Foundations of Financial Economics
DSGE: two-period Arrow-Debreu economy

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Topics

Two period Arrow-Debreu exchange economy

- Contracts and markets
- The household problem
- The dynamic stochastic general equilibrium (DSGE) for a general economy
- The dynamic stochastic general equilibrium (DSGE) for a representative agent economy (RAE)
- Characterizing the DSGE for the (RAE)

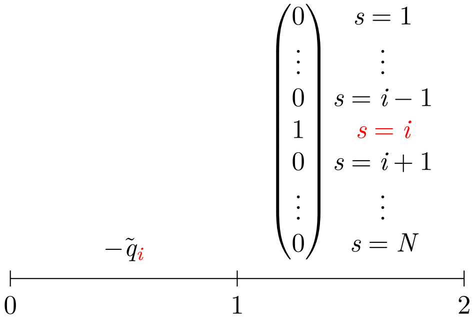
Contracts and markets

AD exchange economy: contracts

AD contract: is a **real forward contract** such that

- for a price associated to state $s = i$, $\tilde{q}_i$ paid at period $t = 0$
- there is delivery of a contingent good at period $t = 1$ at state $s = i$

$$x_{1,i} = \begin{cases} 1, & \text{if } s = i \\ 0, & \text{if } s \neq i \end{cases}$$



This allows to extend the static GE theory to the present intertemporal and stochastic economy context

AD exchange economy: markets

Existing markets:

- 1 spot market operating at period $t = 0$, where the price p_0 is set
- N markets for AD contracts operating at period $t = 0$, where the price vector $\tilde{Q}$ clears the market.

We can **characterize AD markets** by the payoff sequence $(\tilde{Q}, X_1)$ where

- prices are

$$\tilde{Q} = (\tilde{q}_1, \ldots, \tilde{q}_s, \ldots, \tilde{q}_N)$$

- and the deliveries are

$$X_1 = (x_{1,s})_{s=1}^{N} = \begin{pmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & 1 \end{pmatrix}$$

AD exchange economy: spot market

Transactions in the spot market:

the net demand: z_0.

then total expenditure: $p_0 z_0$

AD exchange economy: Arrow-Debreu markets

Transactions in every AD market:

- The number of contracts is

$$Z_1 = (z_{1,1}, \ldots, z_{1,s}, \ldots, z_{1,N})^\top$$

  where

  - if the agent is a buyer of the k-contract, then $z_{1,k} > 0$, and
    - **pays** $\tilde{q}_k z_k$ at $t = 0$
    - **receives** z_k units of the good at $t = 1$ if the state k occurs and 0 otherwise
  - if the agent is a seller of the l-contract, then $z_{1,l} < 0$, and
    - **receives** $\tilde{q}_l z_l$ at $t = 0$ and
    - **delivers** z_l units of the good at $t = 1$ if the state l occurs and 0 otherwise

- Then total net expenditure in all AD markets is

$$\tilde{Q}.Z_1 = \sum_{s=1}^{N} \tilde{q}_s z_{1,s}$$

Household problem

AD exchange economy: consumption financing

- Household i receives a sequence of endowments

$$\{Y^i\} = \{y_0^i, Y_1^i\}$$

- Which finance the (random) sequence of consumption, $\{C^i\} = \{c_0^i, C_1^i\}$, out of his endowment, such that
  - in the period $t = 0$

$$c_0^i = z_0^i + y_0^i$$

  - in period $t = 1$, contingent on the information available and contracts done at time $t = 0$

$$C_1^i = Z_1^i + Y_1^i = \begin{pmatrix} c_{1,1}^i \\ \dots \\ c_{1,s}^i \\ \dots \\ c_{1,N}^i \end{pmatrix} = \begin{pmatrix} z_1^i \\ \dots \\ z_s^i \\ \dots \\ z_N^i \end{pmatrix} + \begin{pmatrix} y_{1,1}^i \\ \dots \\ y_{1,s}^i \\ \dots \\ y_{1,N}^i \end{pmatrix}$$

AD exchange economy: consumer's budget constraint

As

$$\begin{cases} c_0^i - y_0^i = z_0^i, & \text{for } t = 0 \\ c_{1,s}^i - y_{1,s}^i = z_{1,s}^i, & \text{for } t = 1, \text{ for every } s = 1, \ldots, N \end{cases}$$

i.e. for every period and for any state of nature total income is equal to total expenditure

then the budget constraint at time $t = 0$ (i.e., in the beginning of period 0) is

$$p_0 \left(c_0^i - y_0^i \right) + \tilde{Q}.(C_1^i - Y_1^i) = p_0 \left(c_0^i - y_0^i \right) + \sum_{s=1}^{N} \tilde{q}_s \left(c_{1,s}^i - y_{1,s}^i \right) = 0$$

