{2}\mathbb{V}_t[\ln R^j_{t+1}] - \\
&\quad -\zeta\mathrm{Cov}_t\left[\ln(1+\gamma_{t+1}), \ln R^j_{t+1}\right] = \\
&= -\ln R^f_{t+1} + \ln\mathbb{E}_t\left[R^j_{t+1}\right] - \zeta\mathrm{Cov}_t\left[\ln(1+\gamma_{t+1}), \ln R^j_{t+1}\right].
\end{aligned}$$

(end of proof)

# Equilibrium equity premium

## Hansen-Jaganathan bounds

- Let us write the **Equity premium** for asset risky $j$ as: $R^j_{t+1} - R^f_{t+1}$
- Expected premium and standard deviation

$$\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right],\ \sigma_t\left[R^j_{t+1} - R^f_{t+1}\right]$$

- **Equilibrium equity premium** for risky asset $j$ satisfies, under the assumptions of the model:

$$\mathbb{E}_t\left[M_{t+1|t}\left(R^j_{t+1} - R^f_{t+1}\right)\right] = 0$$

- Then,

$$\boxed{\left|\frac{\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right]}{\sigma_t\left[R^j_{t+1}\right]}\right| \leq \frac{\sigma_t[M_{t+1|t}]}{\mathbb{E}_t[M_{t+1|t}]}} \qquad (2)$$

the l.h.s is called the Sharpe ratio and r.h.s. the Hansen-Jaganathan bounds

# Equilibrium equity premium

## Hansen-Jaganathan bounds

- Proof:
- From a standard result on the covariance between two random variables

$$\begin{aligned} & \mathbb{E}_t\left[M_{t+1|t}\left(R^j_{t+1} - R^f_{t+1}\right)\right] = \\ = \; & \mathbb{E}_t\left[M_{t+1|t}\right]\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right] + \text{Cov}_t\left[M_{t+1|t}, \left(R^j_{t+1} - R^f_{t+1}\right)\right] = \\ = \; & 0 \end{aligned}$$

- But

$$\text{Cov}_t\left[M_{t+1|t}, \left(R^j_{t+1} - R^f_{t+1}\right)\right] = \rho_{M_{t+1|t}, R^j_{t+1} - R^f_{t+1}} \sigma_t(M_{t+1|t}) \sigma_t\left(R^j_{t+1} - R^f_{t+1}\right) \quad (3)$$

where $\rho_{M_{t+1|t}, R^j_{t+1} - R^f_{t+1}}$ is the correlation coefficient between $M_{t+1|t}$ and $R^j_{t+1} - R^f_{t+1}$

# Equilibrium equity premium

## Hansen-Jaganathan bounds (cont.)

- Then

$$\frac{\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right]}{\sigma_t\left[R^j_{t+1}\right]} = \rho_{M_{t+1|t},R^j_{t+1}-R^f_{t+1}} \frac{\sigma_t[M_{t+1|t}]}{\mathbb{E}_t[M_{t+1|t}]}$$

- We use the fact $|\rho_{M_{t+1|t},R^j_{t+1}-R^f_{t+1}}| \in [0,1]$

# Equilibrium equity premium

## Example

- If we assume that $Y_{t+1} = (1+\gamma_{t+1})\, Y_t$, the utility function is homogeneous, and $R^f_{t+1} \approx 1/\beta$ then

$$\left| \frac{\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right]}{\sigma_t\left[R^j_{t+1}\right]} \right| \leq \sigma_t\left[u'(1+\gamma_{t+1})\right]$$

- if the utility function is homogeneous, from the equilibrium arbitrage condition

$$\beta \mathbb{E}_t[u'(1+\gamma_{t+1})] R^f_{t+1} = 1$$

- if $R^f_{t+1} \approx 1/\beta$ then

$$\mathbb{E}_t[u'(1+\gamma_{t+1})] = 1$$

# Equilibrium equity premium: example

- If we ssume a CRRA utility function

$$u(C) = u(C) = \frac{C^{1-\zeta} - 1}{1 - \zeta}$$

Then

$$M_{t+1|t} = \beta(1 + \gamma_{t+1})^{-\zeta}$$

$$\sigma_t \left[ u'(1 + \gamma_{t+1}) \right] = \sigma_t \left[ (1 + \gamma_{t+1})^{-\zeta} \right]$$

The higher $\eta$ the lower $\sigma_t[M_{t+1|t}]$ is.

# Equilibrium equity premium puzzle

- **Equity premium puzzle**: if we set $\zeta \approx 2$, we find excessive risk premium in the data:

$$\text{Sharpe ratio} = 0.37 > \frac{\sigma_t[M_{t+1|t})]}{\mathbb{E}_t[M_{t+1|t}]} \approx \frac{0.002}{0.96}$$

- This means that the data displays a higher risk premium than the model would predict (or consumption displays a lower relative volatility than the model predicts)

# Equilibrium equity premium puzzle

- This has led to a whole research program (still going on) for macro finance: see `http://academicwebpages.com/preview/mehra/pdf/FIN200201.pdf` for a survey, by introducing in the model:
  - changes in preferences: habit formation, non-additive preferences concerning risk
  - transactions costs, taxes, etc
  - distributions
  - imperfectly competitive environments
- The basic change we have to introduce should do the following: consumption (and investment) should have a smoother behaviour than the model predicts, which means that the reaction of portfolios to changes in asset prices is more rigid, which implies a higher variation in prices to unpredicted shocks.

