Macroeconomics with financial frictions

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1 Introduction

This note deals with general stochastic general equilibrium models in continuous time. In particular, we deal with macroeconomic finance variables as interest rates, asset prices, net wealth, and leverage, with a view of dealing with macroeconomic fluctuations. The two major dimensions of the analysis are related to the source of uncertainty and to the way it is propagated in the economy.

First, we assume there are aggregate stochastic shocks which hit the economy frequently. That is, we will not deal with rare big shocks, but with continuous, small, and imperfectly observed random perturbations in the fundamentals of the economy. The two branches of financial economics, financial microeconomics and macroeconomics, can be distinguished by the type of risk they tend to address. While microfinance deals mainly with idiosyncratic risk, and the insurance or sharing mechanisms provided by asset markets, macrofinance is more concerned with the existence of aggregate risk and the way it is absorbed, shared, amplified or smoothed out by the operating of asset markets. In this note we will assume there is an exogenous source of risk which is introduced through supply shocks taking the form of dividend or productivity random changes.

Second, the propagation of shocks may depend on the existence of frictions in the economy. We take as a benchmark a frictionless, representative agent, endowment economy (see section 2) and introduce progressively several frictions. In the literature followed in this note, there are frictions if there is some type of heterogeneity among agents. They can be rooted in differences in preferences, participation in financial markets, rates of return on assets, information, for instance. In this note we consider only one type of friction: limited participation in 3. We will study the distortions introduced by this type of friction in a benchmark exchange economy and in a production economy in which there are costs of adjustment of capital (in section 4). Brunnermeier and Sannikov (2016) call this case technological illiquidity. We will see that in some cases there is an effect on the wealth distribution on the asset returns.

The objective of this note is mainly pedagogical to make accessible some relevant recent research, by providing some detail on the construction of the models and on the way they are related. We will focus on the following papers: Basak and Cuoco (1998), Brunnermeier and Sannikov (2014) and the survey Brunnermeier and Sannikov (2016).

In section 2 we present the benchmark frictionless economy. In 3 we present a model with limited asset market participation and in section 4 one model with adjustment costs in investment.

2 Frictionless economies

2.1 Primitives: preferences and endowments

The general equilibrium for a stochastic dynamic economy requires introducing assumptions regarding four elements: information environment, technology available, preference structure, and existing market institutions. Next, we specify those elements for the simplest financial market economy: the endowment or exchange economy.

First, the information structure is given by the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where Ω is the space of admissible realizations, $\mathcal{F}$ is the set of all events belonging to the the space Ω, and the filtration, $\mathbb{F} = \{\mathcal{F}(t) : t \in \mathbb{R}_+\}$, is a flow of non-anticipating events such that $\mathcal{F}(s) \subset \mathcal{F}(t)$, for $s < t$, and $\lim_{t\to\infty} \mathcal{F}(t) = \mathcal{F}$. The filtration is generated by a standard Wiener process, which implies that, associated to the filtration, there is a flow of unconditional probabilities $\{P(t) : \ t \in \mathbb{R}_+\}$ where $P(t) = P(t, \omega(t))$ such that $P(t + dt, \omega^{'}) - P(t, \omega) \sim N(0, dt)$. This information structure is common knowledge.

The flow of consumption and endowments, $C = \{C(t), t \in \mathbb{R}\}$ and $Y = \{Y(t), t \in \mathbb{R}\}$, are adapted stochastic processes to the filtration $\mathbb{F}$. This means that $C(t)$ and $Y(t)$ are $\mathcal{F}_t$-measurable random variables for every $t \in (0, \infty)$

Second, we will deal mainly with either endowment economies, or production economies, in which the aggregate supply, or productivity, is driven by a time-varying exogenous shock that is not perfectly observed by agents. There is only one perishable good in the economy. We will consider cases in which there are some production and investment decisions for firms. In any case, that exogenous forcing variable, endowment or productivity, is assumed to follow a linear stochastic differential equation (SDE)

$$\frac{dY(t)}{Y(t)} = gdt + \sigma dW(t),\ Y(0) = y_0 \text{ given} \tag{1}$$

where, we assume for simplicity that the drift, g, and the volatility, σ, coefficients are constant and known. The initial value of the endowment $Y(0) = y_0$ is also known. Therefore, the sample paths of the endowment have the following statistics

$$\mathbb{E}\left[Y(t)\right] = y_0 e^{gt},\ \mathbb{V}\left[Y(t)\right] = y_0 e^{gt}\left(e^{\sigma^2 t} - 1\right),$$

meaning that: if $g > 0$ then $\lim_{t\to\infty} \mathbb{E}\left[Y(t)\right] = \lim_{t\to\infty} \mathbb{V}\left[Y(t)\right] = \infty$, and the endowment is both non-stationary and displays increasing volatility; or, if $g = 0$, then $\mathbb{E}\left[Y(t)\right] = y_0$ for each $t \in [0, \infty)$ but $\lim_{t\to\infty} \mathbb{V}\left[Y(t)\right] = \infty$.

