Foundations of Financial Economics
Two-period DSGE: introduction

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Topics

Two period General Equilibrium pricing of intertemporal contracts:
to set up a model we need assumptions regarding:

- The economic environment: information tree, real part of the economy
- The market environment: available contracts
- The variables defining the general equilibrium depend on those two categories.

We will study two models: Arrow-Debreu economy and Finance (or Radner) economy

Environments and general equilibrium

Common assumptions: regarding the economic environment

1. the time-information structure;
2. the real part of the economy: intertemporal preferences and availability of resources

Different assumptions regarding the market environment

1. simultaneous markets' opening (Arrow-Debreu economy);
2. sequential markets' opening (Finance economy);

Lead to **different definitions of GE** (general equilibrium) (that may be **equivalent or not**)

The time-information tree

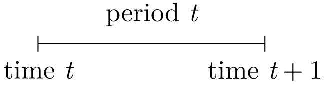
The time-information tree

This refers

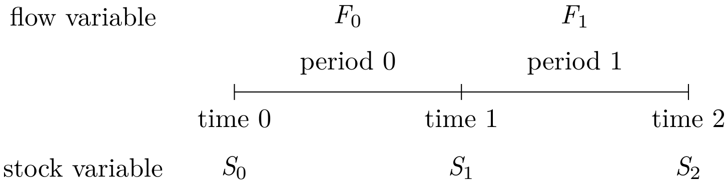
- to the moments in which markets open
- to the timing of the decisions
- the information agents have

In discrete time we have to distinguish between

- time: the timing for **stocks** and prices of stocks
- periods: the timing for **flows** and prices of flows



Two period: The timing for flow and stock variables



Flow and stock variables: refer to prices and/or quantities

For flow variables

We assume:

- $t \in \mathbb{T} = \{0, 1\}$ where $\mathbb{T}$ refer to periods
- information changes along time, from the perspective of period $t = 0$.

Most variables are 2-period random sequences

$$X = \{X_0, X_1\}$$

are determined on the basis of the information known at period $t = 0$:

- at period $t = 0$, they are observed

$$X_0 = x_0$$

- for period $t = 1$, they are contingent on the information available at period $t = 0$

$$X_1(\omega),\ \omega \in (\Omega, \mathcal{F}, \mathbb{P})$$

X_1 is a random variable

Information for a flow variable

The information at period $t = 0$ is:

- If Ω is discrete and there are N elementary events, the information regarding period $t = 1$ we have

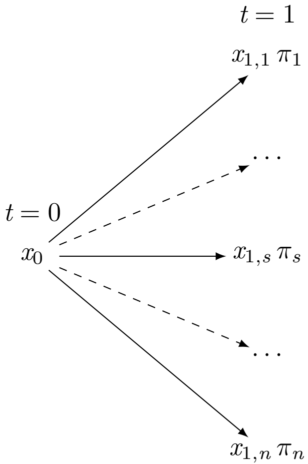
$$X_1 = (x_{1,1}, \ldots, x_{1,s}, \ldots, x_{1,N})^\top$$
$$P_1 = (\pi_1, \ldots, \pi_s, \ldots, \pi_N)^\top$$

  where $x_{1,s}$ is the **outcome** if event s realizes and π_s its probability

- and the sequences of possible outcomes and related probabilities are

information	$x_0, 1$	$\begin{pmatrix} x_{1,1} \\ \ldots \\ x_{1,s} \\ \ldots \\ x_{1,N} \end{pmatrix}, \begin{pmatrix} \pi_1 \\ \ldots \\ \pi_s \\ \ldots \\ \pi_N \end{pmatrix}$
periods	0	1

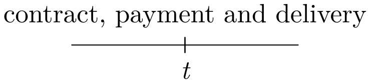
The time-information tree



Timing of contracts: for stocks

We distinguish:

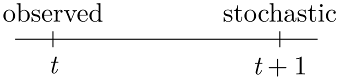
- **spot** contracts: contract, delivery and payment done in the same period



- **intertemporal or forward** contracts: contract and payment in one period, delivery in a future period

They differ along two dimensions:

- the **timing** (which may be relevant if there is , v.g., impatience, depreciation)
- the **information** set associated to the several actions (and prices) involved



Timing of contracts: for flows

- **spot contracts**

contract, payment and delivery

t | $t+1$

- **forward contracts**

contract and payment | delivery

t | $t+1$

- **information**

observed | stochastic

t | $t+1$

The real part of the economy

The real part of the economy

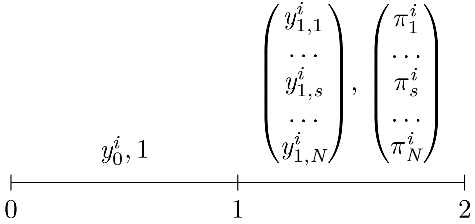
Refers to:

- technology: the type of availability of resources
  - **exchange** economies: the availability of the resources is independent of decisions throughout time,
  - **production** economies: availability of resources is dependent on decisions in previous periods
- preferences: choice among random sequences of cconsumption
- distribution of agents: they can be homogenous or heterogenous regarding
  - endowments or technology
  - preferences
  - information

Technology

If we consider a flow of resources for agent i:

- The resource for agent i is a process $\{Y^i\} = \{y_0^i, Y_1^i\}$ where $y_{t,s}^i$ is the endowment of agent i at time t for the state of nature s, with possible realizations and probabilities



- in an exchange economy

$$Y_1^i \text{ independent of } y_0^i$$

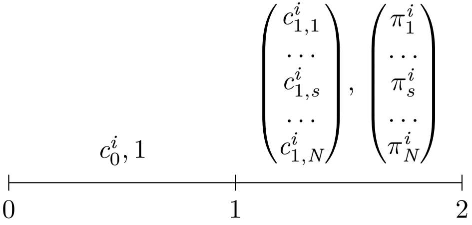
- in a production economy

$$Y_1^i = F_1^i(y_0^i) \text{ dependent on } y_0^i$$

Preferences

Agent i **chooses** among:

- Sequences of consumption $\{C^i\} = \{c_0^i, C_1^i\}$ is the consumption flow for agent i



  where the probabilities can be objective or subjective, exogenous or endogenous, homogeneous or heterogeneous

- Evaluated by an **intertemporal utility functional**

$$U^i(\{C^i\}) = U^i\big(c_0^i, C_1^i\big)$$

Preferences

We can calculate several marginal utilities

- marginal utility for a change of consumption at period $t = 0$

$$U_0 = \frac{\partial U(\{C\})}{\partial c_0}$$

- marginal utility for a change of consumption at period $t = 1$ for state of nature s

$$U_1(s) = \frac{\partial U(\{C\})}{\partial c_{1,s}}, \text{ for } s \in \{1, \dots, N\}$$

- the intertemporal marginal rate of substitution is a random variable

$$IMRS_{0,1}(s) = \frac{U_0}{U_1(s)}, \text{ for } s \in \{1, \dots, N\}$$

Preferences

The Allen-Uzawa elasticities over time:

- "own elasticities" for period $t = 0$

$$\varepsilon_0 = -\frac{\frac{\partial U_0}{\partial c_0}}{U_0}\, c_0,$$

- crossed intertemporal elasticities

$$\varepsilon_{0,1}(s) = -\frac{\frac{\partial U_0}{\partial c_{1,s}}}{U_0}\, c_{1,s}, \text{ for } s \in \{1, \ldots, N\}$$

- the elasticity of intertemporal substitution is also a random variable

$$IES_{0,1}(s) = \frac{c_0\, U_0 + c_{1,s}\, U_1(s)}{c_{1,s}\, U_1(s)\, \varepsilon_0 - 2c_0\, U_0\, \varepsilon_{0,1}(s) + c_0\, U_0\, \varepsilon_1(s,s)}, \text{ for } s \in \{1, \ldots, N\}$$

Preferences

The Allen-Uzawa elasticities over states of nature:

- the AU own-state elasticity for $t = 1$

$$\varepsilon_1(s,s) = -\frac{\frac{\partial U_1(s)}{\partial c_{1,s}}}{U_1(s)}\, c_{1,s},\ \text{for } s \in \{1,\dots,N\}$$

- is usually equal to the **coefficient of relative risk aversion**

$$\varepsilon_1(s,s) = \varrho_r(s)$$

- the AU inter-state elasticity for period $t = 1$

$$\varepsilon_1(s,s') = -\frac{\frac{\partial U_1(s)}{\partial c_{1,s'}}}{U_1(s)}\, c_{1,s'},\ \text{for } s \neq s' \in \{1,\dots,N\}$$

- displays **independence** ($\varepsilon_1(s,s') = 0$) or not ($\varepsilon_1(s,s') \neq 0$) among states of nature

Benchmark preferences: von-Neumman Morgenstern

- The most common utility functional is the discounted time-additive von-Neumann Morgenstern functional

$$U(\{C^i\}) = u^i(c^i_0) + \beta^i \mathbb{E}^i[u^i(C^i_1)] = u^i(c^i_0) + \beta^i \sum_{s=1}^{N} \pi^i_s u^i(c^i_{1,s})]$$

  where $0 \leq \pi_s \leq 1$ and $\sum_{s=1}^{N} \pi^i_s = 1$;
- or, equivalently

$$U(\{C^i\}) = \mathbb{E}^i_0 \left[\sum_{t=0}^{t=1} (\beta^i)^t u^i(c^i_{t,s}) \right]$$