# Non-zero initial wealth

# Equilibrium asset pricing: non-zero initial wealth

**Assumption** assume that the level of initial net wealth is different from zero

**Implications**:

- we cannot transform the household problem into a household problem for an AD economy
- the household problem becomes a stochastic optimal control problem where financial wealth, $A_t$, is the state variable and consumption, $C_t$ is the control variable

# Equilibrium asset pricing: non-zero initial wealth

## Timing and information sequence

- When we introduce the stock of financial wealth $A_t$, we have to be careful as regards the stock-flow accounting and the exact timing of information should be specified. From now on, we assume the information timing for the end period $t$, is taken **after** observing $C_t$ and $Y_t$ and **before** observing $S_t$ and $\mathbb{V}_t$.
- That is, in period $t$:
    1. at the start of period $t$, between times $t$ and $t+1$ we know the portfolio $\boldsymbol{\theta}_t$ and we observe $S^j_{t-1}$ (set at the end of period $t-1$);
    2. along period $t$ we observe the flow $Y_t$ and we decide over the flow $C_t$;
    3. close to the time $t+1$, we observe the payoffs $V^j_t$ for $j = 1, \ldots, K$ and the asset markets open and draw $S^j_t$;
    4. we buy a new portfolio $\theta^j_{t+1}$ at the market prices $S^j_t$
    5. And so on.

# Equilibrium asset pricing: non-zero initial wealth

## Sequential budget constraints

- The stock of asset $j$ at time $t$ is

$$A_t^j = \theta_t^j S_{t-1}^j$$

- and the return of asset $j$ computed at the end of period $t$ is

$$R_t^j = \frac{S_t^j + V_t^j}{S_{t-1}^j} = 1 + r_t^j$$

- Then the period budget constraint for period $t$ is

$$\sum_j R_t^j A_t^j + Y_t = C_t + \sum_j A_{t+1}^j$$

- This is also equivalent to

$$C_t = Y_t + Z_t^\theta$$

# Equilibrium asset pricing: non-zero initial wealth

## Sequential budget constraints

- The **total** wealth at the beginning of period $t$ is

$$A_t = \sum_{j=1j}^{K} A_t^j$$

- If we define the weight of asset $j$ in total wealth as

$$w_t^j \equiv \frac{A_t^j}{A_t}, \; \sum_{j=1}^{K} w_t^j = 1$$

then

$$R_t^A A_t - A_{t+1} = (1 + r_t^A) A_t - A_{t+1}$$

where the average return is the rate of return for the whole portfolio

$$R_t^A = \sum_j w_t^j R_t^j, \; r_t^A = \sum_j w_t^j r_t^j,$$

# Equilibrium asset pricing: non-zero initial wealth

## Sequential budget constraints

- Then the period budget constraint for period $t$ is

$$A_{t+1} = Y_t - C_t + R_t^A A_t,\ t = 0, \dots, T,$$

- Off course, given $A_t = a_t$ at the beginning of period $t$ (starting a time $t$ and ending at time $t+1$) $A_{t+1}$ is a distribution

$$A_{t+1} = (a_{t+1,1}, \dots, a_{t+1,N_{t+1|t}})^\top$$

  where $N_{t+1|t}$ is the number of nodes at $t+1$ subsequent to the node $s_t$ at time $t$;

- We are assuming that all the possible realizations of the budget constraint are of the form:

$$a_{t+1,s_{t+1}|s_t} = y_{t,s_t} - c_{t,s_t} + R^A_{t,s_{t+1}|s_t} a_{t,s_t},\ \text{for } s_t = 1, \dots, N_t,$$

  and $s_{t+1}|s_t = 1, \dots, N_{t+1|t}$

# Equilibrium asset pricing: non-zero initial wealth

- at time $t$ the household observes $A_t$
- along period $t$ he gets $Y_t$ and decides on consumption $C_t$
- at the the end of period he receives a signal $R_t^A$ and decides on the portfolio composition $\theta_{t+1}$ such that $A_{t+1} = \sum_{j=1}^{K} \theta_{t+1}^j S_t^j$
- in our case only the savings decision (not the financial decision over $\{\theta_t\}_{t=1}^T$) matters for the determination of the equilibrium stochastic discount factor

Conditional evolution of the asset position

$$\begin{pmatrix} Y_{t,1} - C_{t,1} + R_{t,1}^A\, A_t \\ \dots \\ Y_{t,N_{t+1|t}} - C_{t,N_{t+1|t}} + R_{t,N_{t+1|t}}^A\, A_t \end{pmatrix}$$

