Foundations of Financial Economics
Two period GE: heterogeneous households

Paulo Brito

[1] pbrito@iseg.ulisboa.pt
University of Lisbon

May 4, 2022

Topics for today

- Types of heterogeneity
- AD equilibrium with heterogeneous households
  - with heterogeneity in endowments
  - with heterogeneity in endowments and preferences
- Insurance in a finance economy

Types of heterogeneity

Types of heterogeneity
Sources

- Consider an economy with I-households (or different types of households) $\mathcal{I} = \{1, \ldots, I\}$
- **Sources of heterogeneity**:
  there is heterogeneity if there are at least two households j and l in $\mathcal{I}$ such that they differ in:
  - **information**: their probability spaces are different $(\Omega^j, P^j) \neq (\Omega^l, P^l)$
  - **preferences**: their degree of impatience differ, $\beta^j \neq \beta^l$, or attitudes towards risk , $u^j(.) \neq u^l(.)$, differ
  - **endowments**: their endowments differ: $y^j = \{y_0^j, Y_1^j\} \neq y^l = \{y_0^l, Y_1^l\}$ (by level, distribution over time or distribution across states of nature)

Types of heterogeneity

Heterogeneity in endowments: idiosyncratic and aggregate

We distinguish:

- Household (or idiosyncratic) endowment process

$$y^i = \{y_0^i, Y_1^i\}, \text{ where } Y_1^i = \left(y_{11}^i, \dots, y_{1N}^i\right)^\top$$

- Aggregate endowment process

$$y = \{y_0, Y_1\}, \text{ where } Y_1 = \left(y_{11}, \dots, y_{1N}\right)^\top$$

where $y_0 = y_0^1 + \ldots + y_0^I$ and

$$Y_1 = \begin{pmatrix} y_{1,1}^1 + \ldots + y_{1,1}^I \\ y_{1,s}^1 + \ldots + y_{1,s}^I \\ \cdots \\ y_{1,N}^1 + \ldots + y_{1,N}^I \end{pmatrix} = \begin{pmatrix} y_{1,1} \\ \cdots \\ y_{1,s} \\ \cdots \\ y_{1,N} \end{pmatrix}$$

Types of uncertainty in endowments

Types of uncertainty: related with state-dependency

- **idiosyncratic certainty**: if $Y_1^j = y_1^j$ and $Y_1^l = y_1^l$ are state-independent
- **idiosyncratic uncertainty**: if Y_1^j and Y_1^l are state-dependent and $Y_1^j \neq Y_1^l$
- **aggregate certainty**: if $Y_1 = \sum_{i=1}^{I} Y_1^i$ **is state-independent**, i.e., $y_{1,s} = \bar{y}_1$ for all $s = 1, \ldots, N$
- **aggregate uncertainty**: if $Y_1 = \sum_{i=1}^{I} Y_1^i$ **is state-dependent**, i.e., there is a pair of components of Y_1 such that $y_{1,s} \neq y_{1,s'}$ for all $s, s^{'} = 1, \ldots, N$

Types of uncertainty in endowments

Possible cases

idiosyncratic \ aggregate	certainty	uncertainty
certainty	(1)	
uncertainty	(2)	(3)

(1) Deterministic (full information)

(2) Stochastic at the household level but deterministic at the aggregate level

(3) Stochastic at both the household and the aggregate level

the other case does not exist

Types of uncertainty in endowments

GE features

Characteristics of the GE:

(1) **idiosyncratic and aggregate certainty**: the GE is deterministic (both consumption at $t = 1$ and discount factor are deterministic)
(2) **idiosyncratic uncertainty and aggregate certainty**: the GE is partially stochastic (consumption at $t = 1$ can be stochastic or deterministic and the market discount factor is deterministic). Complete insurance is possible
(3) **idiosyncratic and aggregate uncertainty**: the GE is stochastic (both consumption at $t = 1$ and the market discount factor are stochastic). There can only exist partial insurance (at best)

Heterogeneity and uncertainty in endowments

(1) can exist with **homogeneous or heterogenous** households' economies, but equilibrium may differ in the two cases

(3) can exist with **homogeneous or heterogenous** households' economies, but equilibrium may differ in the two cases

(2) can only exist in **heterogenous** households' economies

AD equilibrium with heterogeneous households: general case

DSGE for an heterogenous AD economy

Definition 1

General equilibrium (GE): Is the sequence of ***distributions*** *of consumption* $\{(c_0^{*1}, \dots c_0^{*I}), (C_1^{*1}, \dots C_1^{*I})\}$ *and prices* Q *such that:*

