Foundations of Financial Economics
Two period financial markets

Paulo Brito

[1]pbrito@iseg.ulisboa.pt
University of Lisbon

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Topics

- Financial assets
- Financial market
- State prices
- Arbitrage opportunities
- Completeness
- Characterization of a financial market
- Asset pricing implicit probabilities and the stochastic discount factor
- Portfolios
- Arbitrage pricing theory
- Equity premium

Financial contracts and assets

Information

- Assume again that there is **complete** information regarding time $t = 0$, but **incomplete** information regarding time $t = 1$.
- Until now we have (mostly) assumed that the information for $t = 1$

$$\Omega = \{\omega_s\}_{s=1}^{N}, \text{ and } \mathbb{P} = \{\pi_s\}_{s=1}^{N}$$

  is provided by **nature**.
- But next we start assuming that information regarding time $t = 1$ is provided by the **financial market**. A financial market is defined by a **collection of contingent claims**.

The **general idea**: under some conditions, we can extract an **implicit probability distribution** of the states of nature from financial market data.

Financial contracts or assets

Prices and payoffs

A financial contract or financial asset or contingent claim: is defined by a **price and payoff** pair (S_j, V_j) (for asset j) where:

- S_j is a price which is **observed** (deterministic) at time $t = 0$
- V_j is a payoff which is **state-contingent** (stochastic) at time $t = 1$,

$$V_j = (V_{j,1}, \ldots, V_{j,s}, \ldots, V_{j,N})^\top$$

$$V_{j,s} = \text{payoff of asset } j \text{ in the state of nature } s \in \{1, \ldots, N\}$$

- This information refers to time $t = 0$

Financial assets

Returns and rates of return

- The return of asset j in the state of nature s is defined as the ratio between the payoff at state s and the price

$$R_{j,s} = \frac{V_{j,s}}{S_j},\ s \in \{1, \ldots, N\}$$

- The rate of return r_j of asset j in the state of nature s is defined from

$$r_{j,s} = \frac{V_{j,s} - S_j}{S_j},\ s \in \{1, \ldots, N\}$$

- Therefore the return and the rate of return are related as

$$R_{j,s} = 1 + r_{j,s},\ s \in \{1, \ldots, N\}$$

Financial assets

Returns and rates of return

- Therefore, in a two-period case, the return R_j and rate of return r_j of asset j are two random variables which are related

$$R_j = \frac{V_j}{S_j} = \begin{pmatrix} \frac{V_{j,1}}{S_j} \\ \dots \\ \frac{V_{j,s}}{S_j} \\ \dots \\ \frac{V_{j,N}}{S_j} \end{pmatrix} = \begin{pmatrix} 1 + r_{j,1} \\ \dots \\ 1 + r_{j,s} \\ \dots \\ 1 + r_{j,N} \end{pmatrix}$$

- Or, more compactly

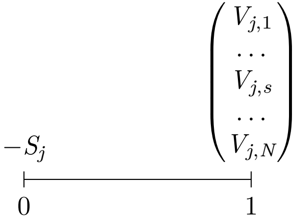
$$R_j = 1 + r_j,$$

Financial assets

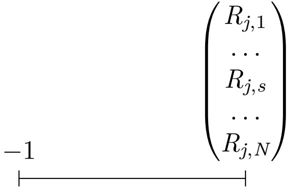
Timing, information and flow of funds

Two alternative ways of representing (net) income flows (for a buyer) of asset j:

- as a **price-payoff** sequence: $\{S_j, V_j\}$



- as an **investment-return** sequence: $\{1, R_j\}$



Financial assets

Statistics

- From the information, at time $t = 0$, on the probabilities for the states of nature at time $t = 1$ we can compute:
- **Expected payoff** for asset j at time $t = 1$, from the information at time $t = 0$

$$\mathbb{E}[V_j] = \sum_{s=1}^{N} \pi_s V_{j,s} = \pi_1 V_{j,1} + \ldots \pi_s V_{j,s} + \ldots + \pi_N V_{j,N}$$

- **Variance of the payoff** for asset j at time $t = 1$, from the information at time $t = 0$

$$\mathbb{V}[V_j] = \sum_{s=1}^{N} \pi_s (V_{js} - \mathbb{E}[V_j])^2 = \pi_1 (V_{j,1} - \mathbb{E}[V_j])^2 + \ldots \pi_N (V_{j,N} - \mathbb{E}[V_j])^2$$

- Observation: an useful relationship

$$\mathbb{V}[V] = \mathbb{E}[V^2] - (\mathbb{E}[V])^2$$

Financial assets

Statistics

- From the information, at time $t = 0$, on the probabilities for the states of nature at time $t = 1$ we can compute:
- **Expected return** for asset j at time $t = 1$, from the information at time $t = 0$

$$\mathbb{E}[R_j] = \sum_{s=1}^{N} \pi_s R_{j,s} = \pi_1 R_{j,1} + \ldots \pi_s R_{j,s} + \ldots + \pi_N R_{j,N}$$

- **Variance of the return** for asset j at time $t = 1$, from the information at time $t = 0$

$$\mathbb{V}[R_j] = \sum_{s=1}^{N} \pi_s (R_{js} - \mathbb{E}[R_j])^2 = \pi_1 \left(R_{j,1} - \mathbb{E}[R_j]\right)^2 + \ldots \pi_N \left(R_{j,N} - \mathbb{E}[R_j]\right)^2$$

- Observation: an useful relationship

$$\mathbb{V}[R] = \mathbb{E}[R^2] - \left(\mathbb{E}[R]\right)^2$$

Financial assets

Statistics

- Relationship between the expected return and the expected payoff

$$\mathbb{E}[R_j] = \frac{\mathbb{E}[V_j] - S_j}{S_j}$$

- Relationship between the variance of the return and the variance of the payoff

$$\mathbb{V}[R_j] = \frac{\mathbb{V}[V_j]}{S_j^2}$$

Classification of assets

Risk classification

As regards **risk**:

- risk-less or **risk-free** asset: payoff is state-independent (non-contingent)

$$V^\top = (v, \dots, v)^\top$$

- risky asset: payoff is state-dependent (contingent)

$$V^\top = (v_1, \dots, v_N)^\top$$

  with at least two different elements, i.e. there are at least two elements, v_i and v_j such that $v_i \neq v_j$ for $i \neq j$

Classification of assets

Types of assets

Particular **types** of assets as regards the income flows:

- one period **bonds** with unit facial value:

$$S = \frac{1}{1+i}, \; V^\top = (1, 1, \ldots, 1)^\top$$

  i yield to maturity (or risk-free interest rate)

- **deposits** or banking credit:

$$S = 1, \; V^\top = (1+i, 1+i, \ldots, 1+i)^\top$$

- **equity**: the payoff are state-dependent dividends

$$S_e, \; V_e^\top = (d_1, \ldots, d_N)^\top$$

Classification of assets

Types of assets

Derivatives over an **underlying** asset with payoff $V^\top = (v_1, \ldots, v_N)^\top$:

- **forward** contract on an underlying asset with offered price p:

$$S_f, \; V_f^\top = (v_1 - p, \ldots, v_N - p)^\top$$

- **european call option** with exercise price p:

$$S_c, \; V_c^\top = (\max\{v_1 - p, 0\}, \ldots, \max\{v_N - p, 0\})^\top$$

- **european put option** with exercise price p:

$$S_p, \; V_p^\top = (\max\{p - v_1, 0\}, \ldots, \max\{p - v_N, 0\})^\top$$

Financial market

Financial market

Definition 1

*A **financial market** is a collection of K traded assets.*
*It can be **characterized** by the structure of prices and payoffs of all K assets: i.e by the pair $(\mathbf{S}, \mathbf{V})$.*

Information: the participants observe $\mathbf{S}$ and have common beliefs $\mathbf{V}$

Financial market

Characterization: :

- vector of observed prices, at time $t = 0$ (we use row vectors for prices)
$$\underset{(1\times K)}{\mathbf{S}} = (S_1, \dots, S_K),$$
- and a matrix of contingent (i.e., uncertain) payoffs, at time $t = 1$
$$\underset{(N\times K)}{\mathbf{V}} = \begin{pmatrix} V_{11} & \dots & V_{K1} \\ \vdots & & \vdots \\ V_{1N} & \dots & V_{KN} \end{pmatrix}$$

Realization: ex-post only one row of **V** will be realized (i.e, will be cast by nature)

Financial market

The payoff matrix contains the following information:

- each column represents the **beliefs** for the payoff of **asset** $j = 1, \ldots, K$

$$\mathbf{V} = \begin{pmatrix} V_{11} & \ldots & \mathbf{V_{j1}} & \ldots & V_{K1} \\ \vdots & & \vdots & & \vdots \\ V_{1s} & \ldots & \mathbf{V_{js}} & \ldots & V_{Ks} \\ \vdots & & \vdots & & \vdots \\ V_{1N} & \ldots & \mathbf{V_{jN}} & \ldots & V_{KN} \end{pmatrix}$$

- each row represents the **outcome** (in terms of payoffs) if **state** of nature $s = 1, \ldots, N$ is realized

$$\mathbf{V} = \begin{pmatrix} V_{11} & \ldots & V_{j1} & \ldots & V_{K1} \\ \vdots & & \vdots & & \vdots \\ \mathbf{V_{1s}} & \ldots & \mathbf{V_{js}} & \ldots & \mathbf{V_{Ks}} \\ \vdots & & \vdots & & \vdots \\ V_{1N} & \ldots & V_{jN} & \ldots & V_{KN} \end{pmatrix}$$

Financial market

- Equivalently, we can charaterize a financial market by the matrix of returns

$$\underset{(N\times K)}{\mathbf{R}} = \begin{pmatrix} R_{11} & \dots & R_{K1} \\ \vdots & & \vdots \\ R_{1N} & \dots & R_{KN} \end{pmatrix}$$

  where $R_{j,s} = \dfrac{V_{j,s}}{S_j} = 1 + r_{j,s}$

- Therefore

$$\underset{(N\times K)}{\mathbf{R}} = \underset{(N\times K)}{\mathbf{V}} \big(\underset{(K\times K)}{\text{diag}(\mathbf{S})}\big)^{-1} \iff \text{diag}(\mathbf{S})\,\mathbf{R} = \mathbf{V}$$

  where

$$\underset{(K\times K)}{\text{diag}(\mathbf{S})} = \begin{pmatrix} S_1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & S_j & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & S_K \end{pmatrix}$$

State prices

State prices

Implicit state valuation: the modern approach to financial economics characterizes asset markets by the prices of the states of nature **implicit** in the relationship between present prices S and future payoffs V, i.e.

$$Q := \{X : \mathbf{S} = X\mathbf{V}\}$$

- if Q is unique it is a vector

$$\underset{(1\times N)}{Q} = (q_1, \ldots, q_N)$$

satisfying

$$\underset{(1\times K)}{\mathbf{S}} = \underset{(1\times N)}{Q}\underset{(N\times K)}{\mathbf{V}} \iff \underset{(K\times 1)}{\mathbf{S}^\top} = \underset{(K\times N)}{\mathbf{V}^\top}\underset{(N\times 1)}{Q^\top}$$

State prices

Definition 2

Q is a state price vector if it is a positive implicit state valuation:

$$i.e\ q_s > 0 \text{ for all } s \in \{1, \ldots, N\}$$

In this case q_s is called the price of state of nature s.

Arbitrage opportunities

Arbitrage opportunities

Definition 2

If there is at least one (implicit state valuation) $q_s \leq 0,\ s = 1, \ldots, N$ *then we say* ***there are arbitrage opportunities***

Definition 3

We say ***there are no arbitrage opportunities*** *if* $q_s > 0$*, for all* $s = 1, \ldots, N$

Intuition:

- existence of arbitrage (opportunities) means there are free or negatively valued states of nature
- absence of arbitrage means every state of nature is costly and therefore, positively priced.

