Foundations of Financial Economics *2021/22*
Problem set 6: two-period GEAP

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1 Two-period equilibrium asset prices

1. Consider a financial economy characterized by (S, V) where

$$S = (1/(1+i), s),\ V = \begin{pmatrix} 1 & v_1 \\ 1 & v_2 \end{pmatrix}$$

for $v_1 < 1 < v_2$. Agents are homogeneous and we consider an endowment economy with an endowment process $\{y_0, y_1\}$ where $y_{1,s} = (1+\gamma_s)y_0$.

   (a) characterize the asset market as regards the existence of arbitrage opportunities and completeness;

   (b) define the equilibrium, determine the asset prices, and interpret the results obtained, for the following utility functions:

      i. a logaritmic utility function, $u(c) = \ln(c)$;

      ii. a quadratic utility function, $u(c) = ac - b/2c^2$, $a > 0$;

      iii. an exponential utility function, $u(c) = -\frac{e^{-\lambda x}}{\lambda}$, $\lambda > 0$;

      iv. a power utility function, $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, $\theta > 0$;

   (c) assume that an european call option is issued, at time 0 for exercise at time 1 with the price k, such that $v_1 < k < v_2$. Determine a replicating portfolio, and find the equilibrium price for option.

2. Consider a financial economy characterized by (S, V) where

$$S = (s_1, s_2),\ V = \begin{pmatrix} 2v_1 & v_1 \\ 2v_2 & v_2 \end{pmatrix}$$

Agents are homogeneous and we consider an endowment economy with an endowment process $\{y_0, y_1\}$ where $y_{1,s} = (1+\gamma_s)y_0$.

(a) characterize the asset market as regards the existence of arbitrage opportunities and completeness;

(b) define the equilibrium, determine the asset prices, and interpret the results obtained, for the following utility functions:

i. a logaritmic utility function, $u(c) = \ln(c)$;

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iv. a power utility function, $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, $\theta > 0$;

3. Let information be given by a two-period binomial tree, with equal probabilities for the two states of nature, and assume an endowment finance economy in which there are no arbitrage opportunities and agents are homogeneous. Further, assume that agents have a standard intertemporal utility functional (i.e, time additive and von-Neumann-Morgenstern) with logarithmic Bernoulli utility function and assume that the endowment at time 1 is $Y_1 = ((1-\gamma)\, y_0, (1+\gamma)\, y_0)$ where $0 < \gamma < 1$. The financial market is characterized by the existence of a bond with unit face value (to be paid at time $t = 1$) and a risky asset with price S and payoff $(2\, v\, S, v\, S)$.

(a) Find the stochastic discount factor.

(b) Find the Sharpe index. Justify your reasoning.

(c) Find the Hansen-Jagannathan bound. Explain its meaning.

(d) For which values of γ the equity premium puzzle holds ? Provide an intuition for your result.

Solution

(a) $M = (\beta(1-\gamma)^{-1}, \beta(1+\gamma)^{-1})$

(b) Returns: risk-free asset $R^f = (1+i, 1+i)$ risky asset $R = (1+r_1, 1+r_2) = (2v, v)$ risk premium $= R - R^f = (r_1 - i, r_2 - i)$. Expected risk premium $= \mathbb{E}[R - R^f] = (3/2)v - 1 - i$ Standard deviation: $\sigma[R - R^f] = v/2$. Sharpe index $(3v - 2(1+i))/v$ where i is the rate of return for the riskless asset.

(c) $\mathbb{E}[M] = \beta\left(\frac{(1-\gamma}{2} + \frac{(1+\gamma}{2}\right) = \frac{\beta}{(1-\gamma^2)}$, $\sigma(M) = \frac{\beta\gamma}{(1+\gamma)(1-\gamma)}$ then the HJ bound $\sigma[M]/\mathbb{E}[M] = \gamma$

d) The equity premium puzzle is verified if $\gamma \geq (3v - 2(1+i))/v$.

