Foundations of Financial Economics
Two period GE: limited participation

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Differences with the benchmark model

- The participation in the risky asset market is not proportional to household wealth: U.S data similar shape for different countries
- Potential explanations: differences in patience, risk aversion, information, wealth (if there are fixed costs in gathering information for participating), knowledge
- **In this lecture**: difference in beliefs together with a **friction** (households cannot have short positions in assets)
- Wealth takes the form of financial wealth only
- There is positive net wealth (external finance: external money and another risky asset in positive net supply)

Possible extensions and applications

- The model is in the other extreme of the benchmark model we studied until this point (free positions vs no short positions)
- A half-way model would consider that internal finance (short positions) is possible, but it is constrained by **collateral constraints**: short positions are limited by the existence of a long position in another asset that should be offered as collateral (for instance money)
- This partly explains
  - the demand for liquidity, for instance by firms,
  - the characteristics of the 2008 and the Euro crises (a crisis may be brewing without signs in the behavior of interest rates)
  - and the increasing consideration of the so-called **balance-sheet effects** in macro-finance models.

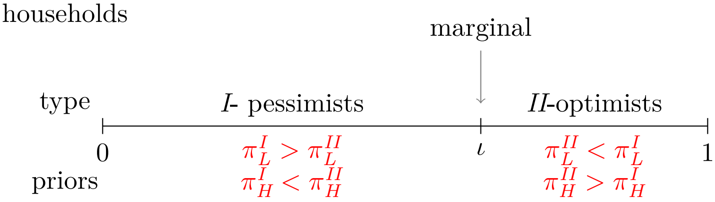
Topics

- Environment
- Types of households: participation in the risky asset market
- Endogenous market participation (related to priors on the likelihood of the good and bad state: pessimists and optimists)
- The equilibrium interest rate depends on the distribution of the participation in the risky asset market
- Interest rate responds to good and bad news in an asymmetric way
- **Welcome to financial economics post 2008 !**

Environment: fundamentals

- Information: two-period binomial tree with two states $s = L, H$
  - **good state** $s = H$ (high return for the risky asset)
  - **bad state** $s = L$ (low return for the risky asset)
- There is a continuum of households with mass equal to one, $i \in \mathcal{I} = [0, 1]$
- **Heterogeneity in priors regarding the states of nature**. There are two groups of households
  - **pessimists** $i \in [0, \iota]$ giving more weight to $s = L$
  - **optimists** $i \in [\iota, 1]$ giving more weight to $s = H$
  - marginal household, indexed by $i = \iota$, is neither pessimist nor optimist (very small group).
- ι can also be seen as the proportion of pessimists in the population

Environment: fundamentals



- where π_s^i is the belief of a household of type i as regards the stats of nature s
- off course: $\pi_L^I + \pi_H^I = \pi_L^{II} + \pi_H^{II} = 1$

Markets

- I assume a finance economy in which there are two assets: **money** and a **risky asset**
- The asset prices and payoffs are

$$\mathbf{S} = (1, S_2), \ \mathbf{V} = \begin{pmatrix} 1 & v_{2L} \\ 1 & v_{2H} \end{pmatrix}$$

  L is a bad state and H is a good state: $v_{2L} < v_{2H}$
- Therefore, the return matrix is

$$\mathbf{R} = (R_1, R_2) = \begin{pmatrix} 1 & R_{2L} \\ 1 & R_{2h} \end{pmatrix}$$

  where $R_{2,s} = v_{2s}/S_2$ for $s = L, H$
- Assume there are **no arbitrage opportunities**:

$$R_{2L} < 1 < R_{2H}$$

Distributions

- We need to consider a given initial distribution of wealth among types of households and among assets

asset / agent	risk-free	risky	wealth
pessimists	$a_{0,1}^{I}$	$a_{0,2}^{I}$	$a_0^I = a_{0,1}^I + S_2\, a_{0,2}^I$
pessimists	$a_{0,1}^{II}$	$a_{0,2}^{II}$	$a_0^{II} = a_{0,1}^{II} + S_2\, a_{0,2}^{II}$
total for asset	$a_{0,1}$	$a_{0,2}$	