Arbitrage opportunities

Proposition 1

Given $(\mathbf{S}, \mathbf{V})$*, there are no arbitrage opportunities if and only if* Q *is a vector of state prices.*

Completeness

Completeness of a financial market

Definition 4

If Q ***is unique****, for a given $(\mathbf{S}, \mathbf{V})$, then we say markets are* ***complete***

Definition 5

if Q ***is not unique****, for a given $(\mathbf{S}, \mathbf{V})$, then we say markets are* ***incomplete***

Intuition:

- completeness: there is an unique valuation for each state of nature (i.e, every state of nature can be uniquely priced).
- incompleteness: there is not an unique valuation for every state of nature (i.e, there are states of nature whose price is uncertain).

Completeness of a financial market

Conditions for completeness

- **Assessing completness**: we can assess completeness of a market by comparing:
  the **number of assets with independent payoffs** with the **number of states of nature**:
- Possible cases:
  1. if $K = N$ and $\det(\mathbf{V}) \neq 0$ then markets are complete and all assets are independent;
  2. if $K > N$ and $\det(\mathbf{V}) \neq 0$ then markets are complete and there are N independent and $K - N$ **redundant** assets;
  3. if $K < N$ or $K \geq N$ and $\det(V) = 0$ then markets are incomplete.

Completeness of a financial market

Conditions for completeness

Proposition 2

Given $(\mathbf{S}, \mathbf{V})$*, markets are complete if and only if* $dim(\mathbf{V}) = N$

$\dim(\mathbf{V})$ = number of linearly independent columns (i.e., assets)

Characterizing financial markets

Characterizing financial markets

Possible cases

We will study next the following cases:

1. Exact completeness: $K = N$ and $\det \mathbf{V} \neq 0$
2. Incompleteness (insufficient number of assets): $K < N$
3. Incompleteness (lack of return independence): $K = N$ and $\det \mathbf{V} = 0$
4. Completeness and redundancy: $K > N$ and $\det \mathbf{V} \neq 0$

Alternative computations

An example

- Next we show **four alternative ways of computing** the state prices
- In dealing with an example we can use any of the following four methods
- Next we consider the **simplest case**:
  the financial market data $(\mathbf{S}, \mathbf{V})$ satisfies

$$K = N \text{ and } \det(\mathbf{V}) \neq 0$$

- It should be adapted to the particular case under study (see next)

Alternative computations

First

- We defined state prices from $\mathbf{S} = Q\mathbf{V}$
- As $K = N$ and $\det(\mathbf{V}) \neq 0$ then $\mathbf{V}^{-1}$ exists and is unique
- Therefore, we obtain uniquely

$$\boxed{\underset{(1\times N)}{Q} = \underset{(1\times K)}{\mathbf{S}}\,\underset{(K\times N)}{\mathbf{V}^{-1}}}$$

that is

$$(q_1, \ldots, q_N) = (S_1, \ldots, S_K)\begin{pmatrix} V_{1,1} & \ldots & V_{1,K} \\ \ldots & \ldots & \ldots \\ V_{N,1} & \ldots & V_{N,K} \end{pmatrix}^{-1}$$

Alternative computations

Second

- Because

$$\mathbf{V}^\top Q^\top = \mathbf{S}^\top$$

- If $\det(\mathbf{V}) \neq 0$ then

$$\boxed{\underset{(N\times 1)}{Q^\top} = \underset{(N\times K)}{\left(\mathbf{V}^\top\right)^{-1}} \underset{(K\times 1)}{\mathbf{S}^\top}}$$

that is

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} V_{1,1} & \dots & V_{N,1} \\ \dots & \dots & \dots \\ V_{1,K} & \dots & V_{N,K} \end{pmatrix}^{-1} \begin{pmatrix} S_1 \\ \vdots \\ S_K \end{pmatrix}$$

Alternative computations

Third

- As $R_{js} = V_{js}/S_j$ equivalently,
- we can write

$$\underset{(1\times N)}{Q}\ \underset{(N\times K)}{\mathbf{R}} = \underset{(1\times K)}{\mathbf{1}^\top}$$

expanding

$$(q_1, \dots, q_N)\begin{pmatrix} R_{1,1} & \dots & R_{1,K} \\ \dots & \dots & \dots \\ R_{N,1} & \dots & R_{N,K} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

- then

$$\boxed{\underset{(1\times N)}{Q} = \underset{(1\times K)}{\mathbf{1}^\top}\ \underset{(K\times N)}{\mathbf{R}^{-1}}}$$

Alternative computations

Fourth

- Transposing the last equation $\mathbf{R}^\top Q^\top = \mathbf{1}$
- then

$$\boxed{\underset{(N\times 1)}{Q^\top} = \underset{(N\times K)}{(\mathbf{R}^\top)^{-1}} \underset{(K\times 1)}{\mathbf{1}}} \qquad (1)$$

matricially

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} R_{1,1} & \dots & R_{1,K} \\ \dots & \dots & \dots \\ R_{N,1} & \dots & R_{N,K} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

- This is the one I will use next (but any of the previous methods can be used instead)

Characterizing financial markets

Arbitrage opportunities and completeness

Case 1: Consider a financial market $(\mathbf{S}, \mathbf{V})$, such that

$$K = N \text{ and } \det(\mathbf{V}) \neq 0$$

- Because Q is unique, we say markets are **complete**
- But the value of Q matters:
  - if $Q \gg \mathbf{0}$ we say there are **no arbitrage opportunities**: (meaning: all states of nature are costly to insure)
  - if Q has a zero or negative element we say there **are arbitrage opportunities** (meaning: there are states that have no cost or are pathological (there is a benefit to insure)
- The equation $Q^\top = (\mathbf{R}^\top)^{-1}\,\mathbf{1}$ conveys an important message: the value of Q is related to the structure of $\mathbf{R}$), i.e., the relationship between beliefs on $\mathbf{V}$ related to observed prices $\mathbf{S}$

Example: case 1 - no arbitrage opportunities

- Let $\mathbf{S} = (1, 1)$ and $\mathbf{V} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- As $\det \mathbf{V} = 2$ then $K = N = 2$
- Using the fourth approach we determine

$$\mathbf{R} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \mathbf{R}^\top$$

- Therefore, using equation (1)

$$Q^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

- Conclusion: markets are complete (Q is unique) and there are no arbitrage opportunities ($Q \gg \mathbf{0}$)

Example case 1 - existence of arbitrage opportunities

- Let $\mathbf{S} = (1, 1)$ and $\mathbf{V} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
- then
$$\mathbf{R} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \ \mathbf{R}^\top = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
- Therefore
$$Q^\top = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Conclusion: markets are complete but **there are arbitrage opportunities**. The cost to ensure state $s = 2$ is zero. Why ?