Third, the representative household preferences will be represented by the utility functional, displaying additive preferences,

$$\mathbb{E}_0\left[\int_0^\infty u(C(t))e^{-\rho t}dt\right] = \int_{\mathcal{F}(t)}\int_0^\infty u(C(t,\omega(t))e^{-\rho t}dtdP(\omega) \tag{2}$$

where $\rho > 0$ is the rate of time preference and $u(.)$ is the instantaneous utility function, satisfying $u^{'}(.) > 0$, $u^{''}(.) < 0$ and $u^{'''}(.) > 0$. For any realization of the consumption process at time t, $C(t) = c > 0$[1] we define the **coefficients of relative risk aversion** and of **relative prudence** by

$$r_r(u) \equiv -\frac{u^{''}(c)c}{u^{'}(c)},\ \pi_r(u) \equiv -\frac{u^{'''}(c)c}{u^{''}(c)}.$$

If $r_r(u) > 0$ the utility function displays risk aversion and if $r_r(u) = 0$ it displays risk neutrality. In order to be able to derive clear results, and because this is the utility function most commonly used in the literature, we will assume a constant relative risk aversion (CRRA) utility function, for evaluating every realization of the consumption process, $C(t) = c$

$$u(c) = \begin{cases} \dfrac{c^{1-\gamma}}{1-\gamma}, & \text{if } \gamma \neq 1 \\ \ln(c), & \text{if } \gamma = 1 \end{cases} \tag{3}$$

Because,

$$r_r(u) = \gamma, \text{ and } \pi_r(u) = 1 + \gamma.$$

the utility function displays risk aversion if $\gamma > 0$ and risk neutrality if $\gamma = 0$, and, for the logarithmic utility function $\gamma = 1$.

Fourth, the institutional structure, that is, the structure of markets, determines the type of contracts available to the household and, therefore, the way the household allocates resources between states of nature (insurance) or across time (savings and investments).

We consider next **two market structures**, and, therefore, two economies: an **Arrow-Debreu economy** and a frictionless **finance economy** in which there is one riskfree asset (in zero net supply) and a risky asset. In those economies, the household is constrained in its allocations by an intertemporal budget constraint and by a instantaneous budget constraint in the finance economy. We will prove that the equilibrium allocations are equivalent in those two economies. This provides a benchmark to compare equilibrium allocations and prices for finance economies with frictions.

2.2 Arrow-Debreu economy

This is a continuous time version of the Lucas (1978) model.

[1] Except for the case of rates of return uppercase letters refer to random variables (i.e, a multivalued function) and lowercase letters refer to a realization (i.e, a number).

The Arrow-Debreu's economy institutional setting is defined by the existence of a large (in fact infinite) number of contingent claim markets, operating at time $t = 0$, in which agents can contract for delivery of one unit of the good, at any future moment and state of nature. The price of an Arrow-Debreu contract for delivery at $(t, \omega(t))$, denoted by $Q(t, \omega(t))$, is $\mathcal{F}_t$-measurable and satisfies $Q(0) = 1$. In an endowment homogeneous agent economy, the representative agent faces the following constraint for its transactions in all the forward markets open at time $t = 0$,

$$\int_0^\infty \int_{\mathcal{F}(t)} (Y(t,\omega) - C(t,\omega))dQ(t,\omega)dt = 0.$$

If we define the stochastic discount factor as the $\mathcal{F}_t$-adapted process $\{M(t), t \in \mathbb{R}_+\}$ such that $dQ(t,\omega) = M(t,\omega)dP(t,\omega)$, then the budget constraint can be equivalently written as the unconditional expected present value of the discounted excess supply of the good in all times and states of nature

$$E_0\left[\int_0^\infty M(t)(Y(t) - C(t))dt\right] = 0 \tag{4}$$

where $M(0) = 1$.

There are some technical conditions on the processes M, Y and C which basically amount to the guarantee of boundedness: they should be class $\mathcal{H}$ functions.

The representative household problem is to maximize the functional (2) subject to the budget constraint (4) given the endowment process $\{Y(t)\}_{t\in\mathbb{R}_+}$.

Definition 1. *An* ***Arrow-Debreu general equilibrium*** *(also called a simultaneous market equilibrium), for an aggregate risk given by $\{W(t)\}_{t\in\mathbb{R}_+}$, is defined by the processes for consumption, $(C(t))_{t\in\mathbb{R}_+}$, and for the stochastic discount factor (SDF) $(M(t))_{t\in\mathbb{R}_+}$, such that:*

1. *the representative household solves its problem, that is, it maximizes the utility functional* (2) *subject to the intertemporal budget constraint* (4)*, given the SDF and endowment processes,*

2. *markets clear at every time and state of nature, that is $C(t) = Y(t)$ at every time and state of nature.*

The equilibrium condition in the goods market implies $dC(t) = dY(t)$. Therefore, because

$$dC(t) = C(t)\left(gdt + \sigma dW(t)\right),\ t \geq 0. \tag{5}$$

consumption is perfectly correlated with the endowment, which implies that consumption is non-stationary if $g > 0$ and is stationary in average if $g = 0$

$$\mathbb{E}[C(t)] = y_0 e^{gt},$$

although even in the last case the variance increases

$$\mathbb{V}[C(t)] = y_0^2 e^{gt} \left(e^{\sigma^2 t - 1} \right).$$

At the equilibrium, the stochastic discount factor (see the appendix A) is the $\mathcal{F}_t$ adapted stochastic process

$$M(t) = e^{-\rho t} \frac{u^{'}(Y(t))}{u^{'}(y_0)}.$$

Using, Itô's lemma [2] we find that the equilibrium stochastic discount factor also follows a diffusion process

$$dM(t) = -M(t) \left(\mu_m(t)dt + \sigma_m(t)dW(t) \right) \tag{6}$$

where the drift and the volatility are

$$\mu_m(t) = \rho + r_r(Y(t)) \left(g - \frac{1}{2} \pi_r(Y(t)) \sigma^2 \right) \tag{7}$$

$$\sigma_m(t) = r_r(Y(t))\sigma \tag{8}$$

Therefore, the stochastic discount factor (SDF) is governed by a backward diffusion process in which:

- the diffusion coefficient has two terms: the first is the rate of time preference and the second depends on the dynamic properties of the endowment and on the behavior towards risk. The diffusion term increases with the diffusion coefficient and decreases with the volatility coefficient of the endowment process;

- the volatility coefficient is proportional to the volatility of the endowment process.

