Foundations of Financial Economics *2021/22*
Problem set 3 : Choice under uncertainty- the static case

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1. Assume the information set has three equiprobable states of nature. A consumer receives the endowment $Y = (y(1+\epsilon), y, y(1-\epsilon))^\top$, where $y > 0$ and $0 < |\epsilon| < 1$. The consumer has the utility functional $\mathbb{E}[\ln(Y)]$.

   (a) Find the certainty equivalent for Y.

   (b) What would be better: to get Y or a certain amount which would be equal to $\mathbb{E}[Y]$? Justify.

   (c) Assume the agent can be in one of two alternative situations: autarky, or in an exchange economy in which the equilibrium price is state-independent $Q = (\bar{q}, \bar{q}, \bar{q})$. Under which situation would the agent be better off ? Justify.

   Solution

   (a) Let y_c be the certainty equivalent of the endowment Y. Then from $u(y_c) = \mathbb{E}[u(Y)] = \ln(\alpha y)$, where $\alpha = (1-\epsilon^2)^{1/3} \in (0,1)$, we obtain $y_c = \alpha y < y$.

   (b) As $\mathbb{E}[Y] = y$ then $u(\mathbb{E}[Y]) = \ln(y) > \ln(\alpha y) = \mathbb{E}[u(Y)]$ then receiving y is better than receiving Y.

   (c) Utility of the consumer in autarky $U^A(C) = U^A(Y) = \mathbb{E}[u(Y)] = \ln(\alpha y)$. Utility of the consumer when there is trade $U^T(C) = \ln(y)$. To prove this we solve the consumer problem

   $$\max_C \sum_{s=1}^{3} \frac{1}{3} \ln(c_s) \text{ s.t } \sum_{s=1}^{3} \bar{q} c_s = \sum_{s=1}^{3} \bar{q} y_s$$

   There is only a solution if $\bar{q} = 1/3$. With this assumption we get $C = (c_1, c_2, c_3) = (y, y, y)$. Then $U^T(C) = \mathbb{E}[\ln(y)] = \ln(y)$. Then trade is better.

2. Assume the information set has two equiprobable states of nature. The consumer has the utility functional $\mathbb{E}\left[\frac{Y^{1-\theta}}{1-\theta}\right]$, where $\theta \geq 1$, and is entitled to the endowment $Y = \{y(1+\epsilon), y(1-\epsilon)\}$, where $y > 0$ and $0 < |\epsilon| < 1$.

(a) Find the certainty equivalent for Y. Justify.

(b) What would be better: to get Y or a certain amount equal to $\mathbb{E}[Y]$? Justify.

(c) Assume the agent can be in one of two alternative arrangements: autarky, or in an exchange economy in which the equilibrium price is $Q = \{\bar{q}, \bar{q}\}$. Under which arrangement would the agent be better off ? Justify.

3. Consider the set of states of nature is $\Omega = \{\omega_1, \omega_2\}$ with associated probabilities $P(\omega_1) = \pi$ and $P(\omega_2) = 1 - \pi$. A lottery pays $Y(\omega_1) = y + \epsilon$ in the good state and $Y(\omega_2) = y - \epsilon$ in the bad state, where $y > 0$ and $\epsilon > 0$. Assume that the utility function is $u(Y(\omega_s)) = -e^{-Y(\omega_s)}$.

   (a) What would be better, the lottery or a certain outcome that would be equal to the expected value of the lottery ?

   (b) Assume that an agent can be in one of the following two environments: (1) autarky, in which case he/she would get the lottery; or (2) in an exchange economy, in which he/she could trade the lottery for a price $Q(\omega_s) = P(\omega_s)$, for $s = 1, 2$. in which environment would he/she be better ? Supply an intuition for your results.

4. Consider the set of states of nature is $\Omega = \{\omega_1, \omega_2\}$ with associated probabilities $P(\omega_1) = \pi$ and $P(\omega_2) = 1 - \pi$. A lottery pays $Y(\omega_1) = y(1 + \epsilon)$ in the good state and $Y(\omega_2) = y(1 - \epsilon)$ in the bad state, where $y > 0$ and $\epsilon > 0$. Assume that the utility function is $u(Y(\omega_s)) = -e^{-Y(\omega_s)}$.

   (a) What would be better, the lottery or a certain outcome that would be equal to the expected value of the lottery ?

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5. Consider the set of states of nature is $\Omega = \{\omega_1, \omega_2\}$ with associated probabilities $P(\omega_1) = \pi$ and $P(\omega_2) = 1 - \pi$. A lottery pays $Y(\omega_1) = \ln(y(1 + \epsilon))$ in the good state and $Y(\omega_2) = \ln(y(1 - \epsilon))$ in the bad state, where $0 < \epsilon < 1$. Assume that the utility function is $u(Y(\omega_s)) = -e^{-Y(\omega_s)}$.