Generic household problem

- The problem for household of type $i \in [0,1]$ is

$$\max_{c_0^i, C_1^i, \theta^i} u(c_0^i) + \beta \mathbb{E}^i[u(C_1^i)]$$

$$\text{subject to}$$

$$c_0^i + \theta_1^i + S_2 \theta_2^i = y_0^i + a_0^i$$

$$c_{1,s}^i = \theta_1^i + v_{2s} \theta_2^i, \; s = L, H$$

$$\theta_1^i \geq 0 \text{ (friction: no short position allowed)}$$

$$\theta_2^i \geq 0 \text{ (friction: no short position allowed)}$$

where $\mathbb{E}^i[u(C_1^i)] = \sum_{s \in \{H,L\}} \pi_s^i u(c_{1s}^i)$ (remark: different priors)

- Assume that the initial wealth composition can involve positions in the two types of assets for any type of household

$$a_0^i = a_{0,1}^i + S_2 \, a_{0,2}^i$$

with $a_{0,1}^i > 0$ and $a_{0,2}^i > 0$ for any i given.

Generic household problem

- **Simplifying assumptions**: $c_0^i = y_0^i$ and $Y_1^i = \mathbf{0}$ (meaning: consumption in period 1 is only financed by financial returns)
- Then the constraint in period zero simplifies to:

$$\theta_1^i + S_2 \theta_2^i = a_0^i$$

(meaning that it involves a change in the portfolio such that $\theta_1^i - a_{0,1}^i = S_2 \left(\theta_2^i - a_{0,2}^i \right)$)

Generic household problem

The problem for household of type $i \in [0,1]$ simplifies to

$$\max_{\theta_1^i, \theta_2^i} u(y_0^i) + \beta \mathbb{E}^i \big[u(\theta_1^i + V_2 \theta_2^i) \big]$$

$$\text{subject to}$$

$$\theta_1^i + S_2 \theta_2^i = a_0^i$$

$$\theta_1^i \geq 0$$

$$\theta_2^i \geq 0$$

Solving the generic household's problem

- Lagrangean

$$\mathcal{L}^i = u(y_0^i) + \sum_{s\in\{H,L\}} \beta\pi_s^i u\big(\theta_1^i + v_{2s}\theta_2^i\big) + \lambda^i(a_0^i - \theta_1^i - S_2\theta_2^i) + \\ + \mu_1^i\theta_1^i + \mu_2^i\theta_2^i$$

- Optimality conditions

$$\frac{\partial\mathcal{L}^i}{\partial\theta_1^i} = 0 \Leftrightarrow \lambda^i = \beta\left(\sum_{s\in\{H,L\}} \pi_s^i u'(c_{1s}^i)\right) + \mu_1^i$$

$$\frac{\partial\mathcal{L}^i}{\partial\theta_2^i} = 0 \Leftrightarrow S_2\lambda^i = \beta\left(\sum_{s\in\{H,L\}} \pi_s^i u'(c_{1s}^i)v_{2s}\right) + \mu_2^i$$

- Complementary slackness conditions

$$\mu_1^i\,\theta_1^i = 0,\ \mu_1^i \geq 0,\ \theta_1^i \geq 0$$

$$\mu_2^i\,\theta_2^i = 0,\ \mu_2^i \geq 0,\ \theta_2^i \geq 0$$

Behavior of household of type I (pessimist):