1. *every consumer* $i = 1, \dots, I$ *determines the optimal sequence* $\{c_0^i, C_1^i\}$ *by solving the problem*

$$\max_{\{c_0^i, C_1^i\}} \mathbb{E}_0^i \left[u^i(c_0^i) + \beta^i u^i(C_1^i) \right]$$

$$c_0^i - y_0^i + q(C_1^i - {Y^i}_1) = 0$$

given Q *and* $\{y_0^i, Y_1^i\}$*;*

2. *the good market clears in every period:*

$$C_t = Y_t \iff \sum_{i=1}^{I} C_t^i, = \sum_{i=1}^{I} Y_t^i, \text{ for } t = 0, 1$$

Case 1: General equilibrium with heterogeneity in endowments

DSGE for an AD economy with heterogeneous households

Definition 1

*General equilibrium (GE): Is the sequence of **distributions** of consumption* $\{(c_0^{*1}, \dots c_0^{*I}), (C_1^{*1}, \dots C_1^{*I})\}$ *and prices* Q *such that:*

1. *every consumer* $i = 1, \dots, I$ *determines the optimal sequence* $\{c_0^i, C_1^i\}$ *by solving the problem*

$$\max_{\{c_0^i, C_1^i\}} \mathbb{E}_0 \left[u(c_0^i) + \beta u(C_1^i) \right]$$

$$c_0^i - y_0^i + q(C_1^i - Y^i{}_1) = 0$$

   given q *and* $\{y_0^i, Y_1^i\}$*;*

2. *the good market clears in every period:*

$$C_t = Y_t \iff \sum_{i=1}^{I} C_t^i, = \sum_{i=1}^{I} Y_t^i, \; for \; t = 0, 1$$

Assumption and discussion

Assumptions: logarithmic preferences, and **idiosyncratic** uncertainty as regards endowments Y_1^i.

Question: what are the properties of the equilibrium stochastic discount factor ?

Method of determination: we may have to solve explicitly the consumers' problems (exception: if the only source of inequality is related to endowments)

Determination: household optimality conditions

1. household' $i \in \mathcal{I}$ problem

$$\max_{c_0^i, c_{11}^i, \dots, c_{1N}^i} \ln\left(c_0^i\right) + \beta \sum_{s=1}^{N} \pi_s \ln\left(c_{1s}^i\right)$$

subject to

$$c_0^i + \sum_{s=1}^{N} q_s c_{1s}^i \leq h^i \equiv y_0^i + \sum_{s=1}^{N} q_s y_{1s}^i$$

where q_s is given to the consumer.

2. optimal consumption of household $i \in \mathcal{I}$ (without satiation)

$$\begin{aligned} c_0^i &= \frac{1}{1+\beta} h^i \\ c_{1s}^i &= \frac{\pi_s \beta}{q_s(1+\beta)} h^i \end{aligned}$$

Determination: aggregation

1. Aggregate supply

$$y_0 = \sum_{i=1}^{I} y_0^i$$

$$y_{1,s} = \sum_{i=1}^{I} y_{1,s}^i, \; s = 1, \ldots, N$$

2. Aggregate demand

$$c_0 = \sum_{i=1}^{I} c_0^i = \frac{1}{1+\beta} h$$

$$c_{1,s} = \sum_{i=1}^{I} c_{1,s}^i = \frac{\beta \pi_s}{q_s (1+\beta)} h, \; s = 1, \ldots, N$$

3. Aggregate wealth

$$h = \sum_{i=1}^{I} h^i = \sum_{i=1}^{I} \left(y_0^i + \sum_{s=1}^{N} q_s \, y_{1s}^i \right) = y_0 + \sum_{s=1}^{N} q_s y_{1,s}$$

Determination: market clearing

1. Market clearing conditions

$$c_0 = y_0 \Leftrightarrow \frac{1}{1+\beta} h = y_0$$
$$c_{1,s} = y_{1,s} \Leftrightarrow \frac{\beta \pi_s}{q_s(1+\beta)} h = y_{1,s}, \; s = 1, \dots, N$$

2. Then: the equilibrium AD price is

$$q_s = \beta\, \pi_s \, \frac{y_0}{y_{1,s}} = \frac{\beta\, \pi_s}{1 + \gamma_{1,s}}, \text{for } s = 1, \dots, N$$

where $y_{1,s} = (1 + \gamma_s) y_0$ and γ_s is the common belief on the rate of growth of the aggregate endowment in state s.