- the transmission of the growth and volatility of the endowment to the SDF is proportional to the relative risk aversion coefficient.

2.2.1 Isoelastic utility function

Because it is an homogeneous function, the CRRA function is particularly convenient because it features, as we saw, constant coefficients of relative risk aversion and prudence. It yields, for

[2]This equation is of type $Y = f(t, X)$ where $dX = X(\mu^x dt + \sigma^x)dW$. Then

$$\begin{aligned} dY &= f_t(t, X)dt + f_x(t, X)dX + \frac{1}{2} f_{xx}(t, X) \\ &= \left(f_t(t, X)dt + f_x(t, X)\mu^x X + \frac{1}{2} f_{xx}(t, X)(\sigma^x)^2 \right) dt + f_x(t, X)\sigma^x X dW \end{aligned}$$

the general case and for the logarithmic utility the drift component of the SDF, a constant drift coefficient

$$\mu_m = \begin{cases} \rho + \gamma g - \frac{1}{2}\gamma(1+\gamma)\sigma^2, & \text{if } \gamma \neq 1 \\ \rho + \gamma g - \sigma^2, & \text{if } \gamma = 1 \end{cases} \tag{9}$$

and a constant volatility coefficient

$$\sigma_m = \begin{cases} \gamma\sigma, & \text{if } \gamma \neq 1 \\ \sigma, & \text{if } \gamma = 1. \end{cases} \tag{10}$$

If there is risk neutrality, that is, if $\gamma = 0$, the stochastic discount factor is deterministic: $\mu_m = \rho$ and $\sigma_m = 0$.

The statistics for M are the following

$$\mathbb{E}[M(t)] = e^{-\mu_m t},\ \mathbb{V}[M(t)] = e^{-2\mu_m t}\left(e^{\sigma_m^2 t - 1}\right)$$

and

$$\mathbb{C}ov[C(t), M(t)] = y_0 e^{(\rho+(1+\gamma)(g-\frac{\gamma}{2}\sigma^2))t}\left(e^{\gamma\sigma^2 t - 1}\right).$$

This means that consumption and the stochastic discount factor are negatively correlated:

$$\text{Corr}[C, M(t)] = \frac{e^{-\gamma\sigma^2 t} - 1}{\sqrt{\left(e^{\gamma\sigma^2 t} - 1\right)\left(e^{\sigma^2 t} - 1\right)}} < 0$$

We already concluded that, at the equilibrium: (1) consumption is perfectly correlated with the endowment; (2) consumption and the stochastic discount factor are negatively correlated.

The distributional dynamics of consumption inherits those of the endowment. However, the distributional dynamic properties of the stochastic discount factor depend upon the sign of g, which depend on the relationship between the parameters of the model, in particular on the magnitude of the elasticity γ. We can see that here is a value for γ,

$$\gamma_c = \frac{g}{\sigma^2} - \frac{1}{2} + \left[\left(\frac{g}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}\right]^{\frac{1}{2}}$$

such that:

- if the risk-aversion is low such that $0 \leq \gamma < \gamma_c$ then $\mu_m > 0$. This implies

$$\lim_{t\to\infty} \mathbb{E}[M(t)] = \lim_{t\to\infty} \mathbb{V}[M(t)] = +\infty$$

  and the solution is non-stationary (there are bubbles);

- if the risk aversion is high such that $\gamma > \gamma_c$ then $\mu_m < 0$ then

$$\lim_{t\to\infty} \mathbb{E}[M(t)] = \lim_{t\to\infty} \mathbb{V}[M(t)] = 0$$

  and the solution is stationary and the $M(t)$ converges to zero

- if $\gamma = \gamma_c$ then $\mu_m = 0$ and

$$\mathbb{E}[M(t)] = 1, \text{ for all } t \in [0, \infty), \lim_{t\to\infty} \mathbb{V}[M(t)] = \infty$$

  the process for M is stationary in average but the volatility increases unboundedly.

2.3 Finance economy

In a finance economy the allocations through time and states of nature can be done by trading in asset markets. In the present continuous time framework markets are continuously open and trade is continuous. We assume that there are two assets, one risk free and one risky asset, and therefore there are two asset markets. While the risky asset is in positive aggregate net supply, the risk free is in aggregate zero net supply. We consider a representative agent economy in which the representative agent can take long or short positions in every asset.

The value of the risk free asset, at time t, is denoted by $B(t)$ and its rate of return, r, follows the deterministic process

$$dr(t) = r(t)dt.$$

Although this process is given for the individual agent, we will determine it endogenously at the general equilibrium.

The risky asset has a value, at time t, denoted by $S(t)$, and it entitles to a gain process $G(t) = S(t) + \int_0^t D(s)ds$, where D is the dividend. We assume that the dividend is exogenous and follows a diffusion process

$$\frac{dD(t)}{D(t)} = gdt + \sigma dW(t). \tag{11}$$

The similarity with the endowment process in the Arrow-Debreu economy of the previous section is introduced in order to make the two economies comparable, as will be seen latter in this section.

