Foundations of Financial Economics
Revisions on utility theory

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Topics of the lecture

- Why utility theory ?
- Marginalist concepts which are frequent in economics
- Basic utility theory

Utility from Aristotle (384- 322 BCE) to the XIX century

A brilliant summary of 23 centuries of knowledge accumulation (see (Kauder, 1953, p. 650))

> *1. Value is dependent on utility and scarcity. (Adopted from Aristotelism and Thomism by Davanzatti, Montanari, Galiani, etc.)*
> *2. Essential for economic valuation is the concrete and not the abstract utility. (Galiani in opposition to the Doctors.)*
> *3. Concrete utility is determined by the law of diminishing utility. (Mentioned earlier by Aristotle but much more clearly expressed by Davanzatti, Montanari, Galiani, etc.)*
> *4. The value of the last piece is the marginal utility. (Bernoulli.)*
> *5. Utility and scarcity decide not only the value of consumer goods but also the value of the factors of production. The consumer's choice is the final factor of economic action. (Galiani, Condillac.)*
> *6. Prices are the outcome of individual comparison between goods. (Turgot.)*

Galiani (1728-1787), Daniel Bernoulli (1700-1782), Turgot (1727-1781)

After the XIX century

- XVIII: Daniel Bernoulli and utility theory under uncertainty (St Petersburg paradox)
- XIX century: development of the marginalist framework
- After 1940:
  - O. von-Neumann (1903-1957): game theory and new insights on the theory of choice under uncertainty
  - K. Arrow (1921-2017) and G. Debreu (1921-2004): general equilibrium and extension of utility theory to the theory of choice over time
  - K. Arrow, D. Kreps (1950-) and Porteus and the state-price theory of finance
  - Extensions of utility theory to several directions: time consistency, ambiguity, etc
  - Empirical and experimental contributions

The state of the art

In the second part of the XX century a consensus among economists emerged:

- utility theory should be stripped from their philosophical, psychological and other dimensions (but you can take a pick on the philosophical perspective on expected utility theory)
- it should be seen as a mathematical theory of rational choice
- it should be seen as a positive theory rather than a normative theory
- an impressive body of formal thinking relating consumption behavior with the axioms of choice has emerged
- it has been tested empirically and in the laboratory (experimental economics)
- unless the developments in other branches of science develop (i.e., neuroscience) it still provides the better framework for modelling the rational human economic decision making, in particular financial ones.

Marginalist concepts

The approach in this course

- We will not present the axioms of choice and their relationship with utility functions
- We go strait to presenting utility functions relating their mathematical properties with the underlying assumptions
- And try to relate them with economic decisions
- The key points are:
  - assessment of the value of different goods,
  - allocation of resources to different goods given their market prices and resources available (”scarcity”),
  - behavior in a world with frictions
- In the rest of the course we apply those concepts to:
  - theory of choice over time
  - theory of choice under uncertainty
  - theory of choice over time when uncertainty increases in the future (the cornerstone of DSGE models)
- And the predictions on the influence of asset prices at the micro level and their determinants at the macro level

Valuation function

- Consider a number of different **objects** indexed by $\mathbb{I} = \{1, \ldots, i, \ldots, n\}$
- The **quantity** of object i is denoted by $x_i \in \mathbb{R}$
- We assume that $x_i \geq 0$, for every $i \in \mathbb{I}$
- A **bundle** is a set of objects. The **composition** of a bundle is the set of quantities of the several objects

$$\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n_+$$

- Sometimes we let the bundle be **represented** by $\mathbf{x}$
- The **value of a bundle** is measured by the (at least twice-) continuously differentiable function

$$F = F(\mathbf{x}) = F(x_1, \ldots, x_i, \ldots, x_n)$$

- We can call $\mathbf{x}$ the **composition** of the bundle
- In economics usually $F(\cdot)$ represents **utility or a production functions**.

Change in the valuation

- The **variation of the value of a bundle** is assessed by the differential (under very weak conditions)

$$dF = F_1(\mathbf{x})\, dx_1 + \ldots + F_i(\mathbf{x})\, dx_i + \ldots = \nabla F(\mathbf{x}) \cdot d\mathbf{x}$$

  where ∇F is the gradient (vector)

$$\nabla F(\mathbf{x}) = \big(F_1(\mathbf{x}), \ldots, F_i(\mathbf{x}), \ldots, F_n(\mathbf{x})\big)^\top$$

- where F_i is the **partial derivative** of F as regards i, that is, using the definition of a derivative

$$F_i(\mathbf{x}) \equiv \frac{\partial F(\mathbf{x})}{\partial x_i} = \lim_{h \to 0} \frac{F(x_1, \ldots, x_i + h, \ldots, x_n) - F(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

  it represents the change in the value of bundle, with composition $\mathbf{x}$; when the quantity of object i changes from x_i to $x_i + h$.