Characterization

Proposition 1

Consider a AD economy in which there is heterogeneity in endowments and homogeneity in preferences and information. Then the equilibrium stochastic discount factor is independent of the distribution of income.

Interpretation

Interpretation: the **equilibrium discount factor** $M = (m_1, \dots m_N)$ where

$$m_s = \frac{\beta}{1+\gamma_s}, \text{ for } s = 1, \dots, N$$

- is independent of the distribution of endowments among households (only depends on the growth factor of the **aggregate endowments**)

Interpretation

For a log utility function the stochastic discount factor is stochastic and depends on the distribution for the rate of growth of the aggregate endowment

$$M = \beta\,(1+\Gamma)^{-1}$$

Recall $(1+\Gamma) = \dfrac{Y_1}{y_0}$

$$1+\Gamma = \begin{pmatrix} \frac{\sum_{i=1}^{I} y_{1,1}^{i}}{\sum_{i=1}^{I} y_{0}^{i}} \\ \dots \\ \frac{\sum_{i=1}^{I} y_{1,s}^{i}}{\sum_{i=1}^{I} y_{0}^{i}} \\ \dots \\ \frac{\sum_{i=1}^{I} y_{1,N}^{i}}{\sum_{i=1}^{I} y_{0}^{i}} \end{pmatrix} = \begin{pmatrix} \frac{y_{1,1}}{y_0} \\ \dots \\ \frac{y_{1,s}}{y_0} \\ \dots \\ \frac{y_{1,N}}{y_0} \end{pmatrix} = \begin{pmatrix} 1+\gamma_{1,1} \\ \dots \\ 1+\gamma_{1,s} \\ \dots \\ 1+\gamma_{1,N} \end{pmatrix}$$

Cases: aggregate uncertainty and aggregate certainty

- **Aggregate uncertainty**:
  if then $Y_1 = y_1$ is state dependent, i.e, $y_{1,1} \neq y_{1,s} \neq y_{1,N}$. Then M is **state-dependent** (stochastic)

$$m_s = \frac{\beta}{1+\gamma_s}, \text{ for } s = 1, \ldots, N$$

- **Aggregate certainty**:
  if $Y_1 = y_1$ is state independent, i.e, $y_{1,s} = y_1$ for every s (even if there is idiosyncratic uncertainty) then M is state-independent (i.e, deterministic):

$$m_s = m = \frac{\beta}{1+\gamma}, \text{ for all } s = 1, \ldots, N.$$

This is the case even if there is idiosyncratic uncertainty.

Existence of perfect insurance

Proposition 2

Consider the previous economy, in which there is idiosyncratic uncertainty but aggregate certainty (i.e, $Y_1 = y_1$ for all states $s = 1, \ldots, N$). Then there is ***perfect insurance*** *(that is consumption at time $t = 1$ is state independent for every household).*

Next we prove that

$$c_{1s}^{*i} = c_1^{*i} = \frac{1+\gamma}{1+\beta} h^{*i}, \ \forall s = 1, \ldots, N, \text{ for any } i \in \mathcal{I}$$

is state-independent if $Y_1 = y_1 = (1+\gamma) y_0$

Case 1: General equilibrium with heterogeneity in endowments

Proof of Proposition 2

- Consumption for $t = 1$ for any household in equilibrium (i.e, for m_s at the equilibrium level)

$$c^i_{1s} = \frac{\beta}{m^*_s(1+\beta)} h^i = \frac{1+\gamma_s}{1+\beta} h^i$$

- If there is no aggregate uncertainty $1+\gamma_s = 1+\gamma$ for every $s = 1, \ldots, N$, then consumption is state-independent

$$c^i_{1s} = \frac{1+\gamma}{1+\beta} h^i, \text{ for any } s = 1, \ldots, N$$

there is **complete insurance** although Y^i_1 is stochastic

Consumption distribution

Proposition 3

In equilibrium, the weight of households' i consumption relative to aggregate consumption is stationary (i.e, time-independent), state independent, and is equal to its share of aggregate wealth.