- households of type I sell their initial stock of the risky asset and invest in money: $\theta_1^I > 0$ and $\theta_2^I = 0$
- Then

$$\theta_1^I = a_0^I = a_{0,1}^I + S_2 a_{0,2}^I$$
$$c_{1s}^I = a_0^I, \text{ for } s = L, H$$

is state-independent

- From complementary slackness: $\mu_1^I = 0$ and $\mu_2^I > 0$. Then

$$\lambda^I = \beta \left(\sum_{s \in \{H,L\}} \pi_s^I u'(c_{1s}^I) \right) > \beta \left(\sum_{s \in \{H,L\}} \pi_s^I u'(c_{1s}^I) R_{2s} \right)$$

- Equivalently $\mathbb{E}^I[u'(C_1^I)] > \mathbb{E}^I[u'(C_1^I) R_2]$

Behavior of household of type I (pessimist)

- Observation: defining the utility weighted prior:

$$\pi_s^{i_u} \equiv \frac{\pi_s^i\, u'(c_{1,s}^i)}{\sum_{s\in\{H,L\}} \pi_s^I u'(c_{1s}^I)}, \text{for } s = L, H$$

- As $\pi_s^{i_u} > 0$ and $\pi_L^{i_u} + \pi_H^{i_u} = 1$
- Then $\mathbb{P}^{i_u} = \left(\pi_H^{i_u}, \pi_S^{i_u}\right)$ is a idiosyncratic probability distribution for household i
- Given the return of asset j, $R_j = (R_{j,H}, R_{j,L})$ then

$$\mathbb{E}^i[u'(C_1^i)R_j] = \mathbb{E}^{i_u}[R_j], \text{ for every asset } j = 1, 2 \text{ for household} i = I, II$$

Behavior of household of type I (pessimist)

- Then $\mathbb{E}^I[u'(C_1^I)] > \mathbb{E}^I[u'(C_1^I)R_2]$ is equivalent to

$$\mathbb{E}^I[R_1] > \mathbb{E}^{I_u}[R_2],$$

- Then pessimists have a prior $(\mathbb{P}^{I_u})$, i.e., a risk- probability distribution, such that

$$\boxed{R_1 = 1 > \mathbb{E}^{I_u}[R_2]}$$

household I invests in the risk-free asset because **it finds** its anticipated return on money (i.e., 1) higher than that of the risky asset.

Behavior of household of type II (optimist)

- households of type II sell their initial stock of money and invest in risky asset: $\theta_1^{II} = 0$ and $\theta_2^{II} > 0$
- Then

$$\theta_2^{II} = \frac{a_0^{II}}{S_2} = \frac{a_{0,1}^{II} + S_2\, a_{0,2}^{II}}{S_2}$$

$$c_{1s}^{II} = \frac{v_{2s}}{S_2}\, a_0^{II} = \frac{a_0^{II}}{R_{2s}}$$

is state-dependent (i.e., risky)

- From complementary slackness: $\mu_1^{II} > 0$ and $\mu_2^{II} = 0$. Then

$$\lambda_0^{II} = \beta\left(\sum_{s\in\{H,L\}} \pi_s^{II} u'(c_{1s}^{II}) R_{2s}\right) > \beta\left(\sum_{s\in\{H,L\}} \pi_s^{II} u'(c_{1s}^{II})\right)$$

Then $\mathbb{E}^{II}[u'(C_1^{II})R_1] = \mathbb{E}^{II}[u'(C_1^{II})] < \mathbb{E}^{II}[u'(C_1^{II})R_2]$.

- Then optimists have a different prior ($\mathbb{P}^{II_u}$), i.e., an equivalent probability distribution such that

$$\boxed{\mathbb{E}^{II_u}[R_2] > 1}$$

Marginal household

- households of type I prefer holding money to holding the risky asset because
$$\mathbb{E}^{I_u}[R_2] < 1$$

- households of type II prefer holding the risky asset rather than money because
$$\mathbb{E}^{II_u}[R_2] > 1$$