   (a) Compute the certainty equivalent of the lottery.

   (b) What would be better, the lottery or a certain outcome that would be equal to the expected value of the lottery ?

   (c) Assume that an agent can be in one of the following two environments: (1) autarky, in which case he/she would get the lottery; or (2) in an exchange economy, in which he/she could trade the lottery for a price $Q(\omega_s) = P(\omega_s)$, for $s = 1, 2$. in which environment would he/she be better ? Supply an intuition for your results.

6. There are two states of nature with equal probabilities and a lottery with payoffs $Y = \left(\frac{1}{\epsilon}, \frac{1}{1-\epsilon}\right)$, where $0 < \epsilon < 1$ and $\epsilon \neq \frac{1}{2}$. Assume that the utility function is $u(y) = 1 - \frac{1}{y}$.

(a) Compute the certainty equivalent of the lottery.

(b) What is better, the lottery or a certain outcome equal to the expected value of the lottery ? Provide an intuition for your result.

(c) Introduce a proportional transfer (a tax or a subsidy) over the certain outcome with the objective of making the agent indifferent between the two choices in (b). Which value should that transfer take ? Justify.

Solution

a) Let $y_c = CE[Y]$ be the certainty equivalent. Then we find that $y_c = 2$

b) Certain outcome $X = \mathbb{E}[Y] = (2\epsilon(1-\epsilon))^{-1} \geq 2$. Three different alternative ways of proving: (1) $X > y_c$; (2) $u(X) - \mathbb{E}[u(Y)] = (1 - 4\epsilon(1-\epsilon))/2 > 0$; (3) by the Jensen inequality $u(X) > \mathbb{E}[u(Y)]$ because $u(y) = 1 - \dfrac{1}{y}$ is concave.

c) We want to find τ such that $u((1-\tau)\mathbb{E}[Y]) = \mathbb{E}[u(Y)]$. We find $\tau = 1 - 4\epsilon(1-\epsilon) > 0$. It is a tax not a subsidy.

7. Let the income tax rate be $0 < t < 1$ and be levied over the reported income $Y - E$, where Y is the true income and E the unreported income. There is a random, from the perspective of the tax-payer, inspection activity which, in case of the existence of un-reported income can charge a penalty, that is a function of the unreported income δE, where $\delta > 0$. The tax-payer assigns a probability of p of being inspected. The flows of consumption are: $C_{no} = Y - t(Y - E)$ in the case of no inspection, and $C_{yes} = Y - t(Y - E) - \delta E$ in the case of inspection. Assume that the tax-payer has a von-Neumann utility functional with a Bernoulli logarithmic utility function. Clearly $0 \leq E \leq Y$.

   (a) What is the optimal reporting behavior by the consumer.

   (b) The effective tax rate is $t(Y - E)/Y$. Find the effective optimal tax from the point of view of the tax-payer

8. Let there be uncertainty characterized by two states of nature with equal probabilities. A lottery has payoffs $Y = (y_1, y_2) = (e^{\epsilon}, e^{-\epsilon})$, where $\epsilon > 0$, and the behavior of an agent is charaterized by a von-Neumann Morgenstern utility functional with a logarithmic Bernoulli utility function.

   (a) Find the certainty equivalent of lottery Y.

   (b) Which is better, the lottery or a certain payoff equal to $\mathbb{E}[Y]$? Describe and give an intuition on the possible approaches to come up with an answer.

   (c) Assume you introduce an flat tax over the certain payoff $\mathbb{E}[Y]$. What would be the level of the tax such that the agent would be indifferent between the penalized certain outcome or the lottery. Provide an intuition.

Solution:

(a) Let $y_c = CE(Y)$ be the certainty equivalent. Then we find $y_c = 1$

(b) The expected value of lottery Y is $\mathbb{E}(Y) = \dfrac{e^{\epsilon} + e^{-\epsilon}}{2}$. Observe that

$$\frac{\partial \mathbb{E}(Y)}{\partial \epsilon} = \frac{e^{\epsilon} - e^{-\epsilon}}{2}, \; \frac{\partial^2 \mathbb{E}(Y)}{\partial \epsilon^2} = \frac{e^{\epsilon} + e^{-\epsilon}}{2} > 0$$

this means that $\mathbb{E}(Y)$ is a convex function of ϵ and reaches a minimum at $\epsilon = 0$. And as $\mathbb{E}(Y|\epsilon = 0) = 2$ then $\mathbb{E}(Y|\epsilon > 0) > 1$. Three ways to compare: (1) $y_c = 1 < \mathbb{E}(Y)$; (2) $\ln(y_c) = 0 < \ln\left((e^{\epsilon} + e^{-\epsilon})/2\right)$; (3) as $u(y_s) = \ln(y_s)$ is concave the Jensen inequality implies $u(\mathbb{E}[Y]) > \mathbb{E}[u(Y)]$

(c) Penalized certain outcome $\mathbb{E}[Y] - T$. Then $T = \frac{e^{\epsilon}+e^{-\epsilon}}{2} - 1 > 0$ for $\epsilon > 0$.

