Foundations of Financial Economics *2021/22*
Problem set 5: two-period APT

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1. Consider a finance economy in which (assets in the columns)

$$S = (1, s)\,,\quad V = \begin{pmatrix} r+\epsilon & r(1-\epsilon) \\ r-\epsilon & r(1+\epsilon) \end{pmatrix}$$

   where $0 < r < 1$, $s > 0$ and ϵ can take any (real) value:

   (a) Under which conditions arbitrage opportunities exist ?

   (b) Can we have absence of arbitrage opportunities and market incompleteness ?

   (c) Compute the market price for an Arrow-Debreu contingent claim, under the assumption of absence of arbitrage opportunities. What is replicating transactions' strategy ? Explain.

2. Consider a finance economy in which

$$S = \left(1, \frac{1}{1+r}\right),\quad V = \begin{pmatrix} r+\epsilon & 1 \\ r-\epsilon & 1 \end{pmatrix}$$

   where $r > 0$ and ϵ can take any (real) value.

   (a) Under which conditions arbitrage opportunities exist ?

   (b) Compute the state prices, by assuming the appropriate existence conditions.

   (c) Under the conditions that you imposed in previous point, can there be market completeness ? How ?

   (d) Compute the market price for an Arrow-Debreu contingent claim, under the assumption of absence of arbitrage opportunities. What is replicating transactions' strategy ? Explain.

3. Assume an asset market represented by the pair of payoff prices (S, V) and consider the linear pricing relationship $S = qV$. Consider a simple case in which the number of assets is equal to the number of states of nature and both equal to 2:

(a) determine an equivalent expression involving a stochastic discount factor and asset returns;

(b) determine a sufficient condition for the existence of arbitrage in terms of the asset returns;

(c) assume that there are no arbitrage opportunities, determine conditions for the existence of completeness in terms of the asset returns;

(d) assume that there are no arbitrage opportunities, determine conditions for the existence of incompleteness in terms of the asset returns;

(e) what are the consequences of the existence of incompleteness on the expected value of asset returns, variances and the covariance with the stochastic discount factor ?

4. Consider the following return matrices ($K \times N$)

$$\begin{pmatrix} 1.1 & 1.2 \\ 1.02 & 1.01 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 1.111 & 1.122 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 1.111 & 1.122 \\ 0.99 & 0.95 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 0.99 & 1.02 \end{pmatrix}.$$

Characterize the asset market as regards the existence of arbitrage opportunities and completeness

5. Consider two (alternative) financial markets, A and B, characterized by the return matrices ($N \times K$)

$$R = \begin{pmatrix} 1 & 1+a & b \\ 1 & 1-a & b \end{pmatrix}, \text{ for financial market } A, \text{ and } R = \begin{pmatrix} 1 & 1 \\ 1+a & 1-a \\ 1-b & 1+b \end{pmatrix}, \text{ for financial market } B$$

where a and b can take any real value.

(a) Under which conditions there are no arbitrage opportunities and there is market completeness in the financial market A

(b) Under which conditions there are no arbitrage opportunities and there is completeness in the financial market B

Solution:

a) There is only a market (in the sense that there is a vector of prices of the state of nature if $b = 1$. In this case there are no arbitrage opportunities and there is market completeness and $q = (1/2, 1/2)$

b) The vector of prices of the states of nature is $q = (q_1, q_2, q_3) = (x, b(1-x)/(a+b), a(1-x)/(a+b)$ for an arbitrary x. There is always market incompleteness. There are no arbitrage opportunities if $x \in (0, 1)$ and $\text{sign}(a) = \text{sign}(b)$

6. Assume there is a financial market with two assets, one risky asset with return $1+r$ and one riskless asset with return $1+i$. Assume there are no arbitrage opportunities. Prove that the Sharpe index verifies
$$\left|\frac{E[r-i]}{\sigma[r]}\right| \leq \frac{\sigma[m]}{E[m]}$$
where m is the stochastic discount factor.

