

Foundations of Financial Economics *2020/21*  
Problem set 5: two-period APT

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1. Consider a finance economy in which (assets in the columns)

$$S = (1, s), \quad V = \begin{pmatrix} r + \epsilon & r(1 - \epsilon) \\ r - \epsilon & r(1 + \epsilon) \end{pmatrix}$$

where  $0 < r < 1$ ,  $s > 0$  and  $\epsilon$  can take any (real) value:

- (a) Under which conditions arbitrage opportunities exist ?
  - (b) Can we have absence of arbitrage opportunities and market incompleteness ?
  - (c) Compute the market price for an Arrow-Debreu contingent claim, under the assumption of absence of arbitrage opportunities. What is replicating transactions' strategy ? Explain.
2. Consider a finance economy in which

$$S = \left(1, \frac{1}{1+r}\right), \quad V = \begin{pmatrix} r + \epsilon & 1 \\ r - \epsilon & 1 \end{pmatrix}$$

where  $r > 0$  and  $\epsilon$  can take any (real) value.

- (a) Under which conditions arbitrage opportunities exist ?
  - (b) Compute the state prices, by assuming the appropriate existence conditions.
  - (c) Under the conditions that you imposed in previous point, can there be market completeness ? How ?
  - (d) Compute the market price for an Arrow-Debreu contingent claim, under the assumption of absence of arbitrage opportunities. What is replicating transactions' strategy ? Explain.
3. Assume an asset market represented by the pair of payoff prices  $(S, V)$  and consider the linear pricing relationship  $S = qV$ . Consider a simple case in which the number of assets is equal to the number of states of nature and both equal to 2:

- (a) determine an equivalent expression involving a stochastic discount factor and asset returns;
  - (b) determine a sufficient condition for the existence of arbitrage in terms of the asset returns;
  - (c) assume that there are no arbitrage opportunities, determine conditions for the existence of completeness in terms of the asset returns;
  - (d) assume that there are no arbitrage opportunities, determine conditions for the existence of incompleteness in terms of the asset returns;
  - (e) what are the consequences of the existence of incompleteness on the expected value of asset returns, variances and the covariance with the stochastic discount factor ?
4. Consider the following return matrices ( $K \times N$ )

$$\begin{pmatrix} 1.1 & 1.2 \\ 1.02 & 1.01 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 1.111 & 1.122 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 1.111 & 1.122 \\ 0.99 & 0.95 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 0.99 & 1.02 \end{pmatrix}.$$

Characterize the asset market as regards the existence of arbitrage opportunities and completeness

5. Consider two (alternative) financial markets,  $A$  and  $B$ , characterized by the return matrices ( $N \times K$ )

$$R = \begin{pmatrix} 1 & 1+a & b \\ 1 & 1-a & b \end{pmatrix}, \text{ for financial market } A, \text{ and } R = \begin{pmatrix} 1 & 1 \\ 1+a & 1-a \\ 1-b & 1+b \end{pmatrix}, \text{ for financial market } B$$

where  $a$  and  $b$  can take any real value.

- (a) Under which conditions there are no arbitrage opportunities and there is market completeness in the financial market  $A$
- (b) Under which conditions there are no arbitrage opportunities and there is completeness in the financial market  $B$

**Solution:**

- a) There is only a market (in the sense that there is a vector of prices of the state of nature if  $b = 1$ . In this case there are no arbitrage opportunities and there is market completeness and  $q = (1/2, 1/2)$ )
- b) The vector of prices of the states of nature is  $q = (q_1, q_2, q_3) = (x, b(1-x)/(a+b), a(1-x)/(a+b)$  for an arbitrary  $x$ . There is always market incompleteness. There are no arbitrage opportunities if  $x \in (0, 1)$  and  $\text{sign}(a) = \text{sign}(b)$

6. Assume there is a financial market with two assets, one risky asset with return  $1 + r$  and one riskless asset with return  $1 + i$ . Assume there are no arbitrage opportunities. Prove that the Sharpe index verifies

$$\left| \frac{E[r - i]}{\sigma[r]} \right| \leq \frac{\sigma[m]}{E[m]}$$

where  $m$  is the stochastic discount factor.

7. Assume there is a financial market with two assets, one risky asset and one riskless asset with prices and payoffs

$$S = \left( \frac{1}{1+i} \quad s \right), \quad V = \begin{pmatrix} 1 & d_1 \\ 1 & d_2 \end{pmatrix},$$

where  $i > 0$  and  $0 < d_1 < d_2$ . Introduce an european call option with exercise price  $d_1 < p < d_2$ . Prove that its price, if there are absence of arbitrage opportunities is

$$S_o = \frac{(r_1 - i)(d_2 - p)}{(1 + i)(r_1 - r_2)}$$

8. Consider a finance economy in which there are three assets with the vector of prices and payoff matrix given by

$$S = \left( p, 1, \frac{1}{1+r} \right), \quad V = \begin{pmatrix} r - \epsilon & r + \epsilon & 1 \\ r + \epsilon & r - \epsilon & 1 \end{pmatrix}$$

where we assume that  $r > 0$ ,  $p > 0$  and  $\epsilon \geq 0$ .

- Under which conditions there are no arbitrage opportunities ?
- Under the conditions that you imposed in the answer to the previous point, what would be the meaning of market completeness in this economy ? Determine the conditions for the existence of market completeness.
- Compute the price for the first asset,  $p$ , by building a replicating transactions' strategy. Explain.

**Solution:**

- There are no arbitrage opportunities if the following conditions hold:  $\epsilon > 1$  and  $\max\{0, (r - \epsilon)/(1 + r)\} < p < (r + \epsilon)/(1 + r)$ . This is the condition that guarantees  $q_s^{1,2} > 0$ ,  $q_s^{1,3} > 0$  and  $q_s^{2,3} > 0$ , for  $s = 1, 2$  where  $q_s^{i,j}$  is the price of state of nature  $s$  for the sub-market composed by assets  $i$  and  $j$ .
- Markets are complete if  $r > 1$  and  $p = (r - 1)/(1 + r)$ . Markets are complete if and only if the prices of the states of nature are unique:  $q_1 = q_1^{1,2} = q_1^{1,3} = q_1^{2,3}$  and  $q_2 = q_2^{1,2} = q_2^{1,3} = q_2^{2,3}$ .
- Let  $\theta_k$  be the number of asset  $k$  in a portfolio. The replicating portfolio is  $\theta_2 = -1$  and  $\theta_3 = 2r$  and the price of asset 1 is, again,  $p = (r - 1)/(1 + r)$ .

9. Consider a two-period finance economy in which the information is given by a binomial tree with objective probabilities  $(\pi_1, \pi_2)$ . The financial market is characterized by the return matrix in the (state  $\times$  asset) form

$$\begin{pmatrix} 1 & R_1 \\ 1 & R_2 \end{pmatrix}.$$

- (a) Provide conditions for the absence of arbitrage opportunities and for the existence of complete markets. Consider those conditions from now on.
- (b) Deduce the Sharpe index (tip: start by proving that the covariance between the two random variables  $X = (x_1, x_2)$  and  $Y = (y_1, y_2)$ , adapted to the previous binomial tree, is  $COV(X, Y) = \pi_1 \pi_2 (x_1 - x_2) (y_1 - y_2)$ ). What are the consequences of the conditions you have derived in (a) on the sign and magnitude of the Sharpe index.
- (c) Find a relationship between the objective and the risk-neutral probabilities such that the expected risk premium is non-negative. Explain.