AD exchange economy: stochastic discount factor

We define:

- the relative price of AD contracts also called the price of the state of nature
$$Q^\top = \Big(q_1, \dots, q_s, \dots, q_N\Big)$$
where
$$q_s \equiv \frac{\tilde{q}_s}{p_0},\ \ s = 1, \dots, N.$$

- the stochastic discount factor is
$$M^\top = \Big(m_1, \dots, m_s, \dots, m_N\Big)$$
where
$$m_s \equiv \frac{q_s}{\pi_s},\ \ s = 1, \dots, N.$$

AD exchange economy: household's problem

Choose a **contingent plan** $\{C^i\} = \{c_0^i, C_1^i\}$:

- that maximizes the **intertemporal utility** functional

$$U^i(\{C^i\}) = U^i\Big(c_0^i, C_1^i\Big) = U^i\Big(c_0^i, \big(c_{1,1}^i, \dots, c_{1,N}^i\big)\Big)$$

- subject to the **intertemporal (instantaneous) budget constraint**

$$c_0^i + \sum_{s=1}^{N} q_s\, c_s^i = y_0^i + \sum_{s=1}^{N} q_s\, y_s^i$$

- given: the AD prices and endowments $(Q, \{Y^i\})$,

  We define the **wealth of the consumer** by the value of the endowments at $t = 0$

$$h_0^i \equiv y_0^i + \sum_{s=1}^{N} q_s\, y_s^i$$

AD exchange economy: household's problem

- Formally the problem is

$$\boxed{\begin{aligned} &\max_{c_0^i,\, C_1^i} U^i\left(c_0^i, C_1^i\right) \\ &\qquad \text{subject to} \\ &\qquad c_0^i + Q \cdot C^1 = h_0^i \end{aligned}}$$

- Particular case: If the utility functional is vNM we have

$$\boxed{\begin{aligned} &\max_{c_0^i,\, C_1^i} U^i\left(c_0^i, C_1^i\right) = u^i(c_0^i) + \beta\, \mathbb{E}^i[u^i(C_1^i)] \\ &\qquad \text{subject to} \\ &\qquad c_0^i + Q \cdot C^1 = h_0^i \end{aligned}}$$

- We consider potential idiosyncratic differences in wealth (h^i), information ($\mathbb{E}^i$), in patience (β^i) and in aversion to risk (u^i)

DSGE: general definition

AD exchange economy: general equilibrium

Definition 1

*The DSGE for an endowment AD economy is **defined** by the sequence of distribution of consumption over time and across agents,* $(C^{i*})_{i=1}^{I}$, *where* $(C^{i*})_{i=1}^{I} = \left(\{c_0^{i*}, C_1^{i*}\}\right)_{i=1}^{I}$, *and by the AD prices,* Q^*, *given a distribution of endowments* $\left(\{y_0^i, Y_1^i\}\right)_{i=1}^{I}$, *such that:*

- *every consumer* $i \in \mathcal{I}$ *determines the optimal sequence of consumption, taking* Y^i *and* Q *as given*

$$\{C^{*i}\} = arg\ max\left\{U^i(c_0^i, C_1^i)\ s.t.\ c_0^i + Q \cdot C_1^i \leq h_0^i\right\}$$

- *and markets clear:*

$$\sum_{i=1}^{I} c_0^i = \sum_{i=1}^{I} y_0^i,$$

$$\sum_{i=1}^{I} c_{1,s}^i = \sum_{i=1}^{I} y_{1,s}^i, for\ each\ s = 1, \ldots, N$$

DSGE: representative agent economy

AD exchange and homogeneous economy: general equilibrium

Assume agents are homogeneous: same preferences, same information, same endowments

AD exchange and homogeneous economy: general equilibrium

Definition 2

The DSGE for representative agent exchange AD economy is ***defined*** *by the sequence of consumption and prices* $(\{c_0^*, C_1^*\}, Q^*)$ *such that:*