- By continuity, **the marginal household (with wealth weight of zero)** has a own-probability distribution $\mathbb{P}^{\iota} = \left(\pi^{\iota}, 1 - \pi^{\iota}\right)$ such that
$$\mathbb{E}^{\iota}[R_2] = 1 \Leftrightarrow \boxed{S_2 = \pi^{\iota} v_{2L} + (1 - \pi^{\iota}) v_{2H}} \tag{1}$$

Aggregate demand and supply of assets

- Aggregate demand of the two assets

$$\iota\theta_1^I + (1-\iota)\theta_1^{II} = \iota\, a_0^I + (1-\iota)\, 0 \text{ (risk-free asset)}$$
$$\iota\theta_2^I + (1-\iota)\theta_2^{II} = \iota\, 0 + (1-\iota)\, a_0^{II} \text{(risky asset)}$$

where ι is the proportion of pessimists and $1-\iota$ is the proportion of optimists in the total population of households (normalized to 1)
(remember that $\theta_1^{II} = \theta_2^I = 0$ and $\theta_1^I = a_0^I$ and $\theta_2^{II} = a_{0,2}^{II}/S_2$)

- Aggregate supply of the two assets

$$\iota a_{0,1}^I + (1-\iota)\, a_{0,1}^I = a_{0,1} \text{ (risk-free asset)}$$
$$\iota\, S_2\, a_{0,2}^I + (1-\iota)\, S_2\, a_{0,2}^{II} = S_2\, a_{0,2} \text{ (risky asset)}$$

where $a_{0,1}$ is the aggregate stock of the risk-free asset and $a_{0,2}$ is the aggregate stock of the risky asset (in quantities)

Equilibrium in the asset markets

- Market equilibrium conditions

$$\iota a_0^I = a_{0,1} \text{ (risk-free asset)}$$
$$(1-\iota) a_0^{II} = S_2\, a_{0,2} \text{ (risky asset)}$$

- **The equilibrium values for S_2 and ι are jointly determined**: the asset price depends on the rate of participation.

Equilibrium asset price

- **Assumption**: homogeneity in the distribution of wealth among optimists and pessimists, that is $a_0^I = a_0^{II} = \bar{a}$
- Then the **equilibrium price for the risky asset** is

$$\boxed{S_2^{eq} = S_2(\iota) = \left(\frac{1-\iota}{\iota}\right)\frac{a_{0,1}}{a_{0,2}}} \tag{2}$$

- As

$$\frac{\partial S_2}{\partial \iota} = -\frac{a_{0,1}}{\iota^2 \, a_{0,2}} < 0$$

  The asset price decreases with the proportion of non-participation ι (i.e., if there are more pessimists the asset price decreases)
- The asset price increases with the stock of money $a_{0,1}$

Equilibrium participation

- **Assumption**: the probability distribution of the marginal investor, π^{ι}, is a function of their weight in the total population ι. For simplicity let $\pi^{\iota} = \iota$.
- Then, from equations (1) and (2), the equilibrium value $\iota^{eq} = \{\iota \in (0,1) : \mathcal{I}(\iota) = 0\}$ where

$$\mathcal{I}(\iota) \equiv (1-\iota)\, a_{0,1} - (\iota v_{2L} + (1-\iota) v_{2H})\, \iota a_{0,2}$$

Proposition

There is one unique value $\iota^{eq} \in (0,1)$,

$$\iota^{eq} = \frac{v_{2H} a_{0,2} + a_{0,1}}{2(v_{2H} - v_{2L}) a_{0,2}} - \left[\left(\frac{v_{2H} a_{0,2} - a_{0,1}}{2(v_{2H} - v_{2L}) a_{0,2}} \right)^2 + \frac{4 v_{2L} a_{0,1} a_{0,2}}{4(v_{2H} - v_{2L})^2 a_{0,2}^2} \right]^{\frac{1}{2}}$$