Characterization of objects

- We say object i included in a bundle $\mathbf{x}$ is a

$$\begin{cases} \textbf{good} & \text{if } F_i(\mathbf{x}) > 0 \\ \textbf{at a saturation point} & \text{if } F_i(\mathbf{x}) = 0 \\ \textbf{bad} & \text{if } F_i(\mathbf{x}) < 0 \end{cases}$$

- Intuitioon: an object
  - is a good if more of it increases the value of the bundle
  - is at a saturation point if more of it does not change the value of the bundle
  - is a bad if more of it reduces the value of the bundle
- From now on we consider only goods, i.e. objects $i \in \mathbb{I}$ such that $F_i > 0$

Marginal value for goods

Definition 1

*We call **marginal contribution** of good i to the change in the value of a bundle, $F(\mathbf{x})$, if its quantity changes by dx_i*

$$\text{(Definition) } M_i \equiv \frac{dF}{dx_i}$$

- Consider the bundle variation $d\mathbf{x} = (0, \ldots, 0, dx_i, 0, \ldots, 0)$. Then $dF = F_i \, dx_i$ which implies that the marginal contribution is equal to the partial derivative

  $$\text{(Implication) } M_i = F_i$$

  therefore a **good has a positive marginal contribution to the value of a bundle**.
- Observation: M_i represents an economic concept of change in value. When we say $M_i = F_i$ it means that it is measured by the derivative F_i. We could use any other method for measuring value. This measure is the product of around 2 millennia of thought!

Change in marginal contributions

- Observe that $M_i(\mathbf{x}) = F_i(\mathbf{x})$ because F_i is a function of $\mathbf{x}$
- If F is twice-differentiable we can calculate second-order derivatives

$$\text{(own derivative) } F_{ii} \equiv \frac{\partial^2 F(\mathbf{x})}{\partial x_i^2}$$

$$\text{(crossed derivative) } F_{ij} \equiv \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial_j}, \text{ for any } j \neq i \in \mathbb{I}$$

Change in own marginal contributions

Definition 2

*The **variation of the marginal contribution** of object i to the value of the bundle, that is $F(\mathbf{x})$, is*

$$\frac{\partial M_i}{\partial x_i} = F_{ii}$$

Therefore the **marginal contribution** of object i to the value of the bundle is

- is **increasing** if $F_{ii} > 0$
- is **constant** if $F_{ii} = 0$
- is **decreasing** if $F_{ii} < 0$

Change in crossed marginal contributions

Definition 3

Pareto-Edgeworth *relationships: are defined by the variation of the marginal contribution of i, M_i, for a variation in the the quantity of another object j:*

$$\frac{\partial M_i}{\partial x_j} = F_{ij}$$

Uzawa-Allen elasticities: are defined by the elasticities of F

$$\text{(own) } \varepsilon_{ii} \equiv -\frac{F_{ii}\, x_i}{F_i} \text{ (crossed) } \varepsilon_{ij} \equiv -\frac{F_{ij}\, x_j}{F_i}$$

Definition 4

- *goods i and j are* ***complements*** *(in the PE sense)* $F_{ij} > 0 \iff \varepsilon_{ij} < 0$
- *goods i and j are* ***independent*** *(in the PE sense)* $F_{ij} = 0 \iff \varepsilon_{ij} = 0$
- *goods i and j are* ***substitutes*** *(in the PE sense)* $F_{ij} < 0 \iff \varepsilon_{ij} > 0$

Observe that we are assuming that $x_i \geq 0$ and $F_i \geq 0$ (object i is not bad)

Compensated variations

Definition 5

*The **marginal rate of substitution** of good i by good j is the variation in the quantity of good j for a unit variation in good i such that the value of the bundle does not change*

$$MRS_{ij} \equiv -\frac{dx_j}{dx_i}\Big|_{dF=0}$$

Proposition 1

The marginal rate of substitution between object i and j is equal to the ratio of the marginal contributions

$$\textit{(Implication)}\ MRS_{ij}(\mathbf{x}) = \frac{F_i(\mathbf{x})}{F_j(\mathbf{x})}\ \textit{for}\ F(\mathbf{x}) = \textit{constant}$$

Compensated variations

Proof: We want to know what would be dx_j if we change dx_i in such a way as to keep the value F constant

$$d\mathbf{x} = (0, \ldots, 0, dx_i, 0, \ldots, dx_j, 0, \ldots, 0) \text{ such that } dF = 0$$