Consumption distribution

- The **equilibrium distribution** of human wealth for household i is (if we substitute m_s)

$$h^{*i} = y_0^i + \beta \sum_s \frac{\pi_s y_{1,s}^i}{1+\gamma_s} = y_0^i \left(1 + \beta \sum_{s=1}^{N} \pi_s \frac{1+\gamma_s^i}{1+\gamma_s} \right) \quad i = 1, \ldots, I$$

- The equilibrium aggregate human wealth is

$$h^{*} = y_0 + \beta \sum_s \frac{\pi_s y_{1,s}}{1+\gamma_s} = y_0 \left(1 + \beta \sum_{s=1}^{N} \pi_s \frac{1+\gamma_s}{1+\gamma_s} \right) = y_0(1+\beta)$$

- The distribution of consumption at $t=0$ is

$$\frac{c_0^{*i}}{c_0} = \frac{1}{1+\beta} \frac{h^{*i}}{y_0} = \frac{h^{*i}}{h} = \frac{y_0^i}{y_0} \left(\frac{1 + \beta \sum_{s=1}^{N} \pi_s \frac{1+\gamma_s^i}{1+\gamma_s}}{1+\beta} \right)$$

- and at $t=1$ is

$$\frac{c_{1s}^{*i}}{c_{1s}} = \frac{1+\gamma_s}{1+\beta} \frac{h^{*i}}{y_{1s}} = \frac{1}{1+\beta} \frac{h^{*i}}{y_0} = \frac{h^{*i}}{h}, \text{for all } s = 1, \ldots, N$$

Insurance and distribution of consumption

- Observation: the fact that every consumer can perfectly insure (i.e, the distribution of consumption for every consumer is state independent) does not mean that the distribution of consumption among households is symmetric
- The consumption for every household is dependent of their specific (idiosyncratic wealth)

$$c_1^i = \frac{1+\gamma}{1+\beta} h^i$$

- **There is perfect insurance but not perfect equality in consumption.**

Example 1: homogeneous household economy

	$t=0$	$t=1$	
		$s=1$	$s=2$
y^a	50	45	55
y^b	50	45	55
$y = y^a + y^b$	100	90	110
m		**1.089**	**0.891**
c^a	50	45	55
c^b	50	45	55

Table: Two homogeneous households (a and b). Common parameter: $\beta = 1/1.02$. Idiosyncratic and aggregate uncertainty

Example 2: heterogenous households and aggregate uncertainty

	$t=0$	$t=1$	
		$s=1$	$s=2$
y^a	30	27	33
y^b	70	63	77
$y=y^a+y^b$	100	90	110
m		**1.089**	**0.891**
c^a	30	27	33
c^b	70	63	77

Table: Two heterogeneous households (a and b). Common parameter: $\beta = 1/1.02$. Idiosyncratic and aggregate uncertainty. In this case there is no insurance

Example 3: idiosyncratic uncertainty and aggregate certainty

	$t = 0$	$t = 1$	
		$s = 1$	$s = 2$
y^a	50	45	55
y^b	50	55	45
y	100	**100**	**100**
m		**0.98**	**0.98**
c^a	50	50	50
c^b	50	50	50

Table: Two heterogeneous households (a and b). Common parameter: $\beta = 1/1.02$. Idiosyncratic uncertainty and aggregate certainty: **perfect insurance**

Conclusions

- **Summing up**:
  - if there is **aggregate certainty** then:
    the stochastic discount factor is **deterministic** and there is **perfect insurance** c_1^i is state-independent (because γ is state-independent);
  - if there is **aggregate uncertainty** then:
    the stochastic discount factor is **stochastic** and there is **not** perfect insurance c_1^i is state-dependent (because γ is state-dependent);
- Then:
  - only aggregate variables determine the stochastic discount factor;
  - the **distribution of income is irrelevant** for the determination of the stochastic discount factors
  - **Those results extend to a finance economy with complete asset markets.** (see next)

Comparing a representative agent with a heterogeneous agent economy

- In a representative agent economy we can only have two cases
  - Aggregate and individual (idiosyncratic) certainty
  - Both aggregate and individual (idiosyncratic) uncertainty. In this case there is not insurance
- In a heterogeneous household economy we have three cases
  - Aggregate and individual (idiosyncratic) certainty
  - Both aggregate and individual (idiosyncratic) uncertainty. In this case there is some insurance
  - Aggregate certainty and individual (idiosyncratic) uncertainty. In this case there can be **perfect insurance** and redistribution.