Although the change in market value of the risky asset is endogenous, we assume that it follows the linear diffusion

$$\frac{dS(t)}{S(t)} = \mu_s(t)dt + \sigma_s(t)dW(t).$$

If we denote the price-dividend ratio by q, then $S(t) = q(t)D(t)$ then the rate of return for holding risky assets follows the (endogenous) process

$$dr_s(t) = \tilde{\mu}_s(t)dt + \sigma_s(t)dW(t). \tag{12}$$

where $\tilde{\mu}_s(t) = \mu_s(t) + \frac{1}{q(t)}$.

In the literature the following measure of the relative returns of the two assets is called the **Sharpe ratio**

$$\text{Sharpe} = \frac{\tilde{\mu}_s - r}{\sigma_s}$$

The households net wealth is invested in risk free and risky assets. Denoting net wealth , at time t, by $N(t)$, we have $N(t) = B(t) + S(t)$. If the weight of the risky asset is denoted by $w(t) \equiv S(t)/N(t)$, then the budget constraint for the agent is

$$dN(t) = N(t)\left[r(t)(1 - w(t)) + (\tilde{\mu}_s(t)dt + \sigma_s(t)dW(t))\, w(t)\right] - C(t)dt \tag{13}$$

where we assume that wealth at the initial time is deterministic $N(0) = n_0$. Because the risky asset is in aggregate positive supply, we will have $w(t) > 0$, and because agents can take long or short positions in the risk-free asset $w(t)$ has a free upper bound. If agents leverage their position on the risky asset by risk-free borrowing then $w(t) > 1$. If agents cannot take short positions on the risk-free asset then $w(t)$ would be constrained to be smaller than one.

Definition 2. ***General equilibrium for an unconstrained finance economy*** *It is the allocations* $(C^{eq}(t), B^{eq}(t), S^{eq}(t))_{t\in\mathbb{R}_+}$ *and the returns* $(r^{eq}(t), r^{s,eq}(t))_{t\in\mathbb{R}_+}$ *such that, given the aggregate risk process* $(W(t))_{t\in\mathbb{R}_+}$*:*

1. *the representative household solves its problem, that is, it maximizes the utility functional* (2) *subject to the instantaneous budget constraint* (13)*, given the assets' rates of return processes;*
2. *markets clear:* $C^{eq}(t) = D(t)$*,* $B^{eq}(t) = 0$*, and* $S^{eq}(t) = N^{eq}(t)$*, at every time and state of nature.*

In the appendix we derive the optimality conditions for the representative household: First, consumption is proportional to net wealth,

$$C^*(t) = \theta(t)N(t), \text{ where } \theta(t) \equiv \frac{1}{\gamma}\left[\rho + (\gamma - 1)\left(r(t) + \frac{1}{2\gamma}\left(\frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)}\right)^2\right)\right] \tag{14}$$

the optimal portfolio weight of the risky asset is

$$w^*(t) = \frac{1}{\gamma}\left(\frac{\tilde{\mu}_s(t) - r(t)}{(\sigma_s(t))^2}\right). \tag{15}$$

This means that the demand for the risk-free asset is $B^*(t) = (1 - w^*(t))N(t)$ and the demand for the risky asset is $S^*(t) = w^*(t)N^*(t)$.

Substituting the consumption and the portfolio weights, equations (14) and (15), in the budget constraint (13), and using the Itô's formula in equation (14), we find that the rates of growth of consumption and net wealth are perfectly correlated

$$\frac{dC^*(t)}{C^*(t)} = \frac{dN^*(t)}{N^*(t)} = \mu_n(t)dt + \sigma_n(t)dW(t) \tag{16}$$

where

$$\mu_n(t) = \frac{1}{\gamma}\left[r(t) - \rho + \frac{1+\gamma}{2\gamma}\left(\frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)}\right)^2\right]$$
$$\sigma_n(t) = \frac{1}{\gamma}\left(\frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)}\right).$$

The equilibrium for the risk-free market requires that the demand and supply for the risky asset are equal $N(t) = S^*(t) = w^{eq}(t)N(t)$, that is

$$w^{eq}(t) = 1.$$

Therefore, the equilibrium stochastic discount factor (which it is shown in the Appendix B.3 that is equal to the Sharpe ratio) is proportional to the volatility of the risky asset

$$X(t) = \frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)} = \gamma\sigma_s(t).$$

Equivalently, the expected change in the risky asset rate of return is equal to the risk-free rate of return plus $X\sigma_s$,

$$\mathbb{E}[dr^s(t)] = \tilde{\mu}_s(t) = \mu_s(t) + \frac{1}{q(t)} = r(t) + \gamma(\sigma_s(t))^2, \tag{17}$$

Therefore, at every point in time, the ratio $C(t)/N(t)$, in equation (14), is simplified to

$$\gamma\theta = \rho + (\gamma - 1)r + (\frac{1-\gamma}{2})(\gamma\sigma_s)^2,$$

and the diffusion and volatility coefficients in equation (16) become

$$\gamma\mu_n = r - \gamma + (\frac{1+\gamma}{2})(\gamma\sigma_s)^2, \text{ and } \sigma_n = \sigma_s.$$

The goods market equilibrium condition

$$C(t) = \theta(t)N(t) = D(t)$$

has two implications. First, together with the definition of q, from $S = qD$, we have an equilibrium equation for the price-dividend ratio $q\theta = 1$, which is equivalent to

$$q(t)\left(\rho + (\gamma-1)\left(r(t) + \frac{1}{2\gamma}(\gamma\sigma_s(t))^2\right)\right) = \gamma. \tag{18}$$

Second, at the equilibrium the rates of growth of the dividend, of consumption and of the aggregate net wealth are perfectly correlated

$$\frac{dD(t)}{D(t)} = gdt + \sigma dW(t) = \frac{dC(t)}{C(t)} = \frac{dN(t)}{N(t)} = \mu_n(t)dt + \sigma_n(t)dW(t)$$

implying $gdt + \sigma dW(t) = \mu_n(t)dt + \sigma_n(t)dW(t)$.