Equilibrium participation

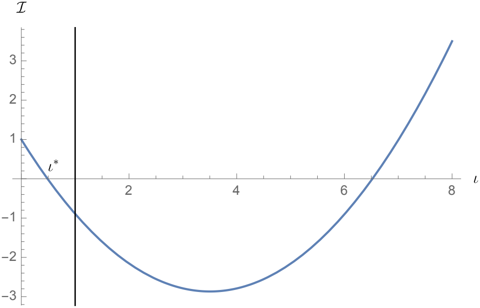


Figure: Proof of Proposition

Equilibrium distribution of households

Proof

- **Proof that $\iota^{eq} \in (0,1)$ exists and is unique.**
Function $\mathcal{I}(\iota)$ is convex in ι (U-shaped) and therefore there can be zero, one or two values of ι such that $\mathcal{I}(\iota) = 0$ for
$-\infty < \iota < \infty$.
However, the domain of ι is $(0,1)$.
It is easy to see that $\mathcal{I}(0) = a_{0,1} > 0$,
$\mathcal{I}^{'}(0) = -(a_{0,1} + v_{2H}a_{0,2}) < 0$ and $\mathcal{I}(1) = -(a_{0,1} + v_{2L}a_{0,2}) < 0$:
therefore, in the interval $(0,1)$ there is one and only one value of ι, ι^{eq} such that $\mathcal{I}(\iota) = 0$.
Therefore, although there are two points $0 < \iota_{-} < 1 < \iota_{+}$ such that $\mathcal{I}(\iota) = 0$, the first one is the solution we are looking for.

Equilibrium distribution of households

Properties

Proposition

The participation rate for the risky asset $(1-\iota^{eq})$ ***increases*** *with the payoff v_{2s} (for any state of nature) and the aggregate stock of the risky asset, $a_{0,2}$, and* ***reduces*** *with the aggregate stock of money, $a_{0,1}$.*

- We showed that $\iota^{eq} = \iota(v_{2H}, v_{2L}, a_{0,1}, a_{0,2})$, and next we prove that

$$\boxed{\frac{\partial \iota^{eq}}{\partial v_{2s}} < 0, \text{ for } s = H, L, \; \frac{\partial \iota^{eq}}{\partial a_{0,1}} > 0, \; \frac{\partial \iota^{eq}}{\partial a_{0,2}} < 0}$$

Equilibrium participation

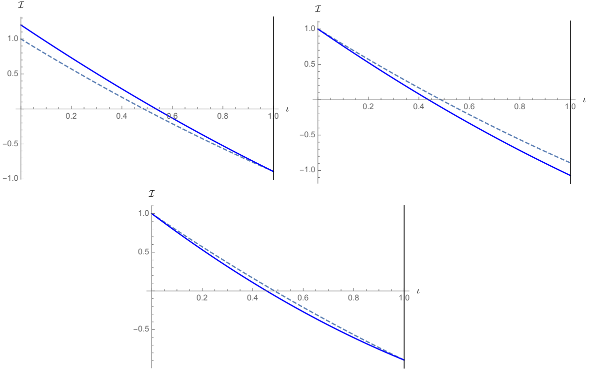


Figure: Change in participation: for variations in $a_{0,1}$, $a_{0,2}$ and $v_{2,H}$

Equilibrium distribution of households
Proof

- **Proof of the sign relationships for** $\dfrac{\partial \iota^{eq}}{\partial v_{2s}}$

  We know that $\mathcal{I}(\iota, v_{2H}, v_{2L}) = 0$. Therefore, the response of ι to the payoffs is
  $$\frac{\partial \iota^{eq}}{\partial v_{2s}} = -\left.\frac{\mathcal{I}_{v_{2s}}}{\mathcal{I}_{\iota}}\right|_{\iota=\iota^{eq}}, \; s = L, H$$