As $dF = \nabla F \cdot d\mathbf{x} = F_i\, dx_i + F_j\, dx_j = 0$ then

$$MRS_{ij} \equiv -\left.\frac{dx_j}{dx_i}\right|_{dF=0} = \frac{F_i}{F_j}$$

Elasticity of substitution

Definition 6

*The **elasticity of substitution** of good i by good j is the relative change in the ratio x_j/x_i for a relative change in the ratio of the marginal rates of subtitution*

$$ES_{ij}(\mathbf{x}) \equiv \frac{d\ln(x_j/x_i)}{d\ln MRS_{ij}(\mathbf{x})}$$

Recall that MRS_{ij} is defined for a constant value of the bundle $\mathbf{x}$ (i.e, for $dF = 0$)

Proposition 2

If F is twice differentiable, then the ES_{ij} is

$$ES_{ij}(\mathbf{x}) = \frac{x_i\, F_i(\mathbf{x}) + x_j\, F_j(\mathbf{x})}{x_j\, F_j(\mathbf{x})\, \varepsilon_{ii}(\mathbf{x}) - 2\, x_i\, F_i(\mathbf{x})\, \varepsilon_{ij}(\mathbf{x}) + x_i\, F_i(\mathbf{x})\, \varepsilon_{jj}(\mathbf{x})}$$

where $x_i\, F_i\, \varepsilon_{ij} = x_j\, F_j\, \varepsilon_{ji}$ and $F_{ij} = F_{ji}$ if F is continuous.

Elasticity of substitution: continuation

- Sketch of the proof: remember we have $F_i\,dx_i + F_j\,dx_j = 0$.
- The numerator is

$$d\ln(x_j/x_i) = d\ln x_j - d\ln x_i = \frac{dx_j}{x_j} - \frac{dx_i}{x_i} =$$
$$= -\frac{dx_i}{x_i\,x_jF_j}\Big(x_iF_i + x_jF_j\Big)\ \text{(because } F_i\,dx_i + F_j\,dx_j = 0\text{)}$$

- The denominator of ES_{ij} is

$$d\ln MRS_{ij} = d\ln\left(\frac{F_i(x_i,x_j)}{F_j(x_i,x_j)}\right) = d\ln F_i - d\ln F_j = \frac{dF_i}{F_i} - \frac{F_j}{F_j}$$

where

$$dF_i = F_{ii}dx_i + F_{ij}dx_j = dx_i\Big(F_{ii} + \frac{dx_j}{dx_i}F_{ij}\Big) = dx_i\Big(F_{ii} - \frac{F_i}{F_j}F_{ij}\Big)$$
$$dF_j = F_{ji}dx_i + F_{jj}dx_j = dx_i\Big(F_{ij} + \frac{dx_j}{dx_i}F_{jj}\Big) = dx_i\Big(F_{ij} - \frac{F_i}{F_j}F_{jj}\Big)$$

- simplify and use the definition of the Uzawa-Allen elasticities.

Example: Cobb-Douglas function

- Let $\mathbf{x} = (x_1, x_2)$, where $x_i \geq 0$ for $i = 1, 2$
- Economists call **Coob-Douglas function** to

$$\boxed{F = F(\mathbf{x}) = x_1^{\alpha}\, x_2^{1-\alpha},\ \text{for } 0 < \alpha < 1,\ x_1 > 0, x_2 > 0}$$

  Mathematically it is a geometric average
- First derivatives:

$$F_1 = \alpha \frac{F}{x_1} > 0,\ F_2 = (1-\alpha)\frac{F}{x_2} > 0$$

  Intuition: both objects 1 and 2 are goods
- Second derivatives:

$$F_{11} = -\alpha(1-\alpha)\frac{F}{(x_1)^2} < 0,\ F_{22} = -\alpha(1-\alpha)\frac{F}{(x_2)^2} < 0,$$

$$F_{12} = F_{21} = \alpha(1-\alpha)\frac{F}{x_1\, x_2} > 0$$

  Intuition: (1) each object has a **decreasing marginal contribution**; (2) the two objects are **PE complements**

Example: Cobb-Douglas function

- The Uzawa-Allen elasticities are

$$\varepsilon_{11} = 1 - \alpha > 0, \ \varepsilon_{22} = \alpha > 0, \ \varepsilon_{12} = -(1 - \alpha) < 0$$

  Intuition: elasticities are constant and independent of the quantities in the bundle, **x**,

- The marginal rate of substitution is

$$MRS_{12} = \frac{F_1}{F_2} = \frac{\alpha}{1 - \alpha} \frac{x_2}{x_1}$$

  Intuition: it is an inverse function of the quantities, for increasing $i = 1$ by one unit we need to reduce $i = 2$ by more if x_2 (x_1) is large (x_1 is small) and if α is large

- The elasticity of substitution is

$$ES_{12} = \frac{x_1 F_1 + x_2 F_2}{x_2 F_2 \varepsilon_{11} - 2 x_1 F_1 \varepsilon_{12} + x_1 F_1 \varepsilon_{22}} = \frac{F}{F} = 1$$