The diffusion and the volatility terms match if and only if

$$g = \mu_n = \frac{1}{\gamma}\left(r - \gamma + \left(\frac{1+\gamma}{2\gamma}\right)(\gamma\sigma_s)^2\right) \tag{19}$$

$$\sigma = \sigma_n = \sigma_s. \tag{20}$$

Solving equations (17), (18), (19), and (20) for q, r, μ_s and σ_s we obtain the equilibrium values for the interest rate of the risk-free asset, as a constant

$$r^{eq}(t) = \begin{cases} \rho + \gamma g - \gamma\left(\frac{1+\gamma}{2}\right)\sigma^2, & \text{if } \gamma > 0 \text{ and } \neq 1 \\ \rho + g - \sigma^2, & \text{if } \gamma = 1 \end{cases} \quad \text{for every } t \in [0,\infty) \tag{21}$$

and the price-dividend ratio

$$q^{eq}(t) = \begin{cases} \left(\rho + (\gamma-1)(g - \gamma\sigma^2)\right)^{-1}, & \text{if } \gamma > 0 \text{ and } \neq 1 \\ \rho^{-1}, & \text{if } \gamma = 1 \end{cases} \quad \text{for every } t \in [0,\infty) \tag{22}$$

is also a constant. Because

$$\mu^{s,eq} = g - \gamma\left(\frac{1-\gamma}{2}\right)\sigma^2, \text{ and } \sigma^{s,eq} = \sigma$$

the equilibrium rate of return for the risky asset, $dr_s^{eq}(t) = \tilde{\mu}^{s,eq}dt + \sigma^{s,eq}dW(t)$ follows a diffusion process

$$dr_s^{eq}(t) = \begin{cases} \left(\rho + \gamma g + \gamma\left(\frac{1-\gamma}{2}\right)\sigma^2\right)dt + \sigma dW(t), & \text{if } \gamma > 0 \text{ and } \neq 1 \\ (\rho + g)\,dt + \sigma dW(t), & \text{if } \gamma = 1. \end{cases} \tag{23}$$

The stochastic discount factor process at the equilibrium (see the Appendix B.3) follows the process

$$dM(t) = -M\left(rdt + XdW(t)\right)$$

where X is the market price of risk:

$$X = \frac{\tilde{\mu}^s - r}{\sigma^s}.$$

If we substitute the equilibrium rates of return for the risk-free and risky asset from equation (21) and (23), respectively, we find that $X^{eq} = \gamma\sigma$

Comparing with the solution we found for the Arrow-Debreu economy, in equations (9) and (10), we observe that $\mu_m = r^{eq}$ and $\sigma_m = X^{eq}$. Therefore, the **implied equilibrium stochastic discount factor for this finance economy is the same as in the Arrow-Debreu economy**, in which the endowment is exogenously supplied to the household.

This means that those two institutional settings provide equivalent market mechanisms to generate an equilibrium intertemporal allocations of resources. This also means that the financial markets in this unconstrained finance economy are complete.

In both economies we conclude that the only source of aggregate risks are fundamental exogenous aggregate risks introduced by $(W(t))_{t\geq 0}$.

3 Heterogeneous market participation

In this section we consider a first type of financial friction originated by limited asset market participation, and, consequently there is heterogeneity on the composition of the net wealth among economic agents. This generates illiquidity, in the sense that some agents cannot fully insure against uncertainty, which has consequences on the risk bearing of agents, and, possibly on the existence of an endogenous source of macroeconomic risks. That endogenous source of risk adds to the exogenous aggregate risks which is the same as in the case of the frictionless economy that we studied in the last section.

Again we assume that there are two assets: a risk-free asset with zero net supply and a risky asset in positive net supply. There are two groups of homogeneous agents: there is a group of households that cannot hold the risky asset (called households in the literature) and other group of agents that participate in both markets (called specialists in several papers). Both groups share the same information and preferences. We continue to assume that the only source of income is financial income. There are several explanations for this difference in participation, v.g., informational constraints or existence of transaction costs.

Because non-participants can only hold the risk-free asset, and because it is in zero net supply, the participants finance their holdings of risky assets by holding short positions (i.e, taking loans) in the risk-free asset. This implies that the fluctuations in fundamentals are completely absorbed by the participants. We will see that the distribution of net wealth varies and drives the dynamics of the equilibrium rates of return, because their level of wealth is one of the determinants of the demand for risky assets thus influencing its price. Because there are no arbitrage opportunities, then the risk free rate of return is also affected.

We will also see that the type of utility function has a relevant effect on the equilibrium properties. This is natural, because the concavity of the utility function is related to the risk aversion behavior. In particular, there is not an endogenous source of risk if the utility function is logarithmic, which is not the case if the utility function displays higher risk aversion.