- Where $\mathcal{I}_{v_{2H}} = -\iota^{eq}(1-\iota^{eq})a_{0,2} < 0$ and $\mathcal{I}_{v_{2L}} = -(\iota^{eq})^2 a_{0,2} < 0$ and
  $$\mathcal{I}_{\iota} = 2(v_{2H}-v_{2L})a_{0,2}\left(\iota^{eq} - \frac{a_{0,1}+v_{2H}a_{0,2}}{2(v_{2H}-v_{2L})a_{0,2}}\right) < 0$$

  because $0 < \iota^{eq} < \dfrac{a_{0,1}+v_{2H}a_{0,2}}{2(v_{2H}-v_{2L})a_{0,2}}$

Equilibrium rate of return for the risky asset

- Equilibrium rate of return of the risky asset is

$$\boxed{R_{2,s}^{eq} = \frac{v_{2s}}{S_2^{eq}(v_{2L}, v_{2H}, .)}, s = L, H} \tag{3}$$

- This means that **if there is an increase in** v_{2s} **generates two effects on** R_{2s}:
  - a direct positive effect (of the payoff in the "own" state)
  - a negative indirect effect, because the prices increases as a result of the change in the participation in the risky asset market: $S_2^{eq}\Big(\iota^{eq}(v_{2L}, v_{2H}, \cdot), \cdot\Big)$:

    we have

$$\frac{\partial S_2^{eq}}{\partial v_{2s}} > 0 \text{ for any } s = H, L$$

    because

$$\frac{\partial S_2}{\partial \iota} < 0, \ \frac{\partial \iota}{\partial v_{2s}} < 0, \ s = H, L$$

  - The **final effect is ambiguous**.

Equilibrium rate of return for the risky asset

- For the case in which there is **no change in participation** we have

$$\frac{d\bar{R}_{2s}}{dv_{2s}} = \frac{1}{\bar{S}_2} > 0, \frac{d\bar{R}_{2s'}}{dv_{2s}} = 0,\ s \neq s^{'} = H, L$$

- The rate of return outcome for a particular state of nature **only changes when the payoff outcome for the same state of nature varies**.

Equilibrium R distribution and news

Proposition

If there is a change in participation, then a change in any of the anticipated outcomes in the payoff distribution will change the rate of return, whatever the state of nature that occurs at time $t = 1$. However, the change will be state-dependent. In particular, we have

	ΔR_{2L}	ΔR_{2H}
Δv_{2L}	+ (+)	- (0)
Δv_{2H}	- (0)	+ (+)

Table: *In parenthesis no change in participation*

Equilibrium rate of return for the risky asset

- Proof: When there is a change in participation we have

$$\frac{\partial R_{2s}}{\partial v_{2s}} = \frac{1-\epsilon_{\iota}^{S_2}\epsilon_{v_{2s}}^{\iota}}{S_2(\iota^{eq})},\ \frac{\partial R_{2s'}}{\partial v_{2s}} = -\frac{v_{2s'}}{v_{2s}}\frac{\epsilon_{\iota}^{S_2}\epsilon_{v_{2s}}^{\iota}}{S_2(\iota^{eq})},\ s \neq s^{'} = L, H$$

  where

  - the elasticity of S_2 to ι is

$$\epsilon_{\iota}^{S_2} = \frac{\partial S_2}{\partial \iota}\frac{\iota}{S_2} - \frac{1}{1-\iota^{eq}} < -1$$

  - the elasticity of ι to v_{2s} is

$$\epsilon_{v_{2s}}^{\iota} = \frac{\partial \iota^{eq}}{\partial v_{2s}}\frac{v_{2s}}{\iota},\ s = H, L$$

  - The rate of return outcome for a particular state of nature changes with **variations in the payoff of any state of nature** due to the change in participation.