7. Assume there is a financial market with two assets, one risky asset and one riskless asset with prices and payoffs
$$S = \left(\tfrac{1}{1+i} \quad s\right), \; V = \begin{pmatrix} 1 & d_1 \\ 1 & d_2 \end{pmatrix},$$
where $i > 0$ and $0 < d_1 < d_2$. Introduce an european call option with exercise price $d_1 < p < d_2$. Prove that its price, if there are absence of arbitrage opportunities is
$$S_o = \frac{(r_1 - i)(d_2 - p)}{(1+i)(r_1 - r_2)}$$

8. Consider a finance economy in which there are three assets with the vector of prices and payoff matrix given by
$$S = \left(p, 1, \frac{1}{1+r}\right), \quad V = \begin{pmatrix} r-\epsilon & r+\epsilon & 1 \\ r+\epsilon & r-\epsilon & 1 \end{pmatrix}$$
where we assume that $r > 0$, $p > 0$ and $\epsilon \geq 0$.
   (a) Under which conditions there are no arbitrage opportunities ?
   (b) Under the conditions that you imposed in the answer to the previous point, what would be the meaning of market completeness in this economy ? Determine the conditions for the existence of market completeness.
   (c) Compute the price for the first asset, p, by building a replicating transactions' strategy. Explain.

Solution:

   (a) There are no arbitrage opportunities if the following conditions hold: $\epsilon > 1$ and $\max\{0, (r-\epsilon)/(1+r)\} < p < (r+\epsilon)/(1+r)$. This is the condition that guarantees $q_s^{1,2} > 0$, $q_s^{1,3} > 0$ and $q_s^{2,3} > 0$, for $s = 1, 2$ where $q_s^{i,j}$ is the price of state of nature s for the sub-market composed by assets i and j.
   (b) Markets are complete if $r > 1$ and $p = (r-1)/(1+r)$. Markets are complete if and only if the prices of the states of nature are unique: $q_1 = q_1^{1,2} = q_1^{1,3} = q_1^{2,3}$ and $q_2 = q_2^{1,2} = q_2^{1,3} = q_2^{2,3}$.
   (c) Let θ_k be the number of asset k in a portfolio. The replicating portfolio is $\theta_2 = -1$ and $\theta_3 = 2r$ and the price of asset 1 is, again, $p = (r-1)/(1+r)$.

9. Consider a two-period finance economy in which the information is given by a binomial tree with objective probabilities (π_1, π_2). The financial market is characterized by the return matrix in the (state × asset) form

$$\begin{pmatrix} 1 & R_1 \\ 1 & R_2 \end{pmatrix}.$$

   (a) Provide conditions for the absence of arbitrage opportunities and for the existence of complete markets. Consider those conditions from now on.

   (b) Deduce the Sharpe index (tip: start by proving that the covariance between the two random variables $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, adapted to the previous binomial tree, is $COV(X, Y) = \pi_1\, \pi_2\, (x_1 - x_2)\, (y_1 - y_2))$. What are the consequences of the conditions you have derived in (a) on the sign and magnitude of the Sharpe index.

   (c) Find a relationship between the objective and the risk-neutral probabilities such that the expected risk premium is non-negative. Explain.

10. For a two period binomial-tree with two states of nature, let a financial market be characterized by the following price vector and $(N \times K)$ payoff matrix

$$\mathbf{S} = \left(1, \frac{1}{R}\right), \text{ and } \mathbf{V} = \begin{pmatrix} R+\epsilon & 1 \\ R-\epsilon & 1 \end{pmatrix},$$

where $R > 1$ and ϵ can take any real value.

   (a) Under which conditions we may have arbitrage opportunities ? Justify.

   (b) From now on assume there are no arbitrage opportunities. Find the state prices.