4. Consider an endowment finance economy where the asset market is characterized by the vector of asset prices and the payoff matrix

$$\mathbf{S} = \begin{pmatrix} \frac{1}{1+i} & s \end{pmatrix} \text{ and } \mathbf{V} = \begin{pmatrix} 1 & v_1 \\ 1 & v_2 \end{pmatrix}$$

where $v_1 < 1 < v_2$ and $s > 0$. Agents are homogeneous and the endowment process is $\{y_0, Y_1\} = \{1, (1+\gamma_1, 1+\gamma_2)\}$, for arbitrary values of γ_1 and γ_2. They value the consumption process $\{c_0, C_1\}$ by a von-Neumann-Morgenstern utility functional with discount factor β and a Bernoulli utility function $u(c) = \dfrac{c^{1-\theta}}{1-\theta}$.

(a) Define the general equilibrium for this economy

(b) Solve the representative agent problem.

(c) Find the equilibrium returns for the two assets.

5. Assume that the financal market data is given in the matrix

$$\begin{pmatrix} 1 & R_1 \\ 1 & R_2 \end{pmatrix},$$

where conditions for the absence of arbitrage conditions and complete markets are satisfied.In this economy there is a representative household who solves the problem

$$\max_{c_0, C_1, \ell, \theta} u(c_0) + \beta\, \mathbb{E}[u(C_1)],\ 0 < \beta < 1$$

subject to the budget constraint $c_0 = y_0 - \ell - \theta$ at time $t = 0$ and $c_{1,s} = \ell + \theta\, R_s + y_{1s}$ at time $t = 1$, for the states of nature $s = 1, 2$. We denote by $\{c_0, C_1\}$ and $\{y_0, Y_1\}$, where $y_{11} \neq y_{12}$, the processes for consumption and (exogenous) endowments, and ℓ and θ the (long) positions on money and the risky asset. Assume that the utility function is $u(c) = \ln(c)$.

(a) Solve the household problem.

(b) Derive the conditions for the existence of full insurance at the household level. Under the previous conditions, find the relationship between the optimal household position in the risky asset and the covariance between the return of the risky asst and the endowment at time $t = 1$, i.e., $COV(R, Y_1)$. Provide an intuition for the two results.

(c) Assume this is a homogeneous agent economy. Is it possible to have complete insurance at the general equilibrium level ? Provide an explanation.

6. Consider an homogeneous agent endowment finance economy in which there is a risk-free asset, with a return equal to $R^f = 1 + r$, and a risky asset with return $R = (1+\varrho, 1-\varrho)$, for $\varrho > 0$. The endowment process is $Y = \{y_0, Y_1\}$ where $Y_1 = ((1+\gamma)y_0, (1-\gamma)y_0)$ for $0 < \gamma < 1$. The representative consumer has the intertemporal utility functional

$$U(c_0, C_1) = \frac{c_0^{1-\theta} - 1}{1-\theta} + \beta \sum_{s=1}^{2} \pi_s \frac{c_{1s}^{1-\theta} - 1}{1-\theta},$$

for $0 < \beta < 1$ and $\theta > 1$.

(a) Find the dynamic stochastic general equilibrium for this economy.

(b) Define the dynamic stochastic general equilibrium for this economy. Find the equilibrium stochastic discount factor.

(c) Find the equilibrium rates of return for the risk free and the risky asset, i.e., r and ϱ. Discuss why it is possible to determine them uniquely.

7. Consider an homogeneous agent endowment finance economy in which there is a risk-free asset, with a return equal to $R^f = 1 + r$, and a risky asset with return $R = (1 + \varrho, 1 - \varrho)^\top$, for $\varrho > 0$. The endowment process is $Y = \{y_0, Y_1\}$ where $Y_1 = ((1 + \gamma)y_0, (1 - \gamma)y_0)^\top$ for $0 < \gamma < 1$. The representative consumer has the intertemporal utility functional

$$U(c_0, C_1) = \log(c_0) + \beta\,\mathbb{E}\big[\log(C_1)\big],\ \text{where } 0 < \beta < 1.$$

(a) Characterize the behavior of the agent which is implicit in the utility functional.

(b) Define the dynamic stochastic general equilibrium for this economy. Find the equilibrium stochastic discount factor.