9. The per capita real growth rates for Portugal for the period 1970-2014 (data: Penn World Table 9.0) are shown in the next figure:

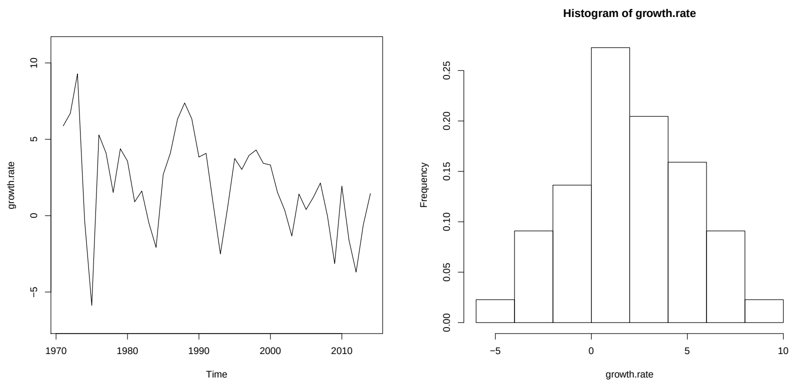

Figure 1: Real per capita growth rates: Portugal 1970-2014

In the next table we gather the breaks in the rates of growth and the absolute frequencies.

growth rate (percent)	$[-6,-4)$	$[-4,-2)$	$[-2,0)$	$[0,2)$	$[2,4)$	$[4,6)$	$[6,8)$	$[8,10)$
frequency (# years)	1	4	6	12	9	7	4	1

The average growth rate was approximately 2.039 per cent.

(a) Assuming a logarithmic utility function determine the certainty equivalent rate of growth (hint: use $1 + g$ in your calculations, where g is the growth rate in decimals).

(b) Determine the certainty equivalent growth rate for CRRA utility functions for the different values of the coefficient of relative risk aversion (example: 2, 3, 4).

(c) Provide an intuition for your results.

10. Consider an economic policy authority (EPA) in charge of assessing and controlling the economic growth of an economy, for the period of one year. It has the following information: it observes the growth factor $g_0 = 1 + \gamma$, at the beginning of the year, and it assumes that the growth factor follows a binomial random variable $G_1 = (g_1, g_2) = (1 + \gamma - \sigma, 1 + \gamma + \sigma)$, for $0 < \sigma < 1 + \gamma$, at the end of the year.

    (a) If the EPA assumes that the process $\{g_0, G_1\}$ is a martingale (tip: a martingale is a process $\{X_t\}_{t=0}^{T}$ such that $\mathbb{E}_t[X_{t+1}] = x_t$), what will be the expected value and the standard deviation for G_1 ?

    (b) The EPA measures the cost of macroeconomic volatility by $C(G_1) = \mathbb{E}[G_1] - CE(G_1)$, where $CE(G_1)$ is the certainty equivalent of the growth factor, assuming an utility function $u(g) = \ln(g)$. Find $C(G_1)$. Explain its meaning.

    (c) Let the EPA have a state-independent instrument $\tau \in (-g_0, g_0)$ that can additively change the growth factor to $\tilde{G}_1(\tau) = (g_1 + \tau, g_2 + \tau)$. If the EPA would use the instrument τ in order minimize $\tilde{G}_1(\tau)$ what would be the minimum cost of volatility that it can achieve ? Can it be zero ? Why ?

11. Consider two potential investments projects, labelled A and B, that generate contingent profits. Let $\Pi_s^i = p^i y_s^i - c^i$ be the profit of project $i \in \{A, B\}$, at state of nature $s \in \{1, 2\}$, where p^i is the selling price, y_s^i is the contingent output and c^i is the cost. Consider the following data: and assume that the two states of nature have equal probabilities, and that

Investments		$i = A$	$i = B$
selling price	p^i	$1 + \psi$	1
cost	c^i	1	1
output in state $s = 1$	y_1^i	$1 - a$	$1 + \epsilon\, a$
output in state $s = 2$	y_2^i	$1 + a$	$1 - \epsilon\, a$

$0 < a < 1$, $0 < \varepsilon < 1$, $\psi > -1$ and $a^2 \left(3 - (1 + a^2)\varepsilon^2\right) < 1$. The projects are ranked by their value, with the value of project i be determined by $V^i = \mathbb{E}[u(\Pi^i)]$.

    (a) Assume that the agent has the utility function $u(\Pi) = \Pi$. How would the investor rank the projects ?

    (b) Now assume that the agent values the projects with the utility function $u(\Pi) = \Pi - \frac{1}{2}\Pi^2$. How would the investor rank the projects in this case?

    (c) Provide an intuition for the results you obtained in (a) and (b).