Characterization of a financial market

Market incompleteness because of insufficient assets

Case 2: Consider a financial market $(\mathbf{S}, \mathbf{V})$, such that

$$K < N \text{ assume that } \det(\mathbf{V}_1) \neq 0.$$

Introduce the partition

$$\underset{(K\times N)}{\mathbf{V}^\top} = \left(\underset{(K\times K)}{\mathbf{V}_1^\top} \mid \underset{(K\times N-K)}{\mathbf{V}_2^\top} \right)$$

- We defined state prices from $\mathbf{V}^\top Q^\top = \mathbf{S}^\top$
- But now we have

$$\begin{pmatrix} \mathbf{V}_1^\top & \mathbf{V}_2^\top \end{pmatrix} \begin{pmatrix} Q_1^\top \\ Q_2^\top \end{pmatrix} = \mathbf{S}^\top$$

where Q_1 is $(1 \times K)$ and Q_2 is $(1 \times N - K)$.

Characterization of a financial market

Market incompleteness

- Then

$$\mathbf{V}_1^\top Q_1^\top + \mathbf{V}_2^\top Q_2^\top = \mathbf{S}^\top$$

- Because $\det(\mathbf{V}_1) \neq 0$ we can make

$$Q_1^\top = (\mathbf{V}_1^\top)^{-1} \left(\mathbf{S}^\top - \mathbf{V}_2^\top Q_2^\top\right)$$

- There are only K independent prices: i.e, Q is indeterminate and the degree of indeterminacy is $N - K$,
- As Q is not uniquely determined (i.e, we can fix arbitrarilty $N - K$ state prices) the market is **incomplete**

Example case 2: market incompleteness because of insufficient number of assets

- Let $\mathbf{S} = (1)$ and $\mathbf{V} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- then

$$\mathbf{R} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

- then we also have

$$(q_1, q_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \iff 2q_1 + q_2 = 1$$

- has an infinite number of solutions

$$Q = \begin{pmatrix} \frac{1-k}{2} \\ k \end{pmatrix}$$

for any k, then markets are incomplete and if $0 < k < 1$ there are no arbitrage opportunities (but they cannot be ruled out)

Characterization of a financial market

Market incompleteness because of insufficient independent assets assets

Case 3: Consider a financial market $(\mathbf{S}, \mathbf{V})$, such that

$$K = N \text{ assume that } \det(\mathbf{V}) = 0$$

- If there are $\tilde{K}$ independent assets (i.e., $\text{char}(\mathbf{V}) = \tilde{K} < N$, then we can apply the previous method by extracting data from the original $\mathbf{V}$ such that $\tilde{\mathbf{V}}$ are the independent columns of $\mathbf{V}$ and having the partition

$$\underset{(\tilde{K} \times N)}{\tilde{\mathbf{V}}^\top} = \left(\underset{(\tilde{K} \times K)}{\tilde{\mathbf{V}}_1^\top} \,|\, \underset{(\tilde{K} \times N - K)}{\mathbf{V}_2^\top} \right)$$

  where $\det(\tilde{\mathbf{V}}_1) \neq 0$

- We obtain the state prices from Q_1 from

$$\begin{pmatrix} \tilde{\mathbf{V}}_1^\top & \tilde{\mathbf{V}}_2^\top \end{pmatrix} \begin{pmatrix} Q_1^\top \\ Q_2^\top \end{pmatrix} = \tilde{\mathbf{S}}^\top$$

  where Q_1 is $(1 \times \tilde{K})$ and Q_2 is $(1 \times N - \tilde{K})$.

Characterization of a financial market

Market incompleteness

- Then

$$\tilde{\mathbf{V}}_1^\top \, Q_1^\top + \tilde{\mathbf{V}}_2^\top \, Q_2^\top = \tilde{\mathbf{S}}^\top$$

- Because $\det(\tilde{\mathbf{V}}_1) \neq 0$ we can make

$$Q_1^\top = (\tilde{\mathbf{V}}_1^\top)^{-1} \left(\tilde{\mathbf{S}}^\top - \tilde{\mathbf{V}}_2^\top \, Q_2^\top \right)$$

- There are only $\tilde{K}$ independent prices: i.e, Q is indeterminate and the degree of indeterminacy is $N - \tilde{K}$,
- As Q is not uniquely determined (i.e, we can fix arbitrarilty $N - \tilde{K}$ state prices) the market is **incomplete**

Example case 3: market incompleteness with non-independent prices

- Let $\mathbf{S} = (1, 2)$ and $\mathbf{V} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ and $\det \mathbf{V} = 0$
- then $\mathbf{R} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{R}^\top = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ two assets have the same returns
- then $\mathbf{R}^\top Q^\top = \mathbf{1}^\top$ becomes

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

has an infinite number of solutions

$$Q^\top = \begin{pmatrix} (1-k)/2 \\ k \end{pmatrix}$$

for an arbitrary k, then markets are incomplete and if $0 < k < 1$ there are no arbitrage opportunities;

Characterization of a financial market

Market completeness with redundant assets

Case 4: Consider a financial market $(\mathbf{S}, \mathbf{V})$, such that

$$K > N \text{ assume that } \det(\mathbf{V}) \neq 0$$

- As $K > N$ we can partition $\mathbf{V}$ as

$$\underset{(N\times K)}{\mathbf{V}} = \left(\underset{(N\times N)}{\mathbf{V}_1} \,\middle|\, \underset{(N\times K-N)}{\mathbf{V}_2} \right)$$

  and $\det(\mathbf{V}_1) \neq 0$

- We defined state prices from $\mathbf{V}^\top Q^\top = \mathbf{S}^\top$
- But now we have

$$\begin{pmatrix} \mathbf{V}_1^\top \\ \mathbf{V}_2^\top \end{pmatrix} (Q^\top) = \begin{pmatrix} \mathbf{S}_1^\top \\ \mathbf{S}_2^\top \end{pmatrix}$$

  where $\mathbf{S}^1$ is $(1 \times N)$ and $\mathbf{S}^2$ is $(1 \times K - N)$.

