Foundations of Financial Economics *2021/22*
Problem set 2: two-period deterministic micro and DGE models

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Disclaimer: This problem set is provided as a help for self study, in an open academic spirit of providing potentially interesting problems. Not all questions have been completely verified. Solving them is not mandatory, however it is advisable because exams' questions will be in a large part similar to some of them. The instructor does not commit himself to provide the solutions of them all, but is available to help solving specific difficulties arising in efforts to **actually** solving them.

Study material: Slide 3 covers section 1 and most (but not all) questions (b) of sections 2 and 3. Slide 4 covers questions (a) and (c) of sections of sections 2 and 3.

1 Intertemporal utility functions

1. Consider the following intertemporal utility functions (IUF)

$$U(c_0,c_1) = \ln(c_0) + \beta \ln(c_1),\ \text{for } 0<\beta<1, \tag{1}$$

$$U(c_0,c_1) = \frac{c_0^{1-\phi}-1}{1-\phi} + \beta\frac{c_1^{1-\phi}-1}{1-\phi},\ \text{for } 0<\beta<1,\ \phi>0, \tag{2}$$

$$U(c_0,c_1) = -\frac{e^{-\zeta c_0}}{\zeta} + \beta\frac{e^{-\zeta c_1}}{\zeta},\ \text{for } 0<\beta<1,\ \zeta>0, \tag{3}$$

$$U(c_0,c_1) = \ln(c_0-\bar{c}) + \beta \ln(c_1-\bar{c}),\ \text{for } 0<\beta<1,\ \bar{c}>0 \tag{4}$$

$$U(c_0,c_1) = \frac{c_0^{1-\phi}-1}{1-\phi} + \frac{\beta}{1-\phi}\Big(\frac{c_1}{c_0}\Big)^{\zeta(1-\phi)},\ \text{for } 0<\beta<1,\ \zeta>0,\ \phi>0 \tag{5}$$

$$U(c_0,c_1) = \ln(c_0) + \beta \ln\left(\Big(\frac{c_1}{c_0}\Big)^{\zeta}\right),\ \text{for } 0<\beta<1,\ \zeta>0 \tag{6}$$

$$U(c_0,c_1) = \ln(c_0) + \beta \ln(c_1-\eta\, c_0),\ \text{for } 0<\beta<1,\ 0<\eta<1 \tag{7}$$

$$U(c_0,c_1) = (1-\beta)\ln(c_0) + \beta \ln(c_1),\ \text{for } 0<\beta<1, \tag{8}$$

$$U(c_0,c_1) = \Big((1-\beta)c_0^{\eta} + \beta c_1^{\eta}\Big)^{\frac{1}{\eta}},\ \text{for } 0<\beta<1,\ \eta>0 \tag{9}$$

$$U(c_0, c_1) = \ln(c_0) + \beta \begin{cases} \frac{c_1^{1-\phi} - 1}{1-\phi}, & \text{if } 0 < c_1 < c_0, \\ \ln(c_1), & \text{if } c_1 \geq c_0, \end{cases} \text{ for } 0 < \beta < 1,\ \phi > 0 \tag{10}$$

For each utility function:

(a) Find the intertertemporal marginal rate of substitution, $IMRS_{0,1}$.

(b) Find the intertemporal elasticity of substitution, $IES_{0,1}$

(c) Discuss the implicit properties of the IUF concerning patience and intertemporal dependence. If needed, impose conditions for the existence of impatience and intertemporal substitution.

(b) Provide an economic intuition for your results.

2 Arrow-Debreu economy

1. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \log c_1,\ 0 < \beta < 1$$

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

c) Find the equilibrium AD price. Provide an intuition.

2. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{c_0^{1-\theta}}{1-\theta} + \beta \frac{c_1^{1-\theta}}{1-\theta},\ 0 < \beta < 1,\ \theta > 0$$

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions Solve the representative agent problem.

c) Find the equilibrium AD price. Provide an intuition.

3. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = -\frac{e^{-\zeta c_0}}{\zeta} + \beta \left(-\frac{e^{-\zeta c_1}}{\zeta} \right),\ 0 < \beta < 1,\ \zeta > 0$$

a) Characterize the (static and dynamic) behavioral assumptions which are implicit in the intertemporal utility function.
b) Solve the representative agent problem. Provide an intuition.
c) Define the general equilibrium. Find the equilibrium AD price. Provide an intuition.

4. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the intertemporal utility function is

$$U(c_0, c_1) = \log(c_0 - \bar{c}) + \beta \log(c_1 - \bar{c}),\ 0 < \beta < 1,\ \bar{c} > 0$$

and the flow of endowment is $\{y_0, y_1\}$. Assume that the flow of endowment for any period is larger than the subsistence consumption $\bar{c}$, that is $\bar{c} < \min\{y_0, y_1\}$.

a) Characterize the (static and dynamic) behavioral assumptions which are implicit in the intertemporal utility function.
b) Solve the representative agent problem. Provide an intuition.
c) Define the general equilibrium. Find the equilibrium AD price. Provide an intuition.

Solution:

(a) Static properties: the Bernoulli utility function is increasing and concave (but not Inada) Dynamic properties: $MRS(c,c) = \beta^{-1} > 1$ and $\varepsilon_{0,1} = 0$ impatience and intertemporally independent preferences, although the $IES_{0,1}(c,c) = 1 - \frac{\bar{c}}{c} \in (0,1)$

(b) $c_0 - \bar{c} = \frac{h - (1+p)\bar{c}}{1+\beta}$, $c_1 - \bar{c} = \frac{h - (1+p)\bar{c}}{p(1+\beta)}$. Intuition the need satisfy the subsistence level may change the consumption as regards the $\log(c)$ case

(c) $p = \beta \left(\frac{y_0 - \bar{c}}{y_1 - \bar{c}}\right)$.

5. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{(c_0 - \bar{c})^{1-\theta}}{1-\theta} + \beta \frac{(c_1 - \bar{c})^{1-\theta}}{1-\theta},\ 0 < \beta < 1$$

a) Specify the agent's problem. Define the general equilibrium.
b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.
c) Provide one condition for the existence of an equilibrium. Find the equilibrium AD price. Provide an intuition.

6. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \log\left(\left(\frac{c_1}{c_0}\right)^{\zeta}\right),\ 0 < \beta < 1$$

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

c) Find the equilibrium AD price. Provide an intuition.

7. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{c_0^{1-\theta}}{1-\theta} + \frac{\beta}{1-\theta}\frac{\left(\left(\frac{c_1}{c_0}\right)^{\zeta}\right)^{1-\theta}}{1-\theta},\ 0 < \beta < 1,\ \theta > 0,\ \zeta > 0.$$

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

c) Find the equilibrium AD price. Provide an intuition.

8. Consider a deterministic, two-period, representative-agent Arrow-Debreu endowment economy where the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \begin{cases} \log c_1 & \text{if } c_1 \geq c_0 \\ \frac{c_1^{1-\theta}}{1-\theta} & \text{if } 0 < c_1 < c_0 \end{cases}$$

for $0 < \beta < 1$ and $\theta > 1$.

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

c) Find the equilibrium AD price. Provide an intuition.

9. Consider a two-period intertemporal utility function, in a deterministic setting, for the consumption sequence $\{c_0, c_1\}$

$$U\big(c_0, c_1\big) = \Big((1-\mu)\, c_0^{\eta} + \mu\, c_1^{\eta}\Big)^{\frac{1}{\eta}}$$

for $0 < \mu < 1$ and $\eta \in (-\infty, \infty)$.

(a) After determining the intertemporal marginal rate of substitution, the Allen-Uzawa elasticities, and the elasticity of intertemporal substitution, characterize the possible types of behavior and therir dependence on the parameters μ and η.