Case 2: General equilibrium with heterogeneity in endowments and preferences

DSGE for an heterogeneous AD economy

Definition 1

General equilibrium (GE): Is the sequence of ***distributions*** *of consumption* $\{(c_0^{*1}, \dots c_0^{*I}), (C_1^{*1}, \dots C_1^{*I})\}$ *and prices* Q *such that:*

1. *every consumer* $i = 1, \dots, I$ *determines the optimal sequence* $\{c_0^i, C_1^i\}$ *by solving the problem*

$$\max_{\{c_0^i, C_1^i\}} \mathbb{E}_0 \left[u(c_0^i) + \beta^i \, u(C_1^i) \right]$$

$$c_0^i - y_0^i + q(C_1^i - {Y^i}_1) = 0$$

   given q *and* $\{y_0^i, Y_1^i\}$*;*

2. *the good market clears in every period:*

$$C_t = Y_t \iff \sum_{i=1}^{I} C_t^i, = \sum_{i=1}^{I} Y_t^i, \text{ for } t = 0, 1$$

Assumptions

- Homogeneous utility function: $u(c) = \ln(c)$
- Heterogeneity in **impatience** (β^i). Let the distribution of psychological discount factors be represented by

  $$B = (\beta^1, \ldots, \beta^i, \ldots \beta^I)$$

  household j is more patient than l: $\beta^j > \beta^l$
- **idiosyncratic uncertainty as regards endowments** Y_1^i

Household's problem

- Assuming an intertemporally additive VNM utility functional
- The consumption problem is now

$$\max_{c_0^i, c_{11}^i, \ldots, c_{1N}^i} \ln\left(c_0^i\right) + \beta^i \sum_{s=1}^{N} \pi_s \ln\left(c_{1s}^i\right)$$

subject to

$$c_0^i + \sum_{s=1}^{N} \pi_s m_s c_{1s}^i \leq h^i \equiv y_0^i + \sum_{s=1}^{N} \pi_s m_s y_{1s}^i$$

Solution to the household i problem

- The optimal consumption process for household i is

$$
\begin{aligned}
c_0^i &= \frac{1}{1+\beta^i} h^i, \ i = 1, \ldots, I \\
c_{1s}^i &= \frac{\beta^i}{m_s(1+\beta^i)} h^i, \ i = 1, \ldots, I
\end{aligned}
$$

Endowment distribution

- Define the process for the shares of household i in the aggregate endowments $\{\phi_0^i, \Phi_1^i\}$,
- At time $t = 0$ we have

$$\phi_0^i = \frac{y_0^i}{y_0} = \frac{y_0^i}{\sum_{i=1}^{I} y_0^i} \text{ for } i = 1, \ldots, I$$

  where $\sum_{i=1}^{I} \phi_0^i = 1$ and
- At time $t = 1$ we have

$$\phi_{1,s}^i = \frac{y_{1,s}^i}{y_{1,s}} = \frac{y_{1,s}^i}{\sum_{i=1}^{I} y_{1,s}^i} \text{ for } s = 1, \ldots, N, \quad i = 1, \ldots, I$$

  where $\sum_{i=1}^{I} \phi_{1,s}^i = 1$ for all $s = 1, \ldots, N$

Wealth distribution

- Then the human wealth of consumer i can be written as

$$h^i = \left(\phi_0^i + \sum_{s=1}^{N} m_s \pi_s (1+\gamma_s) \phi_{1,s}^i\right) y_0, \; i = 1, \ldots, I \tag{1}$$

because $y_0^i = \phi_0^i y_0$ and $y_{1s}^i = \phi_{1s}^i y_{1s} = \phi_{1s}^i (1+\gamma_s) y_0$

Aggregate consumption

- aggregate consumption in period $t = 0$

$$c_0 = \sum_{i=1}^{I} c_0^i = \sum_{i=1}^{I} \frac{h^i}{1+\beta^i}$$

- aggregate consumption in period $t = 1$

$$c_{1,s} = \sum_{i=1}^{I} c_{1,s}^i = \frac{1}{m_s}\left(\sum_{i=1}^{I} \frac{\beta^i h^i}{1+\beta^i}\right) = (1+\gamma_s)y_0,\ \ s = 1,\ldots,N$$

- Observation: now we are summing not only over wealth h^i but also over the distribution of the discount factors β^i (B)