  Intuition: if we want to change the ratio of the two objects and keep the value of the bundle constant, the rate of change of the ratio x_1/x_2 should be equal to the rate of change of the MRS_{12}

Basic utility theory

Marginal utility concepts

Now we apply those concepts and results to the consumer choice

- Consider a bundle containing **two different consumption goods** indexed by $i \in \{1, 2\}$
- The composition of the bundle is given by $\mathbf{c} = (c_1, c_2)$, where $c_i \geq 0$ is the quantity of good i
- The value of the bundle is measured by a **utility function**

  $$U = U(\mathbf{c}) = U(c_1, c_2)$$

  that we assume is continuous and differentiable.
- The contribution of i to the value of the bundle $\mathbf{c}$, is measured by the **marginal utility**

  $$U_i = \frac{\partial U(\mathbf{c})}{\partial c_i} \text{ for } i \in \{1, 2\}$$

- **Assumption** the utility function is increasing in both arguments

  $$\nabla U(\mathbf{c}) = (U_1(\mathbf{c}), U_2(\mathbf{c}))^\top \gg 0$$

- That is **goods** $i = 1, 2$ **are indeed goods** and there is **no satiation**

Marginal utility concepts

- The **marginal utility variations** are

$$U_{ii} = \frac{\partial^2 U(\mathbf{c})}{\partial c_i^2} \text{ for } i \in \{1, 2\}, \text{ and } U_{12} = \frac{\partial^2 U(\mathbf{c})}{\partial c_1 \, \partial c_2}$$

- The Allen-Uzawa elasticities are

$$\varepsilon_{ii} = -\frac{U_{ii} \, c_i}{U_i}, \text{ for} i \in \{1, 2\}$$

and

$$\varepsilon_{12} = -\frac{U_{12} \, c_2}{U_2}$$

- **Assumption**: the utility function is strictly concave, that is it satisfies

$$U_{11} < 0, U_{22} < 0, \text{and} \, U_{11} \, U_{22} - U_{12}^2 \geq 0.$$

- Therefore, we are assuming that:
  - both goods have **decreasing marginal utilities** (remember Aristotle)
  - and the two goods can be
    - PE **complements** if $U_{12} > 0 \iff \varepsilon_{12} < 0$
    - PE **independent** if $U_{12} = 0 \iff \varepsilon_{12} = 0$
    - PE **substitutes** if $U_{12} < 0 \iff \varepsilon_{12} > 0$

Marginal utility theory

- The **marginal rate of substitution** is positive

$$MRS_{12}(\mathbf{c}) = \frac{U_1}{U_2} > 0$$

  Intuition: if we increase the quantity of good $i = 1$ by one unit and keep the value of the bundle constant, we need to sacrifice the MRS_{12} of good $i = 2$

- The elasticity of substitution between the two goods

$$ES_{12}(\mathbf{x}) = \frac{c_1\ U_1(\mathbf{c}) + c_2\ U_2(\mathbf{c})}{c_2\ U_2(\mathbf{c})\,\varepsilon_{11}(\mathbf{c}) - 2\,c_1\ U_1(\mathbf{c})\,\varepsilon_{12}(\mathbf{c}) + c_1\ U_1(\mathbf{c})\,\varepsilon_{22}(\mathbf{c})}$$

  is positive if the two goods are PE complements or independent and is ambiguous if they are PE substitutable. (Hint: recall the interpretation we just made on the meaning of ES)

Example

1. Assume the utility function is of Cobb-Douglas type

$$U = U(c_1, c_2) = c_1^{\alpha}\, c_2^{1-\alpha}, \text{ for } 0 < \alpha < 1$$

2. The marginal utilities are, as expected, positive: both $i = 1, 2$ are goods

$$U_1 = \frac{\partial U}{\partial c_1} = \alpha \frac{U}{c_1} > 0$$
$$U_2 = \frac{\partial U}{\partial c_2} = (1-\alpha) \frac{U}{c_2} > 0$$

3. The derivatives of the marginal utilities are

$$U_{11} = \frac{\partial^2 U}{\partial c_1^2} = -\alpha\,(1-\alpha)\,\frac{U}{c_1^2} < 0$$
$$U_{22} = \frac{\partial^2 U}{\partial c_2^2} = -\alpha\,(1-\alpha)\,\frac{U}{c_2^2} < 0$$
$$U_{12} = \frac{\partial^2 U}{\partial c_1\,\partial c_2} = \alpha(1-\alpha)\frac{U}{c_1\,c_2} > 0$$

Intuition: (1) there is decreasing marginal utilities for the two goods; and (2) the two goods are PE complements