   (c) Consider a worker facing a prospect of unemployment at period $t = 1$ and expecting to earn a contingent wage $Y^{un} = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$ for $\phi > 0$. Assume there is an institution which can insure its income such that his wage can become state independent, that is $Y^{in} = \begin{pmatrix} \phi \\ \phi \end{pmatrix}$. This institution hedges the difference $Y^{in} - Y^{un}$ by building a replicating portfolio. Find the replicating portfolio and the cost of providing insurance. Discuss your result.

11. For a two period binomial-tree with two states of nature, let a financial market be characterized by the following price vector and $(N \times K)$ payoff matrix

$$\mathbf{S} = \left(1, \frac{1}{R}\right), \text{ and } \mathbf{V} = \begin{pmatrix} R+\epsilon & 1 \\ R-\epsilon & 1 \end{pmatrix},$$

where $R > 1$ and ϵ can take any real value.

   (a) Under which conditions we may have arbitrage opportunities ? Justify.

   (b) From now on assume there are no arbitrage opportunities. Find the state prices.

(c) Consider a worker facing a prospect of unemployment at period $t = 1$ and expecting to earn a contingent wage $Y^{un} = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$ for $\phi > 0$. Assume there is an institution which can insure its income such that his wage can become state independent, that is $Y^{in} = \begin{pmatrix} \phi \\ \phi \end{pmatrix}$. This institution hedges the difference $Y^{in} - Y^{un}$ by building a replicating portfolio. Find the replicating portfolio and the cost of providing insurance. Discuss your result.

Abridged solution

(a) The return matrix is $\mathbf{R} = \begin{pmatrix} R+\epsilon & R \\ R-\epsilon & R \end{pmatrix}$. It has determinant $\det(\mathbf{R}) = 2\,R\epsilon$. We know that if $\epsilon = 0$ then the state prices will satisfy $R\,(q_1 + q_2) = 1$, which means that the market will be incomplete and we cannot rule out the existence of arbitrage opportunities, i.e., the existence of $s \in \{1, 2\}$ such that $q_s \leq 0$. In (b) it is shown that if $\epsilon \neq 0$ there will be no arbitrage opportunities.

(b) Let $\epsilon \neq 0$ then $\mathbf{R}^\top Q^\top = \mathbf{1}$ has the solution

$$Q^\top = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{2\,R} \begin{pmatrix} 1 \\ 1 \end{pmatrix} >> \mathbf{0}$$

so there are no arbitrage opportunities and the market is complete.

(c) The insurance company has the potencial payment $Y^{in} - Y^{un} = \begin{pmatrix} 0 \\ \phi \end{pmatrix}$. To hedge it, with the assets in the market it can build a replicating portfolio Θ solving

$$\mathbf{V}\,\Theta = Y^{in} - Y^{un} \iff \Theta = \frac{\phi}{2\,\epsilon} \begin{pmatrix} -1 \\ R+\epsilon \end{pmatrix}.$$

The cost of hedging is $c = \theta_1 + \dfrac{\theta}{R} = \dfrac{\phi}{2R}$.

12. Assume there is a financial market with two assets, one risky asset and one riskless asset with prices and payoffs

$$\mathbf{S} = \begin{pmatrix} \frac{1}{1+i} & s \end{pmatrix},\ \mathbf{V} = \begin{pmatrix} 1 & d_h \\ 1 & d_l \end{pmatrix},$$

where $i > 0$ and $d_h > d_l$ are the payoffs for the risky asset in the two states of nature h, l.

(a) Find the conditions under which there are no arbitrage opportunities and the market is complete. (From now on assume the condition you have just found.)

(b) Introduce a European put option with exercise price d_0, satisfying $d_l < d_0 < d_h$. By constructing a replicating portfolio, find to option's price under the assumption of absence of arbitrage opportunities.

(c) Assume that the two states of nature have equal probabilities. In this model the Sharpe index is equal to the Hansen-Jaganathan bound. Check it and provide an intuition why this is the case