Market clearing conditions

- The market clearing conditions are

$$\begin{aligned} c_0 &= y_0 \Leftrightarrow \sum_{i=1}^{I} \frac{h^i}{1+\beta^i} = y_0 \\ c_{1,s} &= y_{1,s} \Leftrightarrow \frac{1}{m_s} \left(\sum_{i=1}^{I} \frac{\beta^i}{1+\beta^i} h^i \right) = (1+\gamma_s) y_0, \; s = 1, \ldots, N \end{aligned}$$

- Then (using equation (1))

$$m_s (1+\gamma_s) = \sum_{i=1}^{I} \frac{\beta^i}{1+\beta^i} \left(\phi_0^i + \sum_{s=1}^{N} m_s \pi_s (1+\gamma_s) \phi_{1,s}^i \right)$$

Market clearing conditions

- Define

$$\varphi_0 \equiv \varphi_0(B) \equiv \sum_{i=1}^{I} \frac{\beta^i}{1+\beta^i}\phi_0^i,$$

$$\varphi_{1,s} \equiv \varphi_{1,s}(B) \equiv \sum_{i=1}^{I} \frac{\beta^i}{1+\beta^i}\phi_{1,s}^i$$

where B is the distribution of the psychological discount factors

$$B = (\beta^1, \dots \beta^i, \dots, \beta^I)$$

- Then, the equilibrium conditions for $t = 1$ can be written as (check !)

$$m_s(1+\gamma_s) = \varphi_0(B) + \sum_{s=1}^{N} \pi_s m_s(1+\gamma_s)\varphi_{1,s}(B),\ \ s = 1, \dots, N$$

- As the right-hand-side expression is a constant then $m_1(1+\gamma_1) = \dots = m_s(1+\gamma_s) = \dots = m_N(1+\gamma_N)$.

Equilibrium stochastic discount factor

- Then

$$\sum_{s=1}^{N} \pi_s m_s (1+\gamma_s)\varphi_{1,s}(B) = m_s(1+\gamma_s)\mathbb{E}[\varphi_1(B)]$$

  for any s

- Then we determine the **equilibrium discount factor**

$$\boxed{m_s = \tilde{\beta}(B)\frac{1}{1+\gamma_s}, \text{where } \tilde{\beta}(B) \equiv \left(\frac{\varphi_0(B)}{1-\mathbb{E}[\varphi_1(B)]}\right)}$$

- where $\varphi_0(B)$ and $\varphi_{1,s}(B)$ are the endowment-weighted averages of $0 < \dfrac{\beta^i}{1\beta^i} < 1$. Then $0 < \varphi_0(B) < 1$ and $0 < \mathbb{E}[\varphi_1(B)] < 1$

Conclusions

:

- if there is heterogeneity in the psychological discount factor and there is idiosyncratic uncertainty then the **equilibrium stochastic discount factor is formally similar to the homogeneous case**: it multiplies a weighted psychological discount factor with the inverse of the endowment growth factor;
- the weighted psychological discount factor, $\tilde{\beta}$ **depends upon the distribution of income** but is state-independent and constant;
- If there is **no** aggregate uncertainty then the stochastic discount factor m is **state-independent**.

Example 2 bis: heterogenous households and aggregate uncertainty

	$t=0$	$t=1$	
		$s=1$	$s=2$
y^a	30	27	33
y^b	70	63	77
y	100	90	110
m		**1.094**	**0.895**
c^a	30.2	26.8	32.8
c^b	69.8	63.2	77.2

Table: Two heterogeneous households (a and b). Heterogenous preferences: $\beta^a = 1/1.025$ $\beta^b = 1/1.015$. Idiosyncratic and aggregate uncertainty

Example 3 bis: idiosyncratic uncertainty and aggregate certainty

	$t=0$	$t=1$	
		$s=1$	$s=2$
y^a	50	45	55
y^b	50	55	45
y	100	**100**	**100**
m		**0.9804**	**0.9804**
c^a	50.2	49.8	49.8
c^b	49.8	50.2	50.2

Table: Two heterogeneous households (a and b) where b is more patient than a: $\beta^a = 1/1.025$ $\beta^b = 1/1.015$. There is both idiosyncratic uncertainty and aggregate certainty: **perfect insurance**. But as b is more patient the time profile of consumption is different from a which is less patient.