Example

Therefore, there is no surprise that

1. The rate of marginal substitution between $i = 1$ and $i = 2$ is

$$MRS_{1,2}(\mathbf{c}) = \frac{\alpha \, c_2}{(1-\alpha) \, c_1} > 0$$

2. The elasticity of substitution of $i = 1$ by $i = 2$ is

$$ES_{1,2}(\mathbf{c}) = 1 > 0$$

Utility theory

The problem: optimal allocation

- **The problem**: consider an agent with a resource whose level is W that wants to **allocate it optimally** to a bundle of two goods, 1 and 2, given their prices p_1 and p_2, and measures the value of the bundle by the utility function $U(c_1, c_2)$, where the quantities of the two goods are c_1 and c_2.
- **Further assumptions**:
  - The utility function $U(\cdot)$ is: continuous, differentiable, increasing and concave.
  - The endowment is positive and the prices are positive: $W > 0$, $p_i > 0$
- The nominal expenditure is $E = E(c_1, c_2) \equiv p_1 c_1 + p_2 c_2$

Free allocation: optimality

- Assume there are no other constraints with the exception of the resource constraint $E(c_1, c_2) = W$
- The (primal) problem: find the maximum value of the bundle such that the expenditure in obtaining it exhaust the resource level W

$$V(W; p_1, p_2) = \max_{c_1, c_2} \left\{ U(c_1, c_2) : \; E(c_1, c_2) = W \right\}$$

- Function $V(.)$ is called **indirect utility or value function** because it provides a measure of the **value of the endowment** W when it can be spent in the two goods.
- The dual problem is: find the minimum expenditure allowing the value of the bundle to be V

$$W(V; p_1, p_2) = \min_{c_1, c_2} \left\{ E(c_1, c_2) : \; U(c_1, c_2) = V \right\}$$

Optimal free allocation

Finding the solution to the primal problem

- Form the Lagrangean

$$\mathcal{L}(c_1, c_2, \lambda) = u(c_1, c_2) + \lambda(W - E(c_1, c_2))$$

  where λ is the Lagrange multiplier
- The solution (which always exists) $(c_1^*, c_2^*, \lambda^*)$ satisfies the conditions

$$\begin{cases} \mathcal{L}_j = U_j(c_1, c_2) - \lambda p_j = 0, \quad j = 1, 2 \\ \mathcal{L}_\lambda = W - E(c_1, c_2) = 0 \end{cases}$$

Proposition 4

At the optimum the $MRS_{1,2}$ is equalized to the relative prices

$$MRS_{1,2} = \frac{U_1(c_1^*, c_2^*)}{U_2(c_1^*, c_2^*)} = \frac{p_1}{p_2}$$

and

$$E(c_1^*, c_2^*) = p_1 c_1^* + p_2 c_2^* = W$$

Meaning: the relative internal valuation of the two goods is equal to the market (i.e., aggregate) relative valuation

Optimal free allocation

- When there is free allocation, the solution is characterized by the equations,

$$p_2 \, U_1(c_1^*, c_2^*) = p_1 \, U_2(c_1^*, c_2^*) \quad (1)$$
$$E(c_1^*, c_2^*) = W \quad (2)$$

- **Existence and uniqueness of the solution**: if the utility function is strictly concave then with very weak conditions (differentiability) we have an unique interior optimum

Optimal free allocation: graphical representation

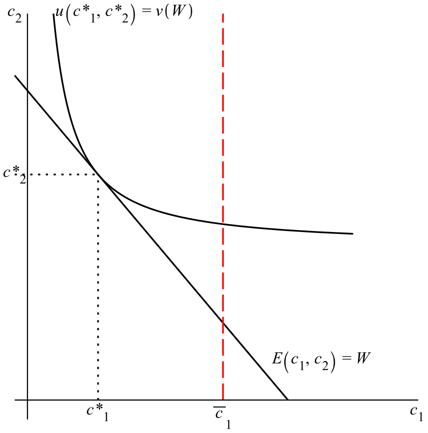

Figure: Interior optimum for a log utility function
$U(c_1, c_2) = \ln c_1 + b \ln c_2$

Optimal free allocation

Value of the resource

Indirect utility function:

- Equation (1) is a first-order partial differential equation with solution (check this)

$$U(c_1^*, c_2^*) = V\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)$$

- from equation (2), in the optimum we have

$$U(c_1^*, c_2^*) = V(w)$$

where $w \equiv \dfrac{W}{p_1}$ is the real level of the resource deflated by p_1.