We present next a version of the Basak and Cuoco (1998) model (see Brunnermeier and Sannikov (2016) for a discussion).

Non-participating households, denoted by an index h, consume C^h and hold net wealth, N^h, by solving the problem

$$\max_{C^h} \mathbb{E}_0 \left[\int_0^\infty \frac{(C^h(t))^{1-\gamma}}{1-\gamma} e^{-\rho t} dt \right]$$

subject to the budget constraint,

$$dN^h(t) = r(t)N^h(t)dt - C^h(t)dt \; t \in [0, \infty)$$

where $N^h(0) = n_0^h$ is given, and $r(t)$ is again the rate of return of the risk free asset, which is taken as given to the household. The first order conditions are, for consumption,

$$C^h(t) = \theta^h(t)N^h(t), \text{ where } \theta^h(t) = \frac{1}{\gamma}\left(\rho + (\gamma-1)r(t)\right) \tag{24}$$

and

$$\frac{dC^h(t)}{C^h(t)} = \frac{dN^h(t)}{N^h(t)} = \left(\frac{r(t)-\rho}{\gamma}\right) dt. \tag{25}$$

Participating, or specialist, households can participate in both asset markets. They solve the problem

$$\max_{C^x, w^x} \mathbb{E}_0 \left[\int_0^\infty \frac{(C^x(t))^{1-\gamma}}{1-\gamma} e^{-\rho t} dt \right]$$

where, C^x is consumption and w^x is the weight of the risky asset in net wealth, for participators, subject to the budget constraint

$$dN^x(t) = \left(r(t)(1-w^x(t)) + \tilde{\mu}_s(t)w^x(t)\right) N^x(t)dt + \sigma_s(t)w^x(t)N^x(t)dW(t) - C^x(t)dt$$

where $N^x(0) = n_0^x$ is given. These households are also price takers in both asset markets, where the rate or return for the risky asset is as in equation (12), which is taken as given by these households. As in the previous section we assume that there are exogenous dividends accruing to the holders of the risky asset following the process given in equation (11).

$$C^x(t) = \theta^x(t)N^x(t), \text{ where } \theta^x(t) \equiv \frac{1}{\gamma}\left[\rho + (\gamma-1)\left(r(t) + \frac{1}{2\gamma}\left(\frac{\tilde{\mu}_s(t)-r(t)}{\sigma_s(t)}\right)^2\right)\right] \tag{26}$$

the portfolio weight of the risky asset is

$$w^x(t) = \frac{1}{\gamma}\left(\frac{\tilde{\mu}_s(t) - r(t)}{(\sigma_s(t))^2}\right) \tag{27}$$

and consumption and net wealth are again perfectly correlated, at the households level,

$$\frac{dC^x(t)}{C^x(t)} = \frac{dN^x(t)}{N^x(t)} = \mu_n^x(t)dt + \sigma_n^x(t)dW(t) \tag{28}$$

where

$$\begin{aligned}
\mu_n^x(t) &= \frac{1}{\gamma}\left[r(t) - \rho + \frac{1+\gamma}{2\gamma}\left(\frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)}\right)^2\right] \\
\sigma_n^x(t) &= \frac{1}{\gamma}\left(\frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)}\right).
\end{aligned}$$

The balance sheet constraint is $N^h(t) = B^h(t)$, for non-participating households, and $N^x(t) = B^x(t) + S(t)$, for participating households, where $B^h(t)$ and $B^x(t)$ are the stocks of bonds and $S(t)$ is the stock of risky assets. All those variables are in nominal terms. We assume that $B^h(t) \geq 0$ meaning that non-participating households are lenders to participating agents. Therefore $B^x(t)$ corresponds to risk-free financing of purchases of risky assets by participating agents, and $B^x(t)/S(t)$ is the leverage ratio. This is equivalent to setting $w^x(t) \geq 1$.

Definition 3. ***General equilibrium for the limited participation finance economy*** *It is the allocations* $(C^{h,eq}(t), C^{x,eq}(t), B^{h,eq}(t), B^{x,eq}(t), S^{eq}(t))_{t\in\mathbb{R}_+}$ *and the returns* $(r^{eq}(t), r^{s,eq}(t))_{t\in\mathbb{R}_+}$ *such that, given the aggregate risk process* $(W(t))_{t\in\mathbb{R}_+}$*:*

1. *non-participating and participating households solve their particular problems, taking as given the rates of return of the risk free and risky assets;*

2. *markets clear: the good's market clearing condition is*

$$C^{eq}(t) = C^{h,eq}(t) + C^{x,eq}(t) = D(t),$$

where $C(t)$ *is aggregate consumption; the risk-free market clearing condition is*

$$B^{h,eq}(t) + B^{x,eq}(t) = 0$$

and the risky asset market clearing condition is

$$S^{eq}(t) = N^{eq}(t)$$

3. *aggregation:* $N(t) = N^h(t) + N^x(t)$ *is the stock of the risky asset, and is equal to the aggregate wealth of the economy.*

We denote the share of the specialists in the total net wealth of the economy by η, that is

$$\eta(t) \equiv \frac{N^x(t)}{N(t)} \in (0,1).$$

It is constrained to be smaller than one because it is equal to the share of the risky asset in the portfolio of the specialists, that is

$$\eta(t) = \frac{1}{w^x(t)} = \frac{N^x}{S},$$

and because the position of the specialists in the risky asset is leveraged, that is $w^x \geq 1$.

