Foundations of Financial Economics *2020/21*
Problem set 2: two-period DGE

Paulo Brito

26.2.2021

Disclaimer: This problem set is provided as a help for self study, in an open academic spirit of providing potentially interesting problems. Solving them is not mandatory, however it is advisable because exams' questions will be in a large part similar to some of them. The instructor does not commit himself to provide the solutions of them all, but is available to help solving specific difficulties arising in efforts to **actually** solving them. Not all questions have been completely verified. However, the solutions that will be provided (and the questions arising in exams) will or have been carefully verified.

1 Intertemporal utility functions

1. Consider the following intertemporal utility functions (IUF)

$$U(c_0,c_1)=\ln(c_0)\ +\beta\ln(c_1),\ \text{for } 0<\beta<1, \tag{1}$$

$$U(c_0,c_1)=\frac{c_0^{1-\phi}-1}{1-\phi}\ +\beta\frac{c_1^{1-\phi}-1}{1-\phi},\ \text{for } 0<\beta<1,\ \phi>0, \tag{2}$$

$$U(c_0,c_1)=-\frac{e^{-\zeta c_0}}{\zeta}\ +\beta\frac{e^{-\zeta c_1}}{\zeta},\ \text{for } 0<\beta<1,\ \zeta>0, \tag{3}$$

$$U(c_0,c_1)=\ln(c_0-\bar{c})\ +\beta\ln(c_1-\bar{c}),\ \text{for } 0<\beta<1,\ \bar{c}>0 \tag{4}$$

$$U(c_0,c_1)=\frac{c_0^{1-\phi}-1}{1-\phi}\ +\frac{\beta}{1-\phi}\Big(\frac{c_1}{c_0}\Big)^{\zeta(1-\phi)},\ \text{for } 0<\beta<1,\ \zeta>0,\ \phi>0 \tag{5}$$

$$U(c_0,c_1)=\ln(c_0)\ +\beta\ln\left(\Big(\frac{c_1}{c_0}\Big)^{\zeta}\right),\ \text{for } 0<\beta<1,\ \zeta>0 \tag{6}$$

$$U(c_0,c_1)=\ln(c_0)\ +\beta\ln(c_1-\eta\,c_0),\ \text{for } 0<\beta<1,\ 0<\eta<1 \tag{7}$$

$$U(c_0,c_1)=(1-\beta)\ln(c_0)\ +\beta\ln(c_1),\ \text{for } 0<\beta<1, \tag{8}$$

$$U(c_0,c_1)=\Big((1-\beta)c_0^{\eta}\ +\beta c_1^{\eta}\Big)^{\frac{1}{\eta}},\ \text{for } 0<\beta<1,\ \eta>0 \tag{9}$$

$$U(c_0,c_1)=\ln(c_0)\ +\beta\begin{cases}\dfrac{c_1^{1-\phi}-1}{1-\phi}, & \text{if } 0<c_1<c_0,\ \text{for } 0<\beta<1,\ \phi>0\\ \ln(c_1), & \text{if } c_1\geq c_0,\end{cases}\tag{10}$$

For each utility function:

(a) Find the intertertemporal marginal rate of substitution, $IMRS_{0,1}$.

(b) Find the intertemporal elasticity of substitution, $IES_{0,1}$

(c) Discuss the implicit properties of the IUF concerning patience and intertemporal dependence. If needed, impose conditions for the existence of impatience and intertemporal substitution.

(b) Provide an economic intuition for your results.

2 Arrow-Debreu economy

1. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0,y_1\}$ and the intertemporal utility function is

$$U(c_0,c_1)=\log c_0+\beta\log c_1,\ 0<\beta<1$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Find the equilibrium AD price. Provide an intuition.

2. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0,y_1\}$ and the intertemporal utility function is

$$U(c_0,c_1)=\frac{c_0^{1-\theta}}{1-\theta}+\beta\frac{c_1^{1-\theta}}{1-\theta},\ 0<\beta<1,\ \theta>0$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions Solve the representative agent problem.

   c) Find the equilibrium AD price. Provide an intuition.

3. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0,y_1\}$ and the intertemporal utility function is

$$U(c_0,c_1)=-\frac{e^{-\zeta c_0}}{\zeta}+\beta\left(-\frac{e^{-\zeta c_1}}{\zeta}\right),\ 0<\beta<1,\ \zeta>0$$

a) Characterize the (static and dynamic) behavioral assumptions which are implicit in the intertemporal utility function.
b) Solve the representative agent problem. Provide an intuition.
c) Define the general equilibrium. Find the equilibrium AD price. Provide an intuition.

4. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the intertemporal utility function is

$$U(c_0, c_1) = \log(c_0 - \bar{c}) + \beta \log(c_1 - \bar{c}),\ 0 < \beta < 1,\ \bar{c} > 0$$

and the flow of endowment is $\{y_0, y_1\}$. Assume that the flow of endowment for any period is larger than the subsistence consumption $\bar{c}$, that is $\bar{c} < \min\{y_0, y_1\}$.

a) Characterize the (static and dynamic) behavioral assumptions which are implicit in the intertemporal utility function.
b) Solve the representative agent problem. Provide an intuition.
c) Define the general equilibrium. Find the equilibrium AD price. Provide an intuition.

Solution:

(a) static properties: the Bernoulli utility function is increasing and concave (but not Inada) Dynamic properties: $MRS(c, c) = \beta^{-1} > 1$ and $\varepsilon_{0,1} = 0$ impatience and intertemporally independent preferences, although the $IES_{0,1}(c, c) = 1 - \dfrac{\bar{c}}{c} \in (0, 1)$

(b) $c_0 - \bar{c} = \dfrac{h - (1+p)\bar{c}}{1+\beta}$, $c_1 - \bar{c} = \dfrac{h - (1+p)\bar{c}}{p(1+\beta)}$. Intuition the need satisfy the subsistence level may change the consumption as regards the $\log(c)$ case

(c) $p = \dfrac{y_0 - \bar{c}}{\beta(y_1 - \bar{c}}$.

5. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{(c_0 - \bar{c})^{1-\theta}}{1-\theta} + \beta \frac{(c_1 - \bar{c})^{1-\theta}}{1-\theta},\ 0 < \beta < 1$$

a) Specify the agent's problem. Define the general equilibrium.
b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.
c) Provide one condition for the existence of an equilibrium. Find the equilibrium AD price. Provide an intuition.

6. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \log\left(\left(\frac{c_1}{c_0}\right)^{\zeta}\right),\ 0 < \beta < 1$$

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

c) Find the equilibrium AD price. Provide an intuition.

7. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{c_0^{1-\theta}}{1-\theta} + \frac{\beta}{1-\theta}\frac{\left(\left(\frac{c_1}{c_0}\right)^{\zeta}\right)^{1-\theta}}{1-\theta},\ 0<\beta<1,\ \theta>0,\ \zeta>0.$$

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

c) Find the equilibrium AD price. Provide an intuition.

8. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \begin{cases} \log c_1 & \text{if } c_1 \geq c_0 \\ \dfrac{c_1^{1-\theta}}{1-\theta} & \text{if } 0 < c_1 < c_0 \end{cases}$$

for $0 < \beta < 1$ and $\theta > 1$.

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

c) Find the equilibrium AD price. Provide an intuition.

3 Finance economy

1. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \log c_1,\ 0<\beta<1$$

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

c) Find the equilibrium asset return. Provide an intuition.

2. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{c_0^{1-\theta}}{1-\theta} + \beta \frac{c_1^{1-\theta}}{1-\theta},\ 0 < \beta < 1,\ \theta > 0$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Find the equilibrium asset return. Provide an intuition.

3. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = -\frac{e^{-\zeta c_0}}{\zeta} + \beta \left(-\frac{e^{-\zeta c_1}}{\zeta} \right),\ 0 < \beta < 1,\ \zeta > 0$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Find the equilibrium asset return. Provide an intuition.

4. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log(c_0 - \bar{c}) + \beta \log(c_1 - \bar{c}),\ 0 < \beta < 1$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Provide one condition for the existence of an equilibrium. Find the equilibrium asset return. Provide an intuition.

5. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{(c_0 - \bar{c})^{1-\theta}}{1-\theta} + \beta \frac{(c_1 - \bar{c})^{1-\theta}}{1-\theta},\ 0 < \beta < 1.$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Provide one condition for the existence of an equilibrium. Find the equilibrium asset return. Provide an intuition.

6. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \log \left(\left(\frac{c_1}{c_0} \right)^{\zeta} \right),\ 0 < \beta < 1$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Find the equilibrium asset return. Provide an intuition.

7. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{c_0^{1-\theta}}{1-\theta} + \frac{\beta}{1-\theta} \frac{\left(\left(\frac{c_1}{c_0} \right)^{\zeta} \right)^{1-\theta}}{1-\theta},\ 0 < \beta < 1,\ \theta > 0,\ \zeta > 0.$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Find the equilibrium asset return. Provide an intuition.

8. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \begin{cases} \log c_1 & \text{if } c_1 \geq c_0 \\ \dfrac{c_1^{1-\theta}}{1-\theta} & \text{if } 0 < c_1 < c_0 \end{cases}$$

   for $0 < \beta < 1$ and $\theta > 1$.

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Find the equilibrium asset return. Provide an intuition.