Example: free allocations

- The problem

$$\max_{c_1, c_2} \{ c_1^{\alpha} \, c_2^{1-\alpha} : \; p_1 \, c_1 + p_2 \, c_2 = W \}$$

- The first order conditions are (see previous slides)

$$\begin{cases} p_2 \, U_1 = p_1 \, U_2 \\ p_1 \, c_1 + p_2 \, c_2 = W \end{cases} \Leftrightarrow \begin{cases} (1-\alpha) p_1 \, c_1 - \alpha p_2 \, c_2 = 0 \\ p_1 \, c_1 + p_2 \, c_2 = W \end{cases}$$

then the **optimal consumption allocation** is, therefore

$$\begin{cases} c_1^* = \alpha \dfrac{W}{p_1} \\ c_2^* = (1-\alpha) \dfrac{W}{p_2} \end{cases}$$

Example:

Case 1: free allocations

Properties:

as

$$c_1^* = c_1^*(p_1, W),\ c_2^* = c_2^*(p_2, W)$$

1. the consumption for each good is **proportional to nominal wealth deflated by its price**
2. there is **no complementarity or substitutability in the Hicksian sense**, i.e. their cross-derivatives relative to the price of the other good are zero

$$\frac{\partial c_1^*}{\partial p_2} = \frac{\partial c_2^*}{\partial p_1} = 0.$$

Substituting in the utility function we get the value of the resource W

$$V(W) = \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} W =$$
$$= \chi(\alpha) \frac{W}{P}$$

where $P \equiv p_1^{\alpha}\, p_2^{1-\alpha}$ is the **consumer price index**

3. The **value of the resource** W, assuming there is an optimal free allocation among the two goods, **is proportional to its real level** deflated by the consumer's own price index (which is a geometrical mean whose weights are given by those of the utility function)

Constrained allocation

- Let us assume that the agent is **constrained in the allocation to good 1**. For instance, assume that $c_i \in [0, \bar{c}_1]$
- The (primal) problem is now

$$V(W; p_1, p_2, \bar{c_1}) = \max_{c_1, c_2} \{ U(c_1, c_2) : \; E(c_1, c_2) = W, \; 0 \leq c_1 \leq \bar{c}_1 \}$$

- Observation: most models of financial frictions introduce constraints of this type
- More generally we could assume there are restrictions in allocation of resources to the two goods and in the total use of the resource W. In this case the problem would become

$$V(W; p_1, p_2, \bar{c_1}, \bar{c_2}) = \max_{c_1, c_2} \{ U(c_1, c_2) : \; E(c_1, c_2) \leq W, 0 \leq c_j \leq \bar{c}_j, j = 1, 2 \}$$

Constrained allocation: optimality

- The Lagrangean is now

$$\mathcal{L}(c_1, c_2, \lambda, \eta_1, \eta_2, \zeta_1, \zeta_2) = u(c_1, c_2) + \lambda(W - E(c_1, c_2)) + \\ + \eta_1\, c_1 + \eta_2\, c_2 + \zeta_1\, (\bar{c}_1 - c_1) + \zeta_2\, (\bar{c}_2 - c_2)$$

  where λ is the Lagrange multiplier and η_i and ζ_i are marginal valuation of the constraints;

- The solution (which always exists) $(c_1^*, c_2^*, \lambda^*, \eta_1^*, \eta_2^*, \zeta_1^*, \zeta_2^*)$ satisfies the Karush-Kuhn-Tucker conditions:
  the optimality conditions

$$\begin{cases} U_j(c_1, c_2) - \lambda p_j + \eta_j - \zeta_j = 0, & j = 1, 2 \\ \lambda(W - E(c_1, c_2)) = 0,\ \lambda \geq 0,\ E(c_1, c_2) \leq W & \end{cases}$$

  together with the complementarity slackness conditions

$$\begin{cases} \eta_j c_j = 0,\ \eta_j \geq 0,\ c_j \geq 0, & j = 1, 2 \\ \zeta_j(\bar{c}_j - c_j) = 0,\ \zeta_j \geq 0,\ c_j \leq \bar{c}_j, & j = 1, 2 \end{cases}$$

Optimal constrained allocation: solution

Corner solution: lower $c_1 = 0$

- Let $c_1^* = 0$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;
- The first-order conditions are

$$\begin{aligned} p_2 U_1(c_1^*, c_2^*) &= p_1 U_2(c_1^*, c_2^*) - p_2 \eta_1^* && (3) \\ E(c_1^*, c_2^*) &= W && (4) \\ \eta_1^* &> 0 \,(\text{because } c_1 = 0) \\ \zeta_1^* &= 0 \,(\text{because } c_1 < \bar{c}_1) \\ \eta_2^* &= \zeta_2^* = 0 \,(\text{because } c_2 \in (0, \bar{c}_2)) \end{aligned}$$

- Now, the MRS is smaller than the relative price

$$MRS_{12} = \frac{U^*_{c_1}}{U^*_{c_2}} = \frac{p_1}{p_2} - \frac{\eta_1^*}{U^*_{c_2}} < \frac{p_1}{p_2}$$

  i.e., there is a **"wedge" between relative prices and the** MRS_{12}
- Equation (3) is a first-order partial differential equation with solution

$$U(c_1^*, c_2^*) = \frac{\eta_1^* c_2^*}{p_1} + V\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)$$

- if we use equation (4) we say **there is misallocation**:

$$U(c_1^*, c_2^*) = -\eta_1^* w + V(w) < V(w)$$

  the resource is not optimally used.