- Observations
  - the utility functional $U(.)$ is doubly additive: linear as regards both time and the states of nature;
  - probabilities may be objective or subjective
  - particular relationship between the intertemporal and the risk aversion properties

Benchmark preferences: von-Neumann Morgenstern

- Write it as $U(c_0, C_1) = u(c_0) + \beta \sum_{s=1}^{N} \pi_s u(c_{1,s})$
- Then the marginal utilities are

$$U_0 = u'(c_0) \text{ and } U_1(s) = \beta\, \pi_s\, u'\big(c_{1,s}\big), \text{ for } s \in \{1, \ldots, N\}$$

- The intertemporal marginal rate of substitution is state-dependent (random variable)

$$IMRS_{0,1}(s) = \frac{u^{'}(c_0)}{\beta\, \pi_s\, u^{'}(c_{1,s})}, \text{ for } s \in \{1, \ldots, N\}$$

Benchmark preferences: von-Neumann Morgenstern

The Allen-Uzawa elasticities are

- For period $t = 0$

$$\varepsilon_0 = -\frac{u^{''}(c_0)}{u^{'}(c_0)}\, c_0,$$

  if ε_0: decreasing marginal utility

- but the intertemporal elasticities are equal to zero

$$\varepsilon_{0,1}(s) = 0, \text{ for all } s \in \{1, \ldots, N\}$$

  (because of the separability between c_0 and C_1)

- Therefore, the elasticity of intertemporal substitution

$$IES_{0,1}(s) = \frac{c_0 u^{'}(c_0) + \beta\pi_s u^{'}(c_{1,s})}{\beta\pi_s u^{'}(c_{1,s}) c_{1,s}\, \varepsilon_0 + c_0 u^{'}(c_0)\, \varepsilon_1(s)}$$

  is also a random variable but has not intertemporal substitution effects

Benchmark preferences: von-Neumann Morgenstern

- AU for state s

$$\varepsilon_1(s) = -\frac{u^{''}(c_{1,s})}{u^{'}(c_{1,s})}\, c_{1,s},\ s = 1, \ldots N$$

- if $\varepsilon_1(s) > 0$ displays risk aversion
- The AU elasticities between states of nature are also equal to zero

$$\varepsilon_1(s, s') = 0, \text{for all } s \neq s' \in \{1, \ldots, N\}$$

- this means that the preferences regarding different states of nature are **independent**

Benchmark preferences: von-Neumann Morgenstern

- If we assume a **CRRA** (constant relative risk aversion) utility function

$$u(c) = \frac{c^{1-\zeta} - 1}{1-\zeta} \Rightarrow u'(c) = c^{-\zeta},\ u''(c) = -\zeta\, c^{-\zeta-1}$$

- where the coefficient of relative risk aversion ϱ_r is constant (i.e., state independent)

$$\varrho_r = -\frac{u^{''}(c)}{u^{'}(c)}\, c = \zeta > 0$$

- then the AU elasticities (own, intertemporal, and inter-state) are

$$\begin{aligned} \varepsilon_0 &= \zeta \\ \varepsilon_{0,1}(s) &= 0,\ \text{for } s \in \{1, \dots, n\} \\ \varepsilon_1(s) &= \zeta = \varrho_r \\ \varepsilon_1(s, s') &= 0 \text{ for } s, s' \in \{1, \dots, n\} \end{aligned}$$

are all state-independent.

Benchmark preferences: von-Neumann Morgenstern

- The elasticity of intertemporal substitution

$$IES_{0,1}(s) = \frac{1}{\zeta}$$

  is:
  (1) **state independent**
  (2) is equal to the inverse of the CRRA

- This means that the we **cannot distinguish the intertemporal and the stochastic properties of preferences**
- This is counterfactual (see Thimme (2017)): empirically impatience can be distinguished from attitudes toward risk

Epstein-Zin preferences

- Are becoming popular among macroeconomists
- They distinguish between the intertemporal preferences and risk aversion by parameterizing them with different parameters
- Most models are multi-period

Epstein-Zin preferences

- A two period version of EZ preferences
- Let $U(c_0, C_1)$ be the intertemporal utility functional
- There is an aggregator $V(c_0, C_1) = u^{-1}\big(U(c_0, C_1)\big)$

$$V(c_0, C_1) = (1-\beta)u(c_0) + \beta u(c_1^c)$$

  where c_1^c is the **certainty equivalent of consumption at period** $t = 1$:

  - intertemporal preferences are represented by $u(c)$, which is increasing and concave $u^{''}(c) < 0 < u^{'}(c)$
  - choice among states of nature is represented by

$$v(c_1^c) = \mathbb{E}[v(C_1)]$$

    is a utility function displaying risk aversion

Epstein-Zin preferences

- Therefore

$$V(c_0, C_1) = (1-\beta)u(c_0) + \beta u\left(v^{-1}\Big(\mathbb{E}[v(C_1)]\Big)\right)$$

Example

- Assume:

$$u(c) = c^{1-\zeta}$$
$$v(c) = \ln(c) \Leftrightarrow c = v^{-1}(v) = e^{v}$$

- (1) build the aggregator $V(c_0, C_1)$
- utility at $t = 0$ is $u(c_0) = c_0^{1-\zeta}$
- utility at $t = 1$ is

$$\mathbb{E}[v(C_1)] = \mathbb{E}[\ln(C_1)] \to v^{-1}\Big(\mathbb{E}[v(C_1)]\Big) = e^{\mathbb{E}[\ln(C_1)]}$$
$$\to u\left(v^{-1}\Big(\mathbb{E}[v(C_1)]\Big)\right) = \left(e^{\mathbb{E}[\ln(C_1)]}\right)^{1-\zeta}$$

- the aggregator is

$$V(\{C\}) = (1-\beta)\, c_0^{1-\zeta} + \beta\, e^{(1-\zeta)\mathbb{E}[\ln(C_1)]}$$

Example

- then

$$U(c_0, C_1) = \left[(1-\beta)\, c_0^{1-\zeta} + \beta\, e^{(1-\zeta)\mathbb{E}[\ln{(C_1)}]}\right]^{1-\zeta}$$

Distribution of agents

Distribution

- The idiosyncratic components defining a consumer are:
  - endowments (Y^i)
  - preferences (β^i, u^i) (impatience, risk aversion)
  - information $\mathbb{P}^i$ (only make sense with subjective probabilities)
- Agents can be homogeneous or heterogeneous regarding one or all of the previous variables and parameters

  in a homogeneous, or **representative agent** economy: endowments, preferences and information are equal, i.e, $Y^1 = Y^I = Y$, etc

  in a heterogeneous economy: **agents differ** in at least one of the three dimensions: endowments ($Y^i \neq Y^j$), preferences ($\beta^i \neq \beta^j$ or $u^i(.) \neq u^j(.)$), or information ($\mathbb{P}^i \neq \mathbb{P}^j$)

The market structure

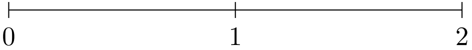
Autarky versus trade economies

The economies are distinguished by the exchanges that agents can make.

- In autarky we will have

$$c^i_{t,s} = y^i_{t,s},\ t = 0, 1,\ s = 1, \ldots, N$$

$$c^i_0 = y^i_0 \qquad \begin{pmatrix} c^i_{1,1} \\ \ldots \\ c^i_{1,s} \\ \ldots \\ c^i_{1,N} \end{pmatrix} = \begin{pmatrix} y^i_{1,1} \\ \ldots \\ y^i_{1,s} \\ \ldots \\ y^i_{1,N} \end{pmatrix}$$

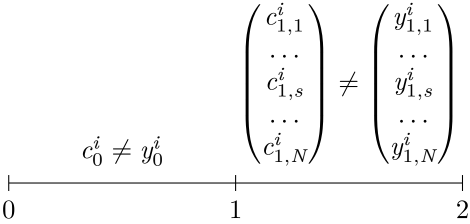


Autarky versus trade economies

- If there are markets for intertemporal transfers of contingent goods, agents can trade and be able to make

$$c^i_{t,s} \neq y^i_{t,s},\ t = 0, 1,\ s = 1, \ldots, N$$

by shifting resources across **time** and **states of nature**.



Real versus financial markets

We distinguish further:

- **real markets**:
  market for goods,
  which can be spot or forward
  prices and deliveries are referred to **periods**
- **financial markets**:
  market on financial instruments,
  which are always forward (in an economic sense)
  and prices and deliveries are referred to **times**

Markets and general equilibrium models

Simultaneous versus sequential market economies

We consider next two economies which are distinguished by the type of intertemporal contracts available:

- **Arrow Debreu economies**:
  there are AD contingent goods traded in spot and forward **real** markets ⇒ there is simultaneous market equilibrium
- **finance economies**:
  Radner economies in which **financial** assets are traded ⇒ there is sequential market equilibrium

They can be **equivalent under some conditions**, i.e., have the same equilibrium allocations

Julian Thimme. Intertemporal substitution in consumption: a literature review. *Journal of Economic Surveys*, 31(1):226–257, February 2017. URL https://ideas.repec.org/a/bla/jecsur/v31y2017i1p226-257.html.