- *the representative consumer determines the optimal sequence*

$$C^* = arg\ max\{U(c_0, C_1)\ s.t.\ c_0 + Q \cdot C_1 = h_0\}$$

  given $Y = \{Y_0, Y_1\}$ *and* Q*,*

- *markets clear*

$$c_0^* = y_0,$$
$$C_1^* = Y_1$$

  or, equivalently

$$c_{t,s}^* = y_{t,s},\ for\ each\ t = 0, 1,\ for\ each\ s = 1, \ldots, N$$

AD exchange and homogeneous economy: general equilibrium

Assume:

- agents are homogeneous: same preferences, same information, same endowments
- agents are characterized by a von-Neumann Morgenstern additive intertemporal utility functional

AD exchange and homogeneous economy: general equilibrium

Definition 3

The DSGE for representative agent exchange AD economy is ***defined*** *is* ***defined*** *by the sequence of consumption and prices* $(\{c_0^*, C_1^*\}, Q^*)$ *such that:*

- *the representative consumer determines the optimal sequence*

$$C^* = arg\ max\{\mathbb{E}_0\left[u(C_0) + \beta u(C_1)\right] s.t. \mathbb{E}_0\left[C_0 + mC_1\right] \leq h_0\}$$

  given $Y = \{Y_0, Y_1\}$ *and* M,

- *markets clear*

$$c_0^* = y_0, \ldots, C_1^* = Y_1$$

  or, equivalently

$$c_{t,s}^* = y_{t,s},\ t = 0, 1,\ s = 1, \ldots, N$$

AD general equilibria: intuition

- Allows for the determination:
  - of the Arrow-Debreu price $Q = (q_1, \dots q_N)$: market price for transactions across time and the states of nature
  - or the stochastic discount factor $M = (m_1, \dots m_N)$: defined as $m_s = \frac{q_s}{\pi_s}$
- In the types of economy
  - Heterogeneous agent economy: dependent upon the preferences, information and the endowments of the economy **and their distribution among agents** (i.e, when there are differences in information, attitudes towards risk and wealth)
  - Homogeneous (representative) agent economy: dependent upon the preferences, information and the endowments of the economy

DSGE RAE: determination

Determination of equilibrium prices

- Assume a benchmark utility functional

  We determine the equilibrium in two steps:

  1. first, determine the optimality conditions

     $$u^{'}(c_0^*)\,q_s = \beta u^{'}(c_{1,s}^*),\ s = 1, \ldots, N$$

     if we assume there is no satiation $u'(c) > 0$;
  2. second, use the market equilibrium conditions

     $$c_{t,s}^* = y_{t,s},\ t = 0, 1,\ s = 1, \ldots, N$$

  or equivalently, the **equilibrium AD price** is

  $$\boxed{q_s^* = \beta \pi_s \left(\frac{u^{'}(y_{1,s})}{u^{'}(y_0)} \right),\ s = 1, \ldots, N}$$

  or, the **equilibrium** stochastic discount factor is

  $$\boxed{m_s^* = \beta \left(\frac{u^{'}(y_{1,s})}{u^{'}(y_0)} \right),\ s = 1, \ldots, N}$$

DSGE RAE: characterization

AD exchange and homogeneous economy

Proposition 1

Assume an endowment homogenous Arrow-Debreu economy in which the utility functional is a time additive von-Neumann Morgenstern utility functional. Then the DGSE is the sequence of consumption $\{c_0^*,\ C_1^*\}$ *and the AD price* Q^* *such that*

$$
\begin{aligned}
c_0^* &= y_0 \textit{ for period } t = 0 \\
c_{1,s}^* &= y_{1,s} \textit{ for period } t = 1, \textit{ and for state } s \in \{1, \ldots, N\} \\
q_s^* &= \beta\, \pi_s \left(\frac{u'(y_{1,s})}{u'(y_0)} \right), \textit{ for } s \in \{1, \ldots, N\}
\end{aligned}
$$

AD exchange and homogeneous economy

Equilibrium consumption

Then the general equilibrium when consumers are homogeneous and there is no satiation :

- consumption is equal to the case in an autarkic economy

$$\{C_t^*\}_{t=0}^1 = \{Y_t\}_{t=0}^1$$

- there is **aggregate uncertainty**: because the endowment Y_1 is stochastic;
- there is **no insurance**: because, in equilibrium, $C_1^* = Y_1$ consumption is stochastic (same distribution of consumption and of endowments)