(c) Find the equilibrium rates of return for the risk free and the risky asset, i.e., r and ϱ. Discuss why it is possible to determine them uniquely.

8. Consider an homogeneous agent endowment finance economy in which there is a risk-free asset, with a return equal to $R^f = 1 + r$, and a risky asset with return $R = (1 + \varrho, 1 - \varrho)^\top$, for $\varrho > 0$. The endowment process is $Y = \{y_0, Y_1\}$ where $Y_1 = ((1 + \gamma)y_0, (1 - \gamma)y_0)^\top$ for $0 < \gamma < 1$. The representative consumer has the intertemporal utility functional

$$U(c_0, C_1) = \log(c_0) + \beta\,\mathbb{E}\big[\log(C_1)\big],\ \text{where } 0 < \beta < 1.$$

(a) Characterize the behavior of the agent which is implicit in the utility functional.

(b) Define the dynamic stochastic general equilibrium for this economy. Find the equilibrium stochastic discount factor.

(c) Find the equilibrium rates of return for the risk free and the risky asset, i.e., r and ϱ. Discuss why it is possible to determine them uniquely.

Abridged solution

a) As

$$IMRS_{0,1s}(c) = \frac{c_{1s}}{\beta\, c_0\, \pi_s},\ s = 1, 2$$
$$\epsilon_{0,0} = \epsilon_{1s,1s} = 1,\ s = 1, 2$$
$$\epsilon_{0,1s} = \epsilon_{1s,1s'} = 0,\ s, s' = 1, 2$$

and $EIS_{0,1s} = 1 > 0$ for any s, the utility functional characterizes a household which displays: (1) impatience $IMRS_{0,1s}(\bar{c}) = \dfrac{1}{\beta\, \pi_s} > 1$ for any s; (2) (Edgeworth) independence both at the intertemporal level and between states of nature, meaning additive separability at both levels; (3) because $\epsilon_{1s,1s} > 0$ risk aversion and (4) intertemporal substitution (independent of the states of nature).

b) The consumer problem

$$\max_{c_0, c_{1,s}} \log(c_0) + \beta\,\mathbb{E}\big[\log(C_1)\big]$$

$$\text{subject to}$$

$$c_0 + \theta^f + \theta^r = y_0$$

$$C_1 = \theta^f\,R^f + \theta^r R + Y_1$$

The DGE is defined by $(\{c_0^{eq}, C_1^{eq}\}, \Theta^{eq}, \mathbf{R}^{eq})$ such that, given y_0, Y_1: (1) the agent solves its problem, given $\mathbf{R} = (R^f, R^r)$; (2) markets clear $c_0 = y_0$, $C_1 = Y_1$ and $\Theta = \mathbf{0}$. Determination: From the f.o.c. of the consumer we have $m_s = \beta \dfrac{c_0}{c_{1,s}}$. But at the equilibrium $c_0^{eq} = y_0$ and $c_{1,s}^{eq} = y_{1,s}$. Therefore the equilibrium SDF is

$$M^{eq} = \begin{pmatrix} m_1^{eq} \\ m_2^{eq} \end{pmatrix} = \begin{pmatrix} \dfrac{\beta}{1+\gamma} \\ \dfrac{\beta}{1-\gamma} \end{pmatrix}.$$

c) As in equilibrium $\mathbb{E}\big[M\,R^f\big] = \mathbb{E}\big[M\,R^r\big] = 1$ then we can find r and ϱ by solving

$$\begin{cases} \pi\,m_1^{eq}(1+r) + (1-\pi)\,m_2^{eq}(1+r) = 1 \\ \pi\,m_1^{eq}(1+\varrho) + (1-\pi)\,m_2^{eq}(1-\varrho) = 1 \end{cases}$$

yielding

$$r = \frac{1 - \mathbb{E}[M^{eq}]}{\mathbb{E}[M^{eq}]}$$

$$\varrho = \frac{1 - \mathbb{E}[M^{eq}]}{\pi\,m_1^{eq} - (1-\pi)\,m_2^{eq}}$$

where $\mathbb{E}[M^{eq}] = \pi\,m_1^{eq} + (1-\pi)\,m_2^{eq}$.