Foundations of Financial Economics *2020/21*
Problem set 9

Paulo Brito

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1 Multiperiod, discrete time models

1.1 Arrow Debreu economies

1. Consider an endowment Arrow-Debreu economy in which there is uncertainty and an infinite number of periods. The endowments are given by an exogenous process $\{Y_t\}_{t=0}^{\infty}$ and the agents are homogeneous. Assume that the representative agent has the intertemporal utility functional
$$E_0\left[\sum_{t=0}^{\infty}\beta^t \ln\left(C_t\right)\right]$$
ă

   (a) Discuss the assumptions underlying the utility functional. Define and find the general equilibrium for this economy.

   (b) Find the recursive stochastic discount factor ($M_{t+1|t}$).

   (c) Assume that $\{Y_t\}_{t=0}^{\infty}$ is a martingale. Find $E_t[M_{t+1|t}]$. Justify and comment your result (hint: to avoid the problem posed by the Jensen inequality show that $\text{Cov}_t[M_{t+1|t}Y_{t+1}] = 0$).

   (d) Under which conditions we may get $V_t[M_{t+1|t}] = 0$?

2. Consider an endowment Arrow-Debreu economy in which there is uncertainty and an infinite number of periods. The endowments are given by an exogenous process $\{Y_t\}_{t=0}^{\infty}$ and the agents are homogeneous. Assume that the representative agent has the intertemporal utility functional
$$E_0\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{C_t^{1-\theta}-1}{1-\theta}\right)\right]$$
ă where $0 < \beta < 1$ and $\theta > 0$.

(a) Discuss the assumptions underlying the utility functional. Define and find the general equilibrium for this economy.

(b) Find the recursive stochastic discount factor ($M_{t+1|t}$).

(c) Assume that

$$Y_{t+1} = (1 + \gamma + \epsilon_{t+1})Y_t, \ E_t[\epsilon_{t+1}] = 0$$

Find $E_t[M_{t+1|t}]$ and a limiting value associated to the Jensen inequality. Justify.

3. Consider an endowment Arrow-Debreu economy in which there is uncertainty and an infinite number of periods. The endowments are given by an exogenous process $\{Y_t\}_{t=0}^{\infty}$ and the agents are homogeneous. Assume that the representative agent has the intertemporal utility functional

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(-\frac{e^{-\eta C_t}}{\eta} \right) \right]$$

ă where $\beta > 0$ and $\eta > 0$.

(a) Discuss the assumptions underlying the utility functional. Define and find the general equilibrium for this economy.

(b) Find the recursive stochastic discount factor ($M_{t+1|t}$).

(c) Assume that $Y_{t+1} = \epsilon_{t+1} + Y_t$ where ϵ_{t+1} follows a time-independent normal distribution $N(0, \sigma^2)$[1]. Find a closed form solution to $E_t[M_{t+1|t}]$. Provide a careful interpretation to your results.

Solution:

(a) The general values for consumption and for the stochastic discount factor are $C_t = Y_t$ and $M_t = \beta^t e^{-\eta(Y_t - Y_0)}$, for $t \in [0, \infty)$ where C_t, Y_t and M_t are $\mathcal{F}_t$-measurable

(b) $\mu^t_{t+1} = \beta e^{-\eta(Y_{t+1} - Y_t)}$.

(c) $E_t[\mu^t_{t+1}] = \beta e^{2(\eta\sigma/2)^2} > \beta$.

1.2 Equilibrium asset pricing

1. In a finance economy the representative consumer has an intertemporal additive utility functional as regards both time and the states of nature and a logarithmic Bernoulli utility function. From the first order conditions we find the following arbitrage condition for financial asset k

$$E_t[M_{t+1|t} R^k_{t+1}] = 1, \ k = 1, \dots, K,$$

for any period $t = 0, \dots, \infty$. The following notation is used: β is the psychological discount factor, R^k_t is the return for asset k and $M_{t+1|t}$ is the conditional stochastic discount factor (or pricing kernel).

[1] Remember that a property of $\epsilon \sim N(\mu, \sigma^2)$ is $E[e^{-a\epsilon}] = e^{-a\mu + a^2\sigma^2/2}$.

(a) Assume that $M_{t+1|t}$ follows a log-normal distribution [2] Find and expression for $\ln R^F_{t+1}$ for the riskless asset.

(b) Assume that, for any risky asset $k \neq F$, R^k_{t+1} follows a log-normal distribution and $M_{t+1|t}R^k_{t+1}$ follows a bivariate normal distribution [3]. Find an expression for $\ln E_t[R^k_{t+1}]$ for the risky asset .

(c) Find the risk premium as the ratio between the expected value of the risky asset as regards the riskless asset. Give an intuition for the previous result, by considering, in particular, different values for $\text{Cov}[\ln(C_{t+1}/C_t), \ln R^k_{t+1}]$.

(d) Empirical evidence on the risk premia, rates of return of riskless assets and their correlations with consumption and compare with the former results, when unconditional probabilities are used, two puzzles emerge: the equity premium puzzle and the risk free rate puzzle. What is the meaning of both puzzles ? Why are they mutually related ?

2. Consider a finance economy under uncertainty and an infinite number of periods. The intertemporal arbitrage condition for the consumer is

$$1 = \beta E_t \left[\frac{C_t R^j_{t+1}}{C_{t+1}} \right], \ t = 0, 1, \dots, \infty$$

ă where R^j_{t+1} is the rate of return for financial asset j at the beginning of period $t+1$, β is the psychological discount factor and C_t is consumption at period t.

(a) Give an intuition for that equation.

(b) Assuming that there are no speculative bubbles, find an equivalent condition in which the price of the asset j, S^j_t, is equal to the expected value of the present value of future payoffs, $\{V^j_s\}^{\infty}_{t=s}$. Justify

[2]If X follows a normal distribution, then $Y = \ln(X)$ follows a log-normal distribution. The following relationship holds: $\ln E[X] = E[Y] + 1/2V[Y]$.

[3]If X_1X_2 follows a bivariate normal distribution and $Y_i = \ln(X_i)$, for $i = 1, 2$ follow log-normal distributions, then $\ln E[X_1X_2] = E[Y_1] + E[Y_2] + 1/2\,(V[Y_1] + V[Y_2] + 2\text{Cov}[Y_1, Y_2])$.