Characterization of a financial market

Market completeness with redundant assets

- Then

$$\begin{cases} \mathbf{V}_1^\top \, Q^\top = \mathbf{S}_1^\top \\ \mathbf{V}_2^\top \, Q^\top = \mathbf{S}_2^\top \end{cases}$$

- Because $\det(\mathbf{V}_1) \neq \mathbf{0}$ we can determine Q uniquely

$$Q^\top = (\mathbf{V}_1^\top)^{-1}\mathbf{S}_1^\top$$

- Which implies that the prices of the remaining $K - N$ assets can be obtained from the prices $\mathbf{S}_1$

$$\mathbf{S}_2^\top = \mathbf{V}_2^\top \, Q^\top = \mathbf{V}_2^\top \, \left((\mathbf{V}_1^\top)^{-1}\mathbf{S}_1^\top\right)$$

- There are $K - N$ **redundant assets** (i.e, they do not add new information on Q)
- As Q is uniquely determined the market is **complete**

Example case 4: market completeness and redundant assets

- Let $\mathbf{S} = (1, 1, 2)$ and $\mathbf{V} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
- then $\mathbf{R} = \begin{pmatrix} 2 & 1 & \frac{3}{2} \\ 1 & 2 & \frac{3}{2} \end{pmatrix}$ and $\mathbf{R}^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}$
- we pick all the combinations of two assets

$$Q^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$Q^\top = \begin{pmatrix} 2 & 1 \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} \\ -1 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

and

$$Q^\top = \begin{pmatrix} 1 & 2 \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{4}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

then markets are complete, there are no arbitrage opportunities and there is one redundant asset

Characterization of a financial market

Suggestion on how to apply this theory

- First: obtain $\mathbf{R}^T$ which is a $(K \times N)$ matrix
- Second: check the number of assets, K and the number of states of nature N
  - if $N > K$ then markets are incomplete
  - if $N \geq K$ check the number of independent rows of R
    - if $\dim(\mathbf{V}) < N$ then markets are incomplete
    - if $\dim(\mathbf{V}) \geq N$ then markets are complete
- Third: obtaining the state prices Q
  - if $K = N$ and $\det \mathbf{R} \neq 0$ compute $Q^\top = (\mathbf{R}^\top)^{-1}\mathbf{1}$
  - if $K > N$ and $\tilde{\mathbf{R}}$ a $N \times N$ partition of $\mathbf{R}$ has $\det \tilde{\mathbf{R}} \neq 0$ then compute $Q^\top = (\tilde{\mathbf{R}}^\top)^{-1}\mathbf{1}$
  - if $K = N$ but $\det \mathbf{R} = 0$, then, solve the independent equations in $\mathbf{R}^\top Q^\top = \mathbf{1}$
  - if $K < N$ solve the independent equations in $\mathbf{R}^\top Q^\top = \mathbf{1}$.

Asset pricing implicit probabilities and the stochastic discount factor

Implicit market probabilities: risk-neutral probabilities

- Let $Q = (q_1, \ldots, q_N)$ be a state-price vector
- If there are no arbitrage opportunities then $q_s > 0$, i.e $Q \gg 0$

Definition 6
*A **Radon-Nikodyn derivative** is defined as*

$$\pi_s^Q = \frac{q_s}{\bar{q}}, \text{ for } s \in \{1, \ldots, N\}$$

where $\bar{q} = \sum_{s=1}^{N} q_s$.

Implicit market probabilities: risk-free probabilities

Proposition 3

*Assume there are no arbitrage opportunities. Then the Radon-Nikodyn derivatives $\mathbb{P}^Q = \{\pi_s^Q\}_{s=1}^N$ define a probability measure. We call them **risk-neutral probabilities**.*

- Proof: if there are no arbitrage opportunities then $\pi_s^Q > 0$ for every $s \in \{1, \ldots, N\}$. As $\pi_s^Q < 1$ and $\sum_{s=1}^N \pi_s^Q = 1$ then $\{\pi_s^Q\}_{s=1}^N$ are formally probabilities.
- Intuition: $\mathbb{P}^Q$ is a probability measure implicit in the financial market $(\mathbf{S}, \mathbf{V})$ (while $\mathbb{P}$ are generated by nature)

Implicit market probabilities: risk-free probabilities

Therefore:

- In complete markets the probability measure $\mathbb{P}^Q$ is unique
- in incomplete the probability measure $\mathbb{P}^Q$ is not unique

And asset pricing

- Using the definition for Q (from $\mathbf{S}^\top = \mathbf{V}^\top Q^\top$)

$$S_j = \sum_{s=1}^{N} q_s \, V_{sj} = \sum_{s=1}^{N} \bar{q} \frac{q_s}{\bar{q}} \, V_{sj}$$

- then, for any asset

$$S_j = \bar{q} \, \mathbb{E}^Q \Big[V_j \Big]$$

Proposition 4

Assume there are no arbitrage opportunities. Then the ***price of any asset is proportional to the expected value of the payoff, using risk-neutral probabilities****.*

Stochastic discount factor

- Let $Q = (q_1, \ldots, q_N)$
- Assume we have a (objective or subjective) probability distribution for the states of nature

$$\mathbb{P} = (\pi_1, \ldots, \pi_N)$$

such that $0 < \pi_s < 1$ and $\sum_{s=1}^{N} \pi_s = 1$

Definition 7

The ***stochastic discount factor*** *is the random variable*

$$\underset{(1 \times N)}{M} = (m_1, \ldots, m_s, \ldots, m_N)$$

such that

$$m_s = \frac{q_s}{\pi_s}, \; s = 1, \ldots, N$$

Stochastic discount factor

And asset pricing

Proposition 5

Let an asset market be characterized by $(\mathbf{S}, \mathbf{V})$*. Then the price of any asset* j *is equal to the expected discounted value of payoffs*

$$S_j = \mathbb{E}\Big[M\, V_j\Big],$$

using an objective probability measure and the market stochastic discount factor.