$t$ — $t+1$

$A_t$

$$\begin{pmatrix} A_{t+1,1} \\ \dots \\ A_{t+1,N_{t+1|t}} \end{pmatrix}$$

# Equilibrium asset pricing: non-zero initial wealth

## The representative agent problem

$$\max_{\{C_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t u(C_t) \right]$$

subject to the **sequence** of random constraints

$$A_{t+1} = Y_t - C_t + R_t^A A_t, \; t = 0, \ldots, T-1$$

given $A_0$ and the non-Ponzi games condition

$$\lim_{t \to \infty} \mathbb{E}_0 \Big[ \prod_{t=1}^{T} R^{-t} A_{t+1} \Big] = 0.$$

# Equilibrium asset pricing: non-zero initial wealth

## Solving the problem using Dynamic Programming

- We define the value function at time $t$

$$V(A_t) = \max_{\{C_\tau\}_{\tau=t}^{T}} \mathbb{E}_t \left[ \sum_{\tau=t}^{T} \beta^{(\tau-t)} u(C_\tau) \right]$$

- If there is an optimal solution, $\{C_t^*\}_{t=0}^{T}$ for the agent's problem, it satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$V(A_t) = \max_{C_t} \{ u(C_t) + \beta \mathbb{E}_t [ V(A_{t+1}) ] \}$$

# Solving the household-investor problem using DP

- Assume there is only one asset: $K = 1$.
- In order to solve the problem:
  1. we derive the first order conditions, for an optimal consumption,
     $$u'(C_t) = \beta \mathbb{E}_t \left[ V'(A_{t+1}) \right]$$
     because $A_{t+1} = Y_t - C_t + R_t A_t$ and $\partial A_{t+1} / \partial C_t = -1$
  2. and we use the envelope condition (taking the derivative to $A_t$)
     $$V'(A_t) = \beta \, \mathbb{E}_t \left[ V'(A_{t+1}) R_t \right]$$

# Equilibrium asset pricing: non-zero initial wealth

- **Optimality condition for the household**: again

$$u'(C_t) = \mathbb{E}_t \left[ u'(C_{t+1}) R_{t+1} \right]$$

- Proof: using the law of iterated expectations, the envelopment theorem and the measurability properties of the variables, we have

$$\begin{aligned} u'(C_t) &= \beta\, \mathbb{E}_t \left[ V'(A_{t+1}) \right] \\ &= \beta\, \mathbb{E}_t \left[ \beta\, \mathbb{E}_{t+1} \left[ V'(A_{t+2}) R_{t+1} \right] \right] \\ &= \beta\, \mathbb{E}_t \left[ \beta\, \mathbb{E}_{t+1} \left[ V'(A_{t+2}) \right] R_{t+1} \right] \\ &= \beta\, \mathbb{E}_t \left[ u'(C_{t+1})\, R_{t+1} \right] \end{aligned}$$

- Now, for any number of assets. $K > 1$
- The first order conditions for an optimal consumption is the same

$$u'(C_t) = \beta \mathbb{E}_t \left[ V'(A_{t+1}) \right]$$

- but we have $A_t = A_t^1 + \ldots + A_t^K$ and $R_t^A A_t = R_t^1 A_t^1 + \ldots + R_t^K A_t^K$
- applying the envelope condition for any asset $j$

$$V'(A_t) \frac{\partial A_t}{\partial A_t^j} = \beta \mathbb{E}_t \left[ V'(A_{t+1}) \frac{\partial A_{t+1}}{\partial A_t^j} \right], \; j = 1, \ldots, K$$

- we get

$$V'(A_t) = \beta \mathbb{E}_t \left[ V'(A_{t+1}) R_t^j \right], \; j = 1, \ldots, K$$

- Then the arbitrage condition for the household choice of asset $j$ is

$$u'(C_t) = \beta \mathbb{E}_t \left[ u'(C_{t+1}) R^j_{t+1} \right], \ j = 1, \ldots, K$$

- or, equivalently

$$u'(C_t) S^j_t = \beta \mathbb{E}_t \left[ u'(C_{t+1}) ( V^j_{t+1} + S^j_{t+1}) \right], \ j = 1, \ldots, K, \ t = 0, \ldots, T$$

# Using stochastic optimal control

- We obtain a similar result using the stochastic Pontriyagin's principle
- Define the hamiltonian, for period $f$

$$H_t = u(C_t) + \Lambda_t \left( Y_t - C_t + R_t\, A_t \right)$$

where $\{\Lambda_t\}_{t=0}^{\infty}$ is an adapted $\mathcal{F}_t$ process, but $\lambda_t$ is conditional on the information at period $t$

- The optimality condition is, conditional on the information at $t$

$$\frac{\partial h_t}{\partial c_t} = 0 \iff u'(c_t) = \lambda_t$$

- The Euler equation is

$$\lambda_t = \beta\, \mathbb{E}_t\left[\frac{\partial H_{t+1}}{\partial A_{t+1}}\right] = \beta\, \mathbb{E}_t\left[\Lambda_{t+1}\, R_{t+1}\right]$$

- Then

$$u'(C_t) = \beta\, \mathbb{E}_t\left[u'(C_{t+1})\, R_{t+1}\right]$$