As the demand for risky assets for the specialists is given by equation (27), then the Sharpe ratio is, at the equilibrium

$$\frac{\tilde{\mu}_s(t) - r(t)}{\sigma^s(t)} = \gamma \frac{\sigma_s(t)}{\eta(t)}. \tag{29}$$

From the good's market clearing condition, $C(t) = D(t)$, the consumption optimality conditions, equations (24) and (26), and the definition of η we obtain

$$\begin{aligned}
C(t) &= C^h(t) + C^x(t) = \\
&= \theta^h(t)N^h(t) + \theta^x(t)N^x(t) = \\
&= \Big(\theta^h(t)\eta(t) + \theta^x(t)(1-\eta(t))\Big) N(t) = \\
&= D(t) = \\
&= \frac{N(t)}{q(t)}
\end{aligned}$$

if we ntroduce, again the price dividend ratio definition, which at the equilibrium satisfies

$$q(t) = \frac{S(t)}{D(t)} = \frac{N(t)}{D}.$$

Therefore,

$$\Big(\theta^h(t)\eta(t) + \theta^x(t)(1-\eta(t))\Big) q(t) = 1.$$

Using the optimal consumption demands, equations (24) and (26), this is equivalent to

$$\left[\rho + (\gamma-1)r(t) + \eta(t)\frac{(\gamma-1)}{2\gamma}\left(\frac{\tilde{\mu}_s(t) - r(t)}{\sigma^s(t)}\right)^2\right] q(t) = \gamma,$$

and, using the Sharpe ratio, (29), yields the dividend-price ratio in equilibrium

$$\frac{1}{q(t)} = \frac{1}{\gamma}\left[\rho + (\gamma-1)r(t) + \gamma\left(\frac{\gamma-1}{2}\right)(\sigma_s(t))^2\left(\frac{1}{\eta(t)}\right)\right]. \tag{30}$$

If we compare with the analogous equation for the frictionless economy, in equation (18), we see that the wealth distribution variable has an influence in the way volatility is transmitted to the dividend-price ratio.

The drift for the rate of return of the risky asset (see equation (12) for the process followed by teh return of the risky asset) in equilibrium

$$\mu_s = \tilde{\mu}_s - \frac{1}{q} = \frac{1}{\gamma}\left[r(t) - \rho + \gamma\left(\frac{1+\gamma}{2}\right)\frac{(\sigma_s(t))^2}{\eta(t)}\right], \tag{31}$$

depends on the rate of interest $r(t)$, the volatility $\sigma_s(t)$, and the asset distribution $\eta(t)$.

The aggregate net wealth change is $dN(t) = dN^h(t) + dN^x(t)$. Substituting the Sharpe ratio in equation (28) and adding equation (**??**) we get the rate of change of aggregate net wealth

$$\frac{dN(t)}{N(t)} = \frac{dN^h(t)}{N^h(t)} + \frac{dN^x(t)}{N^x(t)} = \mu_s(t)dt + \sigma_s(t)dW(t).$$

This allows us to obtain the dynamics for the share of the specialists on total wealth, η, and a dynamic equation for the equity price ratio, q, which can be seen as two state variables driving the equilibrium rates of return.

First, as $\eta(t) = 1 - \dfrac{N^h(t)}{N(t)}$ then, using Itô's formula

$$\begin{aligned} d\eta(t) &= -\frac{N^h(t)}{N(t)}\left[\frac{dN^h(t)}{N^h(t)} - \frac{dN(t)}{N(t)} + \left(\frac{dN(t)}{N(t)}\right)^2\right] = \\ &= -(1-\eta(t))\left[\left(\frac{r(t)-\rho}{\gamma}\right)dt - (\mu_s(t)dt + \sigma_s(t)dW(t)) + (\sigma_s(t))^2 dt\right] = \\ &= -(1-\eta(t))\left[\left(\frac{r(t)-\rho}{\gamma} - \mu_s(t) + (\sigma_s(t))^2\right)dt + \sigma_s(t)dW(t)\right]. \end{aligned}$$

Substituting the drift of the risky asset price process, in equation (31), we obtain one equation for the dynamics of the wealth distribution is driven by the diffusion equation

$$d\eta(t) = (1-\eta(t))\left[\frac{(\sigma_s(t))^2}{\eta(t)}\left(\frac{1+\gamma}{2} - 1\right)dt - \sigma_s(t)dW(t)\right], \tag{32}$$

which can be written as a diffusion process

$$d\eta(t) = (1-\eta(t))\left(\mu_\eta(t,\eta(t))dt - \sigma_s(t)dW(t)\right), \tag{33}$$

where the drift component

$$\mu_\eta(\eta) \equiv \frac{(\sigma_s)^2}{\eta}\left(\frac{1+\gamma}{2} - 1\right)$$

is negative related to η if $\gamma \geq 1$.