(b) Assume a representative-agent Arrow-Debreu (AD) endowment economy, where the flow of endowment is $\{y_0, (1+\gamma)\, y_0\}$ and the price of AD contracts is denoted by q. Solve the representative agent problem. Discuss the response of the optimal consumption c_0 from changes in in q.

(c) Find the equilibrium AD price. Provide an intuition. In particular, discuss the relationship between the behavioral features you discussed in (a) and the equilibrium AD price (tip: observe that q is a discount factor, which means that $1/q = 1+r$ where r can be interpreted as the risk-free interest rate for this economy).

10. Consider a two-period intertemporal utility function, in a deterministic setting, for the consumption sequence $\{c_0, c_1\}$

$$U\big(c_0, c_1\big) = (1-\beta)\, \ln(c_0) + \beta \ln(c_1), \text{ with } 0 < \beta < 1.$$

(a) After determining the intertemporal marginal rate of substitution, the Allen-Uzawa elasticities, and the elasticity of intertemporal substitution, characterize the possible types of household behavior through time.

(b) Assume a representative-agent Arrow-Debreu (AD) endowment economy for which the flow of endowment is $\{y_0, (1+\gamma)\, y_0\}$. Find the general equilibrium. Charaterize the equilibrium state-price under the existence of impatience. Under which conditions equilibrium consumption will be constant through time ?

(c) Assume instead that there are two groups of agents a and b, which are heterogeneous only as regards their sequence of endowments. Those agents have the following flow of endowments, $y^a = \{y_0, 0\}$ and $y^b = \{0, (1+\gamma)\, y_0\}$, with $y_0 > 0$ and $\gamma > 0$, respectively. Find the general equilibrium in this economy and compare with the equilibrium for a representative agent economy in (b). Provide a brief discussion on the difference between the two equilibria.

3 Finance economy

1. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \log c_1,\ 0 < \beta < 1$$

a) Specify the agent's problem. Define the general equilibrium.

b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

c) Find the equilibrium asset return. Provide an intuition.

2. Consider a deterministic, two-period, representative-agent finance economy where the initial financial wealth is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{c_0^{1-\theta} - 1}{1-\theta} + \beta \frac{c_1^{1-\theta} - 1}{1-\theta},\ 0 < \beta < 1,\ \theta > 0$$

a) Characterize the implicit behavioral assumptions.

b) Specify the agent's problem. Solve the representative agent problem.

c) Define the general equilibrium. Find the equilibrium asset return. Provide an intuition.

3. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = -\frac{e^{-\zeta c_0}}{\zeta} + \beta\left(-\frac{e^{-\zeta c_1}}{\zeta}\right),\ 0 < \beta < 1,\ \zeta > 0$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Find the equilibrium asset return. Provide an intuition.

4. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log(c_0 - \bar{c}) + \beta \log(c_1 - \bar{c}),\ 0 < \beta < 1$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Provide one condition for the existence of an equilibrium. Find the equilibrium asset return. Provide an intuition.

5. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{(c_0 - \bar{c})^{1-\theta}}{1-\theta} + \beta\frac{(c_1 - \bar{c})^{1-\theta}}{1-\theta},\ 0 < \beta < 1.$$

   a) Specify the agent's problem. Define the general equilibrium.

   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.

   c) Provide one condition for the existence of an equilibrium. Find the equilibrium asset return. Provide an intuition.

6. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$, where $y_1 = (1+\gamma)y_0$, with $\gamma \geq 0$. The intertemporal utility functional is

$$U(c_0, c_1) = \log c_0 + \beta \log\left(\left(\frac{c_1}{c_0}\right)^{\zeta}\right),\ 0 < \beta < 1$$

(a) Characterize the implicit behavioral assumptions in the utility functional.

(b) Specify and solve the representative agent problem. Characterize and provide an intuition to the savings behavior of the household.

(c) Define the general equilibrium. Find the equilibrium asset return. Characterize and provide an intuition for its properties.

(d) Which type of interest rate theory justifies this model ? Does this explain the historical negative correlation between the risk free interest rate and the rate of economic growth ? Provide an intuition for what is missing in this model to account for that negative correlation.