AD exchange and homogeneous economy

Equilibrium AD price

- The equilibrium relative price for AD contracts is also stochastic

$$Q^* = \left(\beta\pi_1 \left(\frac{u'(y_{1,1})}{u'(y_0)} \right), \ldots, \beta\pi_N \left(\frac{u'(y_{1,N})}{u'(y_0)} \right) \right)^\top$$

  is a function of the **fundamentals** (resources, preferences and information)

- as $q_s^*(y_0, Y_1)$ if the $u(\cdot)$ is concave

$$\frac{\partial q_s^*}{\partial y_0} > 0,\ \frac{\partial q_s^*}{\partial y_{1,s}} < 0,\ \frac{\partial q_s^*}{\partial y_{1,s'}} = 0$$

  increases with y_0, decreases with $y_{1,s}$ and is neutral for $y_{1,s'}$ (no response to the whole distribution)

- and also

$$\frac{\partial q_s^*}{\partial \beta} > 0,\ \frac{\partial q_s^*}{\partial \pi_s} > 0,\ \frac{\partial q_s^*}{\partial \pi_{s'}} = 0$$

  decreases with patience, increases with the probability of the own state but is neutral to the probabilities of the other states

AD exchange and homogeneous economy

Equilibrium AD price

- The equilibrium stochastic discount factor (SDF)

$$M^* = \left(\beta \left(\frac{u^{'}(y_{1,1})}{u^{'}(y_0)} \right), \ldots, \beta \left(\frac{u^{'}(y_{1,N})}{u^{'}(y_0)} \right) \right)^\top$$

  which is again a function of the **fundamentals** (resources and preferences)

- has the same characterization, but is independent from π_s

$$m^*_s = m^*_s\big(\overset{+}{\beta}, \overset{+}{y_0}, \overset{0}{y_{1,1}}, \ldots, \overset{-}{y_{1,s}}, \ldots, \overset{0}{y_{1,N}} \big)$$

- Interpretation: sign $+$ increases in net demand for future consumption; sign $-$ increase in net future supply; 0 consequence of the independence between states of nature assumption in the vNM utility functional $U(c_0, C_1)$

An example with log utility

SDF for state s

Assuming:

- logarithmic Bernoulli utility function

$$u(c) = \ln(c)$$

- stochastic endowment's growth factor

$$y_{1,s} = (1+\gamma_s)y_0,\ s = 1, \ldots, N$$

- How does uncertainty affects the stochastic discount factor and the utility of the consumer ?

An example with log utility

Distribution of the SDF

- the stochastic discount factor is $m^*_s = \frac{\beta}{1+\gamma_s}$

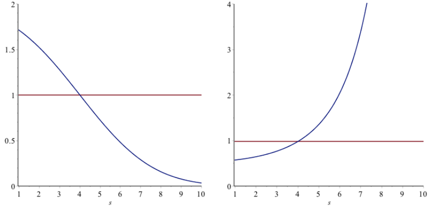


Figure: Growth factor $(1+\Gamma)$ and stochastic the associated discount factor M

- Conclusions:
  1. there is **aggregate uncertainty**
  2. stochastic discount factor is **negatively correlated** with rate of growth

An example with log utility

Sampling the SDF

- the stochastic discount factor is

$$m_s^* = \frac{\beta}{1+\gamma_s}$$

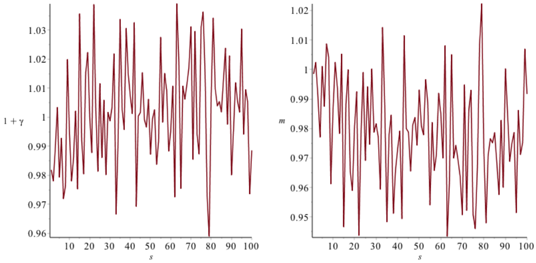


Figure: Sampling from $\gamma \sim N(0, 0.02)$ and the stochastic discount factor

An example with log utility

Aggregate uncertainty and lack of insurance

- The utility for the consumer is (prove it)

$$
\begin{aligned}
U(C^*) &= \ln(c_0^*) + \beta \mathbb{E}_0[\ln(C_1^*)] = \\
&= \ln(y_0) + \beta \mathbb{E}_0[\ln(Y_1)] = \\
&= \ln\left(y_0^{1+\beta}\left(G\mathbb{E}_0[1+\Gamma]\right)^{\beta}\right)
\end{aligned}
$$

  increases with y_0 and with the geometric mean of the growth rate.

- Question: why this looks like the utility in a Robinson-Crusoe economy ?
- Question: what are the consequences of more volatility, to the stochastic discount factor and to consumer's utility ?

References

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