Foundations of Financial Economics *2020/21*
Problem set 4 : Two-period Arrow-Debreu economy under uncertainty

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1. Consider a two-period Arrow-Debreu economy with the data that follows. Define the equilibrium, determine the solution for the consumer problem, and determine the equilibrium AD prices. Interpret the results:
   (a) assume a logaritmic utility function, $u(c) = \ln(c)$, 2, states of nature and generic probability and endowment distributions;
   (b) assume a quadratic utility function, $u(c) = ac - \frac{b}{2}c^2$, $a > 0$, 2, states of nature and generic probability and endowment distributions. Set conditions for the results to make sense;
   (c) assume an exponential utility function, $u(c) = -\frac{e^{-\lambda c}}{\lambda}$, $\lambda > 0$, 2, states of nature and generic probability and endowment distributions;
   (d) assume an isoelastic utility function, $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, $\theta > 0$, 2, states of nature and generic probability and endowment distributions;
   (e) assume a generic HARA utility function, 2, states of nature and generic probability and endowment distributions;
   (f) solve the same problems as before with N states of nature.

2. Assume the following economic environment: (1) there are N states of nature, with an uniform probability distribution, and (2) there is an endowment distribution for the period $t = 1$, $y_{1,s} = y_0\Gamma^{N/2-s}$, $s = 0, \dots, N$, for $0 < \Gamma < 1$. Consider Arrow-Debreu economies with the data that follows. Define the equilibrium, determine the solution for the consumer problem, and determine the equilibrium AD prices. Interpret the results:
   (a) assume a logaritmic utility function, $u(c) = \ln(c)$;
   (b) assume a quadratic utility function, $u(c) = ac - \frac{b}{2}c^2$, $a > 0$. Set conditions for the results to make sense;
   (c) assume an exponential utility function, $u(c) = -\frac{e^{-\lambda c}}{\lambda}$, $\lambda > 0$;

(d) assume an isoelastic utility function, $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, $\theta > 0$;

(e) assume an generic HARA utility function.

3. Consider Arrow-Debreu economies with the data that follows: (1) the information is given by a binomial tree with two periods two periods, $\mathbb{T} = \{0, 1\}$, with probabilities, for period 1, $\pi_s = \zeta \cdot (1+\zeta)^{-s}$ for state $s = 1, \ldots, \infty$, where $\zeta > 0$; (2) the endowment distribution for the period $t = 1$ is $y_{1,s} = y_0 \cdot (1+\zeta)^{-s/\theta}$, for state $s = 1, \ldots, \infty$ and $\theta > 0$; (3) agents are homogenous; (4) the representative agent has a discounted time-additive, von-Neumann-Morgenstern utility functional with a CRRA Bernoulli utility function, $u(C) = \frac{C^{1-\theta}-1}{1-\theta}$;

   (a) Define the equilibrium, and provide an intuition for it.

   (b) Determine the solution for the consumer problem, and provide an intuition for it.

   (c) Determine the equilibrium AD prices. Interpret the results you have obtained

4. Consider Arrow-Debreu economies with the data that follows: (1) the information is given by a binomial tree with two periods, $\mathbb{T} = \{0, 1\}$ and N states of nature for period 1; (2) the endowment distribution for the period $t = 1$ is $y_{1,s} = y_0 \cdot (1 + \gamma_s)$, for state $s = 1, \ldots, N$; (3) agents are homogenous; (4) the representative agent has a discounted time-additive, von-Neumann-Morgenstern utility functional with a CARA Bernoulli utility function,

$$u(C) = -\frac{e^{-\lambda C}}{\lambda}, \ \lambda > 0$$

   (a) Define the equilibrium, and provide an intuition.

   (b) Determine the solution for the consumer problem, and provide an intuition.

   (c) Determine the equilibrium stochastic discount factor. Assuming that $\mathbb{E}[\Gamma] = \gamma > 0$ find a bound to the expected value of the stochastic discount factor by using Jensen's inequality. Provide an intuition for your results.