- Using the definition for Q (from $\mathbf{S}^\top = \mathbf{V}^\top\, Q^\top$)

$$S_j = \sum_{s=1}^{N} q_s\, V_{sj} = \sum_{s=1}^{N} \pi_s \frac{q_s}{\pi_s}\, V_{sj}$$

- This means that the stochastic discount factor combines both market and fundamental information.

The state price approach to asset markets

Summing up:

- The state price translates the information structure implicit in financial transactions
- The relevant information is related to:
  - how costly is the insurance against the states of nature
  - the uniqueness of that insurance cost
  - suggests a method for pricing new assets: **pricing by redundancy** (this is the approah followed in the arbitrage pricing theory (APT))

Portfolios

Portfolios

Definition 7

*A **portfolio** is a vector specifying the positions, θ, in all the assets in the market*

$$\theta^\top = (\theta_1, \ldots, \theta_K)^\top \in \mathbb{R}^K$$

If $\theta_j > 0$ there is a **long** position in asset j

if $\theta_j < 0$ there is a **short** position in asset j

Portfolios

- The stream of income generated by a position θ_j in asset j is

$$
\begin{array}{ccc}
 & & \begin{pmatrix} \theta_j V_{j,1} \\ \dots \\ \theta_j V_{j,s} \\ \dots \\ \theta_j V_{j,N} \end{pmatrix} \\
-S_j\theta_j & & \\
\vdash & \text{———————} & \dashv \\
0 & & 1
\end{array}
$$

  - long position ($\theta_j > 0$): pay $S_j\theta_j$ at time $t = 0$ and receive the contingent payoff $\theta_j V_j$ at time $t = 1$
  - short position ($\theta_j < 0$): receive $S_j\theta_j$ at time $t = 0$ and pay the contingent payoff $\theta_j V_j$ at time $t = 1$

Portfolios

A portfolio generates a **stochastic sequence of income** $\{z_0^\theta, Z_1^\theta\}$, where

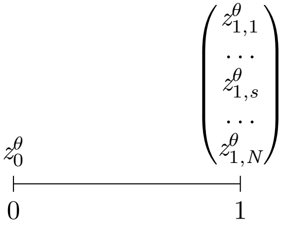
$$\begin{aligned} z_0^\theta &= -C(\theta, \mathbf{S}) = -\underset{(1\times 1)}{\mathbf{S}\,\theta} = -\sum_{j=1}^{K} S_j\,\theta_j \\ \underset{(N\times 1)}{Z_1^\theta} &= \underset{(N\times 1)}{\mathbf{V}\,\theta} = \sum_{j=1}^{K} V_j\,\theta_j \end{aligned}$$

where

$$\underset{(N\times 1)}{Z_1^\theta} = \begin{pmatrix} z_{1,1}^\theta \\ \dots \\ z_{1,s}^\theta \\ \dots \\ z_{1,N}^\theta \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{K} V_{j,1}\,\theta_j \\ \dots \\ \sum_{j=1}^{K} V_{j,s}\,\theta_j \\ \\ \sum_{j=1}^{K} V_{j,N}\,\theta_j \end{pmatrix}$$

Portfolios

The flow of income generated by a portfolio θ is:



Portfolios and arbitrage opportunities

Proposition 6

Assume ***there are arbitrage opportunities****. Then there exists at least one portfolio ϑ such that*

$$z_0^{\vartheta} = 0 \text{ and } Z_1^{\vartheta} > \mathbf{0}$$

or

$$z_0^{\vartheta} > 0 \text{ and } Z_1^{\vartheta} = \mathbf{0}.$$

(Obs. A positive vector $X > 0$ has non-negative elements, and has at least one equal to zero. A strictly positive vector $X \gg 0$ has only positive elements)

Intuition

with a **zero cost** we can get a **positive income** in at least one state of nature.

If we have an **initial income**, we will pay a **zero cost** in the future, in every state of nature.

Example

- We consider Case 1 example 2 again: $\mathbf{S} = (1, 1)$ and $\mathbf{V} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
- We can build a portfolio $\vartheta = (\vartheta_1, \vartheta_2)^\top$ that has a zero cost

$$z_0^\vartheta = -\mathbf{S}\,\vartheta = -(\vartheta_1 + \vartheta_2) = 0 \implies \vartheta_1 = -x, \vartheta_2 = x$$

- The return at time $t = 1$ is

$$Z_1^\vartheta = \mathbf{V}\,\vartheta = \begin{pmatrix} -x + x \\ -x + 2x \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

then if we take a long position in the risky asset $(x > 0)$ we have a **certain** positive income at $t = 1$ with a zero investment.

Portfolios and arbitrage opportunities

Proposition 4

Assume there ***are no arbitrage opportunities****. Then a portfolio with zero cost,* $z_0^{\vartheta} = 0$*, generates a* ***nonpositive*** *income at time 1,* Z_1^{ϑ}*.*

- Non-positive Z_1^{ϑ} means that there is at least one state of nature, s such that $z_{1,s}^{\theta} < 0$.

  Intuition: with a zero cost although we can get a positive income in several states of nature, there will be **at least one state of nature** in which we will have a negative income.

Example 2

- Consider the previous case 1 example 1: $\mathbf{S} = (1, 1)$ and $\mathbf{V} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- We can build a portfolio $(\vartheta_1, \vartheta_2)^\top$ such that $z_0^\vartheta = -S\vartheta = 0$:

$$\vartheta_1 = -x, \vartheta_2 = x$$

- The return at time $t = 1$ is

$$Z_1^\vartheta = \mathbf{V}\,\vartheta = \begin{pmatrix} -2x + x \\ -x + 2x \end{pmatrix} = \begin{pmatrix} -x \\ x \end{pmatrix}$$

then whatever the position, long ($x > 0$) or short ($x < 0$), in the risky asset we will **not have positive income** at $t = 1$ (in some states of the nature we win, but we will lose in others).