Optimal constrained allocation: figure

Corner solution 1

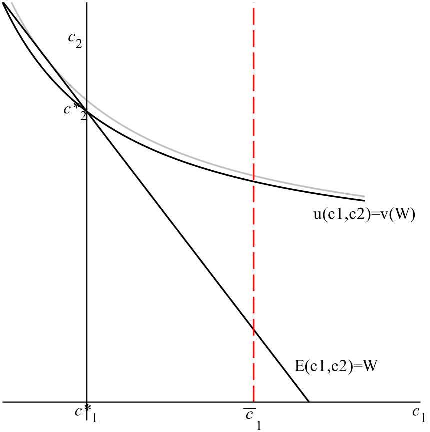

Figure: Corner solution: the indirect utility level is smaller than for the unconstrained case

Example: Cobb-Douglas utility

Case 2: positive allocations to good 1

- The problem

$$\max_{c_1, c_2} \{ c_1^{\alpha}\, c_2^{1-\alpha} :\; p_1\, c_1 + p_2\, c_2 = W,\; c_1 > 0, c_2 \geq 0 \}$$

- the solution is the same as in case 1

$$\begin{cases} c_1^* = \alpha \dfrac{W}{p_1} > 0 \\ c_2^* = (1-\alpha) \dfrac{W}{p_2} \end{cases}$$

- This means that the constraint is **not binding**.

Optimal constrained allocation: solution

Corner solution: upper constraint $c_1 = \bar{c}_1$

- Let $c_1^* = \bar{c}_1$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;
- the first order conditions are

$$
\begin{aligned}
p_2 U_1(c_1^*, c_2^*) &= p_1 U_2(c_1^*, c_2^*) + p_2 \zeta_1^* &\quad (5)\\
E(c_1^*, c_2^*) &= W &\quad (6)\\
\zeta_1^* &> 0 \,(\text{because } c_1 = \bar{c}_1)\\
\eta_1^* &= 0 \,(\text{because } c_1 > 0)\\
\eta_2^* &= \zeta_2^* = 0 \,(\text{because } c_2 \in (0, \bar{c}_2))
\end{aligned}
$$

- There is **again a "wedge"** between the MRS_{12} and the relative price, but now

$$MRS_{12} = \frac{U_{c_1}^*}{U_{c_2}^*} = \frac{p_1}{p_2} + \frac{\zeta_1^*}{U_{c_2}^*} > \frac{p_1}{p_2}$$

- Equation (5) is a first-order partial differential equation with solution

$$U(c_1^*, c_2^*) = -\frac{\zeta_1^* c_2^*}{p_1} + V\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)$$

- if we use equation (4) in the optimum we have

$$U(c_1^*, c_2^*) = -\frac{\zeta_1^* p_1 (w - \bar{c}_1)}{p_2} + V(w) < V(w)$$

But **there is again misallocation** of the resource W

Consumer problem

Corner solution 2

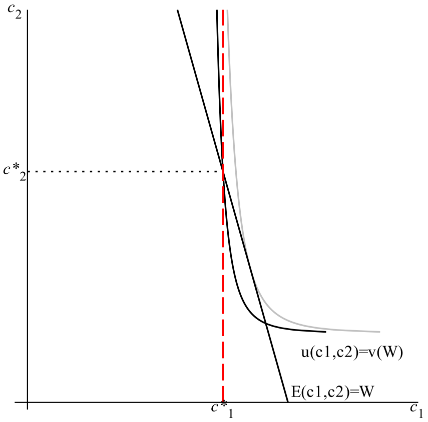

Figure: Corner solution: the indirect utility level is smaller than for the unconstrained case

Equivalent interpretation

- Let the value function in which there are constraints on the consumer be denoted by $\tilde{V}(w)$
- Looking at the previous cases we can write

$$\tilde{v}(w) = V(w) - \delta(w)$$

where $\delta(w) \geq 0$ measures the welfare loss introduced by the constraint $c_1 \in [0, \bar{c}_1]$.