Equilibrium R distribution and news

- Proof (cont): For a change in v_{2H} we have a change in the distribution of R_2
  - if the good state occurs
$$\frac{\partial R_{2H}^{eq}}{\partial v_{2H}} = -\frac{1}{S_2(\iota^{eq})}\left(\frac{2(v_{2H}-v_{2L})a_{0,2}\iota^{eq}(1-\iota_+)}{\mathcal{I}_\iota}\right) > 0$$
  - if the bad state state occurs
$$\frac{\partial R_{2L}^{eq}}{\partial v_{2H}} = -\frac{1}{S_2(\iota^{eq})}\frac{v_{2L}}{v_{2H}}\epsilon_\iota^{S_2}\epsilon_{v_{2H}}^\iota < 0$$
- For a change in v_{2L} we have a change in the distribution of R_2
  - if the good state occurs
$$\frac{\partial R_{2L}^{eq}}{\partial v_{2L}} = -\frac{a_{0,2}}{\mathcal{I}_\iota} > 0$$
  - if the bad state state occurs
$$\frac{\partial R_{2H}^{eq}}{\partial v_{2L}} = -\frac{1}{S_2(\iota^{eq})}\frac{v_{2H}}{v_{2L}}\epsilon_\iota^{S_2}\epsilon_{v_{2L}}^\iota < 0$$

Equilibrium R distribution and news

- A **positive news regarding the good state** v_{2H}, $\Delta v_{2H} > 0$, generates an increase in the rate of return if the good state occurs and a decrease in the rate of return if the bad state occurs:

$$\Delta v_{2H} > 0 \ \Rightarrow \Delta R_{2L} < 0 < \Delta R_{2H}$$

  This is because

$$v_{2H} \uparrow \rightarrow \iota \downarrow \rightarrow S_2 \uparrow \rightarrow \begin{cases} R_{2L} = v_{2L}/S_2 & \downarrow \\ R_{2H} = v_{2H}/S_2 & \uparrow \end{cases}$$

- a **negative news regarding the bad state**, v.g., $\Delta v_{2L} < 0$, there is an increase in the rate of return if the good state occurs and a reduction if the bad state occurs

$$\Delta v_{2L} < 0 \ \Rightarrow \Delta R_{2L} < 0 < \Delta R_{2H}$$

  this is because

$$v_{2L} \downarrow \rightarrow \iota \uparrow \rightarrow S_2 \downarrow \rightarrow \begin{cases} R_{2L} = v_{2L}/S_2 & \downarrow \\ R_{2H} = v_{2H}/S_2 & \uparrow \end{cases}$$

Equilibrium participation

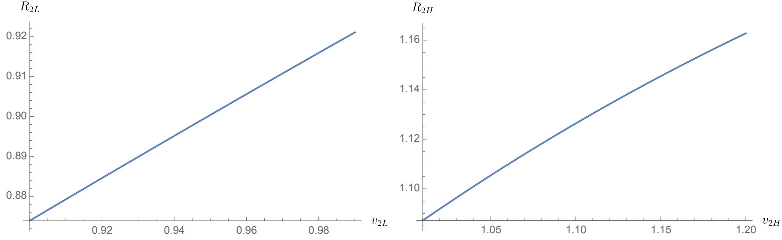


Figure: Reaction to news: R_{2L} to v_{2L} and R_{2H} to v_{2H}

Conclusions

- We showed that when priors differ, and there are participation frictions in the asset market, **asymmetric expected changes in payoffs have an effect on the whole distribution of the rate of return** of risky assets
- Good news regarding the good state or bad news regarding the bad state lead to a kind of an **amplification** response of the rate of return: a higher realized rate of return if the good state realizes and a lower rate of return if the bad state realizes.
- Other results: an expansion in the money supply $M = a_{0,1}$ will increase the rate of return for all states of nature

$$M \uparrow \rightarrow \iota \uparrow \rightarrow S_2 \downarrow \rightarrow \begin{cases} R_{2L} = v_{2L}/S_2 & \uparrow \\ R_{2H} = v_{2H}/S_2 & \uparrow \end{cases}$$

References

This lecture is adapted from Geanakoplos (2010) and Fostel and Geanakoplos (2014).

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