Insurance and asset pricing in a heterogeneous agent finance economy

Perfect insurance in a finance economy

Proposition 4

Assume a finance economy in which asset markets are complete. Assume further that there is idiosyncratic uncertainty regarding endowments and that the aggregate endowment is state independent (i.e., there is no aggregate uncertainty).
Then there is ***perfect insurance****.*

Perfect insurance in a finance economy

Proof

- Assume again that there is **heterogeneity in endowments**
- Remember the problem for household i in a **finance economy**

$$\max_{i_0, C_1^i)\}, \theta^i} u(c_0^i) + \beta \mathbb{E}[u(C_1^i)]$$

subject to

$$\begin{aligned} c_0^i &= y_0^i - S\,\theta^i \\ C_1^i &= Y_1^i + V\theta^i \end{aligned}$$

for $C_1 = (c_{1,s})_{s=1}^{N}$ and $\theta = (\theta_j)_{j=1}^{K}$

- There are two sources of uncertainty: endowments $Y_1 = (\dots, y_{1,s}, \dots)^\top$ and financial $V = \left(v_{j,s}\right)_{j=1,s=1}^{N,K}$
- There is **idiosyncratic uncertainty** if the endowment is uncertain and household-specific

Perfect insurance in a finance economy

Proof, cont.

- If **markets are complete**, then:
  - $\pi_s \hat{m}_s = \hat{q}_s$ and $\hat{Q} = Q = S\,V^{-1}$: the state price is equal to the implicit market state price.
  - and the f.o.c.for household i are

$$\beta u^{'}(c^i_{1,s}) = m_s u^{'}(c^i_0),\ s = 1, \dots, N$$
$$c^i_0 + \mathbb{E}[\hat{M}\,C^i_1] = H^i_0 \equiv y^i_0 + \mathbb{E}[\hat{M}\,Y^i_1]$$

- There is **perfect insurance**, if consumption is the same for every state of nature

$$c_{1,s} = c_1,\ \text{for every } s = 1, \dots, N$$

- Using the f.o.c, we see that **there is perfect insurance if and only if the equilibrium stochastic discount factor is state-independent**, i.e,

$$m_s = m \text{ for every } s = 1, \dots, M$$

Perfect insurance in a finance economy

Proof, cont.

- The equilibrium stochastic discount factor is

$$m_s = \beta \frac{u'(y_{1,s})}{u'(y_0)}, \; s = 1, \dots, N$$

where the **aggregate endowments** are

$$y_0 = \sum_{i=1}^{I} y_0^i$$

$$Y_1 = \sum_{i=1}^{I} Y_{1,s}^i = \begin{pmatrix} \dots \\ \sum_{i=1}^{I} y_{1,s}^i \\ \dots \end{pmatrix} = \begin{pmatrix} \dots \\ y_{1,s} \\ \dots \end{pmatrix}$$

- Therefore $m_s = m$ if and only if the **aggregate endowment is state-independent**, that is

$$y_{1,s} = y_1, \text{for every } s = 1, \dots, N.$$

if there is **no aggregate uncertainty**.

Perfect insurance in a finance economy

- We say there is **perfect insurance** if any consumer although having an uncertain endowment, by trading in the financial markets, he/she can finance a state-independent consumption at time $t = 1$.
- In the previous case $\mathbb{V}[Y_1^i] > \mathbb{V}[C_1^i] = 0$: the variance of consumption is zero while the variance of endowments is positive
- There is **imperfect insurance** if $\mathbb{V}[Y_1^i] > \mathbb{V}[C_1^i] > 0$: although the variance of consumption is positive, it is smaller than the one of endowments. This may be possible even if markets are incomplete.

Equilibrium asset price: aggregate uncertainty

- If markets are complete then (as we saw in this slide) the equilibrium price for asset j is

$$S_j^{eq} = \mathbb{E}[M^{eq}\, V_j] \iff \mathbb{E}[M^{eq}\, R_j^{eq}] = 1$$

- If there is **aggregate certainty** we have

$$M^{eq} = m^{eq} = \beta \, \frac{u'(y_1)}{u'(y_0)}$$

then

$$\boxed{S_j^{eq} = m^{eq}\, \mathbb{E}[V_j] \iff m^{eq} = \frac{1}{\mathbb{E}[R_j^{eq}]}}$$

- the equilibrium asset price is proportional to the expected value of the payoff
- the equilibrium discount factor is equal to the inverse of the rate of return of any asset
- and $\mathbb{E}[R_1^{eq}] = \ldots = \mathbb{E}[R_K^{eq}]$ the average return is equalized for all assets