- We could obtain a similar solution for the consumer problem is instead of considering the endowment level w we consider the resource level

$$\tilde{w} = \{x : (\tilde{v}^{-1})(x) = 0\} < w.$$

- The existence of misallocation implies that the actual level of the resource should be discounted (i.e., the economic level of the resource is smaller than apparent)

Example

Case 3: upper bound on the allocations to good 1

- The problem

$$\max_{c_1, c_2} \{c_1^{\alpha}\, c_2^{1-\alpha} :\ p_1\, c_1 + p_2\, c_2 = W,\ c_1 \in (0, \bar{c}_1], c_2 \geq 0\}$$

- In this case we require that $c_1 \leq \bar{c}_1$. As we know W and p_1 we can write $\bar{c}_1 = \beta\, W/p_1$
- If $\beta < \alpha$ then $\alpha\, W/p_1 > \bar{c}_1$ which means that the constraint is binding: $c_1^* = \bar{c}_1$
- The first order conditions are now (5) and (4) with $c_1 = \bar{c}_1$

$$\begin{cases} \alpha p_2\, c_2 = (1-\alpha) p_1\, \bar{c}_1 + p_2 \bar{c}_1\, c_2 \zeta_1 \\ p_1\, \bar{c}_1 + p_2\, c_2 = W \end{cases}$$

  that we need to solve for c_2 and ζ_1.
- The solution is

$$c_1^* = \bar{c}_1 = \beta \frac{W}{p_1} < \alpha \frac{W}{p_1}$$

$$c_2^* = (1-\beta) \frac{W}{p_2} > (1-\alpha) \frac{W}{p_2}$$

$$\zeta_1^* = \frac{(\alpha - \beta) p_1}{\beta(1-\beta)\, W} > 0$$

  Therefore: the consumption of good 1 (2) will smaller (larger) than in the free allocation case

Example

Case 3: upper bound on the allocations to good 1

- We can calculate the **wedges** (μ_1 and μ_2) regarding the solution of the unconstrained case

$$c_1^* = \bar{c}_1 = \frac{\beta}{\alpha}\,\alpha\frac{W}{p_1} = \mu_1\; c_1^{free}$$

$$c_2^* = \frac{1-\beta}{1-\alpha}(1-\alpha)\frac{W}{p_2} = \mu_2\; c_2^{free}$$

where

$$\boxed{\mu_1 \equiv \frac{\beta}{\alpha} < 1 < \mu_2 \equiv \frac{1-\beta}{1-\alpha}(1-\alpha)}$$

- the (binding) constraint drives the actual consumption of good 1 below and the consumption of good 2 above relative to the consumption that would be optimal if there were no (binding) constraints

Example

Case 3: upper bound on the allocations to good 1

- Again, there is **misallocation**.
- To calculate the value loss

$$
\begin{aligned}
V(W) &= \left(\frac{\beta}{p_1}\right)^{\alpha}\left(\frac{1-\beta}{p_2}\right)^{1-\alpha} W = \\
&= \beta^{\alpha}(1-\beta)^{\alpha}\frac{W}{P} = \\
&= X(\beta)\,\chi(\alpha)\frac{W}{P} < V(W)^{free} = \chi(\alpha)\frac{W}{P}
\end{aligned}
$$

  because $X(\beta) < 1$ for $\beta < \alpha$.
- To prove this, we have

$$
X(\beta) \equiv \left(\frac{\beta}{\alpha}\right)^{\alpha}\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha} > 0
$$

  where we have assumed that $\beta < \alpha$;
- and show that $X(\alpha) = 1$ but $X(\beta) < 1$ because $\beta < \alpha$ and we have locally

$$
\frac{\partial X}{\partial \beta} = \left(\frac{\alpha-\beta}{\beta(1-\beta)}\right) X > 0.
$$

Take away

- Constraints in the allocation of a resource only matter if they are binding
- If there are **no constraints or no binding constraints** in the allocation of an endowment to the purchase of two goods, at the optimum the **internal relative valuation (MRS) of the two goods is equal to the relative market valuation** (provided by their prices)
- If there are **binding constraints** on the free allocation of an endowment to the purchase of two goods
  1. there is a **wedge between the the MRS and the relative prices** (i.e., between the internal and the market valuation)
  2. there is **misallocation** of the resource meaning that there is a an welfare loss relative to free allocation

- This is the core of the mechanism of transfer of resources, through time and states of nature that financial economics deals with
- It also introduces the way financial constrains (or financial frictions) can shape the operation of financial transactions (constraints on consumption/savings, constraints on hedging, etc).

Applications

See Problem set 1

References

- Your favorite microeconomics textbook
- intermediate level textbooks (Varian, 2010, ch. 4, 5, 6) or (Varian, 2014, ch. 4, 5, 6)
- More advanced presentations: The textbook (Mas-Colell et al., 1995, ch 2), or intemediate level mathematical economics textbooks (de la Fuente, 2000, ch 8) or (Chiang and Wainwright, 2005, ch 12, 13).
- History of utility theory: Kauder (1965)

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