Solution

(a) As $U_0 = \dfrac{1-\beta\zeta}{c_0} > 0$ if $0 < \beta\zeta < 1$, $U_1 = \dfrac{\beta\zeta}{c_1} > 0$, $U_{00} = -\dfrac{1-\beta\zeta}{c_0^2} < 0$, $U_{01} = 0$, and $U_{11} = -\dfrac{\beta\zeta}{c_1^2} < 0$, we find $IMRS_{0,1} = \frac{1-\beta\zeta}{\beta\zeta}\dfrac{c_1}{c_0}$ and, for any $\{c_0, c_1\}$, $\epsilon_{00} = \epsilon_{11} = EIS_{0,1} = 1$ and $\epsilon_{01} = 0$. For a constant path $\{c_0, c_1\} = \{c, c\}$ we have $IMRS_{0,1}(c) = \frac{1-\beta\zeta}{\beta\zeta}$. Therefore, (1) threre are positive but decreasing marginal utilities for consumption in every period, (2) there is Edgeworth intertemporal independence; (3) if $\beta\zeta < 1/2$ the utility functional displays impatience; and (4) there is intertemporal substitution in the consumption in both periods and the EIS is constant and equal to one.

(b) The problem (one version)

$$\begin{aligned}
&\max_{c_0,c_1,s} \log c_0 + \beta\zeta(\log c_1 - \log c_0)\\
&\text{s.t}\\
&c_0 + s = y_0\\
&c_1 = R\,s + y_1
\end{aligned}$$

Solution: $c_0 = (1-\beta\zeta)h$, $c_1 = \beta\zeta R h$ and $s = y_0 - (1-\beta\zeta)h$ for $h = y_0 + \frac{y_1}{R}$. Theory behind the savings behavior: intertemporal smoothing of consumption. Behavior: s increases with initial endowment and return and decreases with anticipated endowments: $s = S(y_0, y_1, R)$ where $S_{y_0} > 0$, $S_{y_1} < 0$ and $S_R > 0$

(c) Definition of GE: sequence $\{c_0^{eq}, c_1^{eq}\}$, savings s^{eq} and return R^{eq} such that: (a) $\{c_0, c_1\}$ and s solve the agent problem given R; (b) markets clear $c_0 = y_0$, $c_1 = y_1$ and $s = 0$ (zero net wealth).
Solution: $c_0^{eq} = y_0$, $c_1^{eq} = y_1$, $s^{eq} = 0$ and $R^{eq} = \frac{(1-\beta\zeta)(1+\gamma}{\beta\zeta}$

(d) Interest rate therory related to consumption smoothing behavior. The correlation with γ is positive and not negative as in the historical experience. A possible explanation is that other long-run factor may explain history: technical progress, and the effects of savings in investment and therefore on production.

7. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \frac{c_0^{1-\theta}}{1-\theta} + \frac{\beta}{1-\theta} \frac{\left(\left(\frac{c_1}{c_0} \right)^{\zeta} \right)^{1-\theta}}{1-\theta}, \; 0 < \beta < 1, \; \theta > 0, \; \zeta > 0.$$

   a) Specify the agent's problem. Define the general equilibrium.
   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.
   c) Find the equilibrium asset return. Provide an intuition.

8. Consider a deterministic, two-period, representative-agent finance economy where the initial asset stock is zero, the flow of endowment is $\{y_0, y_1\}$ and the intertemporal utility function is

$$U(c_0, c_1) = \log c_0 + \beta \begin{cases} \log c_1 & \text{if } c_1 \geq c_0 \\ \dfrac{c_1^{1-\theta}}{1-\theta} & \text{if } 0 < c_1 < c_0 \end{cases}$$

   for $0 < \beta < 1$ and $\theta > 1$.

   a) Specify the agent's problem. Define the general equilibrium.
   b) Characterize the implicit behavioral assumptions. Solve the representative agent problem.
   c) Find the equilibrium asset return. Provide an intuition.