Solution

(b)

$$\begin{aligned} c_0^* &= \frac{1}{1+\mathbb{E}[m]} \left(h_0 + \frac{1}{\lambda} \mathbb{E}[m \ln{(m/\beta)}] \right) \\ c_{1,s}^* &= c_0^* - \frac{1}{\lambda} \ln \left(\frac{m_s}{\beta} \right) \end{aligned}$$

(c) We find that the SDF is $m_s = \beta e^{-\lambda y_0 \gamma_s}$. This is a convex function of γ_s. Therefore using the Jensen's inequality for convex functions we have

$$\mathbb{E}[M] = \beta \mathbb{E}\left[e^{-\lambda y_0 \gamma}\right] > \beta e^{-\lambda y_0 \mathbb{E}[\Gamma]} = \beta e^{-\lambda y_0 \gamma}$$

5. Consider endowment economy in which the information be given by a two-period binomial tree, the endowment process, $\{y_0, Y_1\}$, verifies $y_0 = 1$ and $Y_1 = (1-\gamma, 1+\gamma)$ for $0 < \gamma < 1$, the intertemporal utility functional is time additive, discounted and von-Neumann-Morgenstern, with a linear Bernoulli utility function $u(c) = a\,c$, for $a > 0$ constant.

   (a) Define, explicitly, the Arrow-Debreu equilibrium for this economy.

   (b) Write the equilibrium conditions. Under which conditions an equilibrium exists ? Is it unique ? Justify.

   (c) Find the stochastic discount factor and provide an economic intuition for its value.

   Solution

   b) Equilibrium conditions

$$\begin{aligned} a &= \lambda \\ \beta a &= m_s \lambda, \; s = 1, 2 \\ c_0 + \mathbb{E}[MC_1] &= y_0 + \mathbb{E}[MY_1] \\ c_0 &= y_0 \\ C_1 &= Y_1 \end{aligned}$$

   existence conditions $m_1 = m_2 = \beta$; the equilibrium is unique.

   c) $m_1 = m_2 = \beta$: with neutral preferences and homogeneous agents the stochastic discount factor is state-independent (even though there is aggregate uncertainty)

6. Consider a two-period intertemporal utility function, in a stochastic setting, for the consumption sequence $\{c_0, C_1\}$ where $C_1 = (c_{11}, \ldots, c_{1s}, \ldots, c_{1n})$

$$U\big(c_0, c_1\big) = \Big((1-\mu)\, c_0^{\eta} + \mu\, \mathbb{CE}[C_1]^{\eta}\Big)^{\frac{1}{\eta}}$$

   for $0 < \mu < 1$ and $\eta \in (-\infty, \infty)$, where $\mathbb{CE}[C_1]$ is the certainty equivalent of $\mathbb{E}[\ln{(C_1)}]$.

   (a) Discuss the existence of risk aversion (Tip: compare $\mathbb{CE}[C_1]$ with $\mathbb{E}[C_1]$ for the cases in which C_1 is state independent and or it is state-dependent).

   (b) Assume a representative-agent Arrow-Debreu (AD) endowment economy, where the flow of endowment is $\{y_0, (1+\Gamma)\, y_0\}$, where $\Gamma = (\gamma_1, \ldots, \gamma_n)$ is state-dependent. Solve the representative agent problem. Discuss the response of the optimal consumption c_0 to changes in q_s.

   (c) Find the equilibrium stochastic discount factor, M^*. Find the covariance between M^* and $1 + \Gamma$. Which signs this covariance can display ? Do they depend on the behavioral parameters of the model ?

7. Assume a representative-agent Arrow-Debreu (AD) endowment economy, in a stochastic environment, where the flow of endowments is $\{y_0, (\mathbf{1}+\Gamma)y_0\}$ where $\mathbf{1}+\Gamma = (1+\gamma_1, \dots, 1+\gamma_s, \dots, 1+\gamma_n)$.

   (a) Find the dynamic stochastic general equilibrium, assuming that the representative consumer has the intertemporal utility functional
   $$U\big(c_0, c_1\big) = (1-\beta)\,\ln(c_0) + \beta\mathbb{E}\big[\ln(C_1)\big],\ \text{with } 0 < \beta < 1,$$
   over the consumption sequence $\{c_0, C_1\}$, where $C_1 = (c_{11}, \dots, c_{1s}, \dots, c_{1n})$,

   (b) Find the dynamic stochastic general equilibrium, assuming instead that the representative consumer has the intertemporal utility functional
   $$U\big(c_0, c_1\big) = (1-\beta)\,\ln(c_0) + \beta\ln\big(CE[C_1]\big),\ \text{with } 0 < \beta < 1.$$
   where $CE[C_1]$ is the certainty equivalent associated to the utility function $u(c_{1s}) = \dfrac{c_{1s}^{1-\sigma}-1}{1-\sigma}$, with $\sigma \geq 0$.

   (c) Compare the equilibrium stochastic discount factor (ESDF) you have derived in (a) with the one you have derived in (b), addressing specifically the cases in which $\sigma = 0$ and $\sigma > 0$. Provide an intuition.