Arbitrage pricing theory

Replicating portfolios

Definition 9

*Assume we have a contingent income $W = (w_1, \ldots, w_N)^\top$. We call **replicating portfolio** to the portfolio, ϑ, which is build using assets in the market and generates the same (contingent) income W*

$$\vartheta \equiv \{\theta : \; Z_1 = \mathbf{V}\,\theta = W\}.$$

The **cost** of the replicating portfolio is

$$C(\vartheta, \mathbf{S}) = \mathbf{S}\vartheta$$

Arbitrage asset pricing theory (APT)

- Deals with the determination of asset prices **through replication**.
- APT vs GEAP (general equilibrium asset pricing):
  - in APT we take $(\mathbf{S}, \mathbf{V})$ as given ($\mathbf{S}$ is exogenous)
  - in GEAP we take $\mathbf{V}$ as given and want to determine $\mathbf{S}$ from the fundamentals ($\mathbf{S}$ is endogenous)

Both theories have three equivalent formulations:

- using state prices Q (micro view)
- using the stochastic discount factor M (macro view)
- using market or risk-neutral probabilities $\mathbb{P}^Q$ (finance view)

Arbitrage asset pricing

Using state prices

Proposition 5

Assume there is a financial market $(\mathbf{S}, \mathbf{V})$ *such that there are no arbitrage opportunities. Then, given a new asset with payoff* V_k *its price can be determined by redundancy by using the expression*

$$S_k = Q V_k$$

where $Q = \mathbf{S}\,\mathbf{V}^{-1} \gg 0$ *is determined from the market pair* $(\mathbf{S}, \mathbf{V})$*.*

That is

$$S_k = \sum_{s=1}^{N} q_s V_{ks}$$

Arbitrage asset pricing

Using state prices

Proposition 6

If markets are ***complete*** *then the value,* S_k*, is* ***unique****. If markets are* ***incomplete*** *then* S_k *is* ***not unique****.*

Arbitrage asset pricing

Using stochastic discount factors

- If Q is a state price vector and π is the vector or probabilities, we define the stochastic discount factor (SDF) for state s by

$$m_s \equiv \frac{q_s}{\pi_s}$$

Proposition 7

Assume there is a financial market $(\mathbf{S}, \mathbf{V})$ such that there are no arbitrage opportunities. Then, given a new asset with payoff V_k its price can be determined by redundancy by using the expression

$$S_k = \mathbb{E}[MV_k]$$

where $M = (m_1, \ldots, m_N)^\top$ is the stochastic discount factor.

Arbitrage asset pricing

Using stochastic discount factors

- Proof: as $S_k = \sum_{s=1}^{N} q_s V_{ks}$ and using the definition of SDF we get

$$S_k = \sum_{s=1}^{N} q_s V_{ks} = \sum_{s=1}^{N} \pi_s m_s V_{ks}$$

- **Intuition**: if there are no arbitrage opportunities the price of an asset is the expected value of the future payoffs discounted by the stochastic discount factor (which is the market discount for uncertain payoffs)

Arbitrage asset pricing

Using risk-neutral probabilities

- Therefore,

$$S_k = \sum_{s=1}^{N} q_s \, V_{ks} = \bar{q} \sum_{s=1}^{N} \pi_s^Q \, V_{ks} = \bar{q} \, \mathbb{E}^Q[\, V_k]$$

  or compactly

$$\boxed{S_k = \bar{q} \, \mathbb{E}^Q[\, V_k]}$$

- **Intuition**: if there are no arbitrage opportunities there is an equivalent probability measure such that the price of an asset is proportional of the expected value of the future payoffs
- Although $\bar{q}$ is mysterious we can calculate it a intuitive way.

Arbitrage asset pricing

And replicating portfolios

Proposition 8

Assume a financial market $(\mathbf{S}, \mathbf{V})$ *is complete. Then the* ***price of any asset is equal to the cost of its replicating portfolio build with*** N ***independent assets****.*

- Consider an asset with payoff $\mathbf{V}^a = (v_1^a, \ldots\ v_N^a)^\top$ and assume that $\dim(\mathbf{V}) = N$
- Then we can build an unique replicating portfolio, θ^a, such that $\mathbf{V}\,\theta^a = \mathbf{V}^a \iff \theta^a = \mathbf{V}^{-1}\,\mathbf{V}^a$
- Then the price of the asset should be

$$S^a = \mathbf{S}\,\theta^a$$

- This formula can be used to price both existing and new (and therefore redundant) assets

Arbitrage asset pricing

Existence of risk-free asset

Proposition 9

Assume there are no arbitrage opportunities and there is a risk-free bond with price $1/(1+i)$ *and face value* 1*. Then the price of any asset satisfies the relationship*

$$S_j = \frac{1}{1+i}\mathbb{E}^Q[V_j]$$

Exercise: prove this.

Arbitrage asset pricing

Existence of risk-free asset

Proposition 10

Assume there are ***no arbitrage opportunities*** *and there is a risk-free asset. Then* ***there is a (market) probability measure*** *such that the expected returns of every asset is equal to the return of the risk-free asset.*

- Proof. From Proposition 8 we have

$$1 + i = \mathbb{E}^Q[R_j], \text{ for any, } j = 1, \ldots, K$$

As this holds for any asset, therefore π^Q has the property

$$1 + i = \mathbb{E}^Q[R_1] = \ldots = \mathbb{E}^Q[R_K]$$

Application

Arbitrage and completeness with two assets

- Assume that $N = 2$ and there is a risky and a risk-free asset

$$\mathbf{S} = \begin{pmatrix} \frac{1}{1+i} & p \end{pmatrix}, \ \mathbf{V} = \begin{pmatrix} 1 & d_1 \\ 1 & d_2 \end{pmatrix}$$

- and that $r_1 < i < r_2$.

  Then markets are complete and there are no arbitrage opportunities.

  The return matrix is

$$\mathbf{R} = \begin{pmatrix} \frac{1}{\frac{1}{1+i}} & \frac{d_1}{p} \\ \frac{1}{\frac{1}{1+i}} & \frac{d_2}{p} \end{pmatrix} = \begin{pmatrix} 1+i & 1+r_1 \\ 1+i & 1+r_2 \end{pmatrix}$$

Application

Arbitrage asset pricing

- If there are risk-free assets and absence of arbitrage opportunities, then any asset with payoff V_k can be priced by redundancy as

$$S_k = q_1 V_{k,1} + q_2 V_{k,2}$$

where

$$q_1 = \frac{i - r_2}{(1+i)(r_1 - r_2)}, \; q_2 = \frac{r_1 - i}{(1+i)(r_1 - r_2)}.$$

- Proof: assume there is a risk-free and a risky asset such that $Q\mathbf{R} = \mathbf{1}^\top$ becomes

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1+i & 1+i \\ 1+r_1 & 1+r_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and if there are no arbitrage opportunities $r_1 < i < r_2$ implying $q_1 > 0$ and $q_2 > 0$.

Application

Simple Black and Scholes (1973) equation

- Consider an european call option with exercise price p on an underlying asset with payoff
$$V_{underlying}^{\top} = (d_1, d_2)$$

- Then the contingent payoff of the option is
$$V_{option}^{\top} = (\max\{d_1 - p, 0\}, \max\{d_2 - p, 0\})$$

- Question: assuming there are no arbitrage opportunities and there is a risk-free asset what is the market price of the option ?
- Answer: the price for a call option with exercise price p is
$$S_{option} = \frac{(r_2 - i)\max\{d_1 - p, 0\} + (r_1 - i)\max\{d_2 - p, 0\}}{(1+i)(r_1 - r_2)}$$

(prove this)

Equity premium

Equity premium

Definition 10

We call *equity premium* *to the difference in the rates of return between the risky and a risk-free asset*

$$r - i = \begin{pmatrix} r_1 - i \\ r_2 - i \end{pmatrix}$$

Definition 11

The *Sharpe index* *is defined as*

$$\frac{\mathbb{E}[r-i]}{\sigma[r]} = \frac{\sum_{s=1}^{2} \pi_s (r_s - i)}{\sqrt{\sum_{s=1}^{2} \pi_s ((r_s - i) - \mathbb{E}[r-i])^2}}$$

Prove that $\sigma[r-i] = \sigma[r]$ where is the standard deviation of the

Risk neutral probabilities

Proposition 11

If there are no arbitrage opportunities then there is a (market) risk neutral probability distribution such that the expected value of the equity risk premium is zero,

$$E^Q[r - i] = 0$$

This is the reason why π_s^Q are called risk neutral probabilities: $\mathbb{P}^Q = (\pi_1^Q, \ldots \pi_N^Q)$ is a probability measure such that the expected value of the equity premium is zero.

Risk neutral probabilities

- Proof: Let

$$\mathbf{R}^\top = \begin{pmatrix} 1+i & 1+i \\ 1+r_1 & 1+r_2 \end{pmatrix}$$

- Then, because $\mathbf{R}^\top Q^\top = \mathbf{1}$

$$\begin{cases} (1+i)q_1 + (1+i)q_2 = 1 \\ (1+r_1)q_1 + (1+r_2)q_2 = 1 \end{cases}$$

- Then $(1+r_1)q_1 + (1+r_2)q_2 = (1+i)q_1 + (1+i)q_2$.
- Therefore

$$q_1(r_1 - i) + q_2(r_2 - i) = 0$$

Risk neutral probabilities

- Proof (cont)Define $\bar{q} = q_1 + q_2$

$$\frac{q_1}{\bar{q}}(r_1 - i) + \frac{q_1}{\bar{q}}(r_2 - i) = 0$$

- Define again $\pi_s^Q = \frac{q_s}{\bar{q}}$. If there are no arbitrage opportunities, then $q_s > 0$ and $\pi_s^Q > 0$. Because $\sum_{s=1}^{2} \pi_s^Q = 1$ then π_s^Q are probabilities.
- At last

$$\pi_1^Q(r_1 - i) + \pi_2^Q(r_2 - i) = 0$$

The Sharpe index and the SDF

Proposition 12

Assume there are no arbitrage opportunities and there is a risk-free asset. Then the Sharpe index satisfies the relationship

$$\frac{\mathbb{E}[r-i]}{\sigma[r]} = -\rho_{r-i,m}\left(\frac{\mathbb{E}[M]}{\sigma[M]}\right)^{-1}$$

where $\rho_{r-i,m}$ is the coefficient of correlation between the equity premium and the stochastic discount factor.

The Sharpe index and the SDF

- Proof: Using our previous definition of the stochastic discount factor $m_s = q_s/\pi_s$ then

$$\mathbb{E}[M(r-i)] = 0$$

- But $\mathbb{E}[M(r-i)] = Cov[M(r-i)] + \mathbb{E}[M]\,\mathbb{E}[r-i]$
- Using the definition of correlation

$$\rho_{r-i,m} = \frac{Cov[M(r-i)]}{\sigma[M]\,\sigma[r-i]} \in (-1,1)$$

- Then $\rho_{r-i,m}\,\sigma[M]\,\sigma[r-i] + \mathbb{E}[M]\,\mathbb{E}[r-i] = 0$

The equity premium and DGE models

- **Intuition**: the Sharpe index is equal symmetric to the product between coefficient of variation of the stochastic discount factor and correlation coefficient between the equity premium and the stochastic discount factor, $\rho_{r-i,m}$
- The coefficient of variation of the stochastic discount factor contains the aggregate market value of risk.
- $\mathbb{E}[M]/\sigma[M]$ can be derived from simple DSGE models (as we will see next)
- Observe that the Sharpe index satisfies

$$\frac{\mathbb{E}[R - R^f]}{\sigma[R]}$$

where $R^f = 1 + i$ is the risk-free return and R can be taken as a return on a market index. (see `http://web.stanford.edu/~wfsharpe/art/sr/SR.htm`