Second, the dividend-price ratio also follows a diffusion process. Applying Itô's formula we get

$$
\begin{aligned}
d\left(\frac{1}{q(t)}\right) &= \frac{dD(t)}{N(t)} - \frac{D(t)}{N(t)}\frac{dN(t)}{N(t)} - \frac{dD(t)}{N(t)}\frac{dN(t)}{N(t)} + \frac{D(t)}{N(t)}\left(\frac{dN(t)}{N(t)}\right)^2 = \\
&= \frac{D(t)}{N(t)}\left\{\frac{dD(t)}{D(t)} - \frac{dN(t)}{N(t)} - \frac{dD(t)}{D(t)}\frac{dN(t)}{N(t)} + \left(\frac{dN(t)}{N(t)}\right)^2\right\} = \\
&= \frac{1}{q(t)}\left\{gdt + \sigma dW(t) - \mu_s(t)dt - \sigma_s(t)dW(t) - \sigma\sigma_s(t)dt + (\sigma_s(t))^2 dt\right\} = \\
&= \frac{1}{q(t)}\left\{\mu_q(t)dt + \sigma_q dW(t)\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_q(t) &= g + \frac{\rho - r(t)}{\gamma} - \left(\frac{1+\gamma}{2}\right)\frac{(\sigma_s(t))^2}{\eta(t)} + \sigma_s(t)\left(\sigma_s(t) - \sigma\right) \\
\sigma_q(t) &= \sigma - \sigma_s(t)
\end{aligned}
$$

From the last equation we find that the net-wealth volatility can have two components $\sigma_s(t) = \sigma + \sigma_q(t)$, a fundamental and and endogenous component.

In equation (30) we find there is a contemporaneous relationship between the dividend-price ratio and the distributional of wealth η. Taking derivatives of equation (30), and observing that $dr(t) = r(t)dt$ and $d\sigma_s(t) = \sigma_s(t)dt$, we find

$$
d\left(\frac{1}{q(t)}\right) = \left[\left(\frac{\gamma-1}{\gamma}\right)r(t) + (\gamma-1)\left(\sigma_s(t)\right)^2\left(\frac{1}{\eta(t)}\right)\right]dt + \left(\frac{\gamma-1}{2}\right)(\sigma_s(t))^2 d\left(\frac{1}{\eta(t)}\right).
$$

Therefore, $r^{eq}(t)$ and $\sigma_s^{eq}(t)$ are the solutions of the system

$$
\begin{cases}
\mu_q\left(r(t), \sigma_s(t), \eta(t)\right) = \left(\frac{\gamma-1}{\gamma}\right)r(t) + (\gamma-1)\left(\sigma_s(t)\right)^2\left(\frac{1}{\eta(t)}\right) + \left(\frac{\gamma-1}{2}\right)(\sigma_s(t))^2(1-\eta(t))\mu_\eta(t,\eta(t)) \\
\sigma_q\left(\sigma_s(t)\right) = -\left(\frac{\gamma-1}{2}\right)(1-\eta(t))(\sigma_s(t))^3.
\end{cases}
\tag{34}
$$

The system, which does not seem to have closed form solution. However, in the general case, of an isoelastic utility function, the risk-free interest rate and the asset return volatility depend on the distribution of net wealth. This distinguishes this economy from the frictionless case.

As the solution clearly depends on the degree of risk aversion, let us consider first the benchmark case of a logarithmic utility function, i.e., $\gamma = 1$.

The logarithmic case Setting $\gamma = 1$ in system (34), we find that the price dividend ratio is stationary

$$
q(t) = \frac{1}{\rho}, \text{ for any } t \in [0, \infty).
$$

Therefore, there is no endogenous risk, that is $\sigma_s^{eq}(t) = \sigma$ and the rate of return of the risky asset follows the process

$$dr^{s,eq}(t) = (\rho + \gamma)\, dt + \sigma dW(t), \tag{35}$$

which is the same process as for the frictionless economy (see equation (23)).

The interest rate of debt contracts is positively correlated with the distribution of wealth parameter, η

$$r^{eq}(t) = \rho + g - \frac{\sigma^2}{\eta(t)}, \tag{36}$$

meaning that is η is low the demand for leverage by the participants is also low, which reduces the demand for the risk free asset.

Substituting $\gamma = 1$ in equation (33) we determine the process followed by the wealth distribution:

$$\frac{d\eta(t)}{\eta(t)} = \left(\frac{1-\eta(t)}{\eta(t)}\sigma\right)^2 dt + \left(\frac{1-\eta(t)}{\eta(t)}\sigma\right) dW(t). \tag{37}$$

Figure 1 illustrates one sample path for η, in the upper left subfigure, and one hundred sample paths, in the upper right subfigure. In the lower subfigure we substitute the first sample in equation (36) to obtain a sample path for the rate of return of the risk free asset. Although it looks as random, the effect of uncertainty is contemporaneous (as if it were a random variable) and not dynamic as in the equilibrium process for the rate of return of the risk free asset in equation (35). This is the case because the instantaneous change of the wealth distribution is associated to a random change in the demand for risk free bounds to which the rate of interest responds contemporaneously

In 1 we see that the interest rate η converges asymptotically to one. In fact $\eta = 1$, if we look at equation (37) we see that the skeleton (i.e., the diffusion part) has $\eta = 1$ as a steady state. This steady state is an absorbing state because

$$\frac{d}{d\eta}\left(\frac{1-\eta}{\eta}\right)^2 < 0$$

if $0 < \eta < 1$. This means that, asymptotically, all financial wealth will be concentrated on the specialists. This result can be changed if we introduced labor and human capital which would imply that the non-specialists would have positive total wealth. In this case the net wealth of the financially limited agents will converge to their human wealth.

The main difference from the non-financially constrained economy is that the short run interest rate responds to the leverage of the economy , i.e, to $1 - \eta$: we see that the higher the leverage the lower the interest rate. This process discourages further loans from the financially constrained households, which drives up the interest rate and reduces leverage.

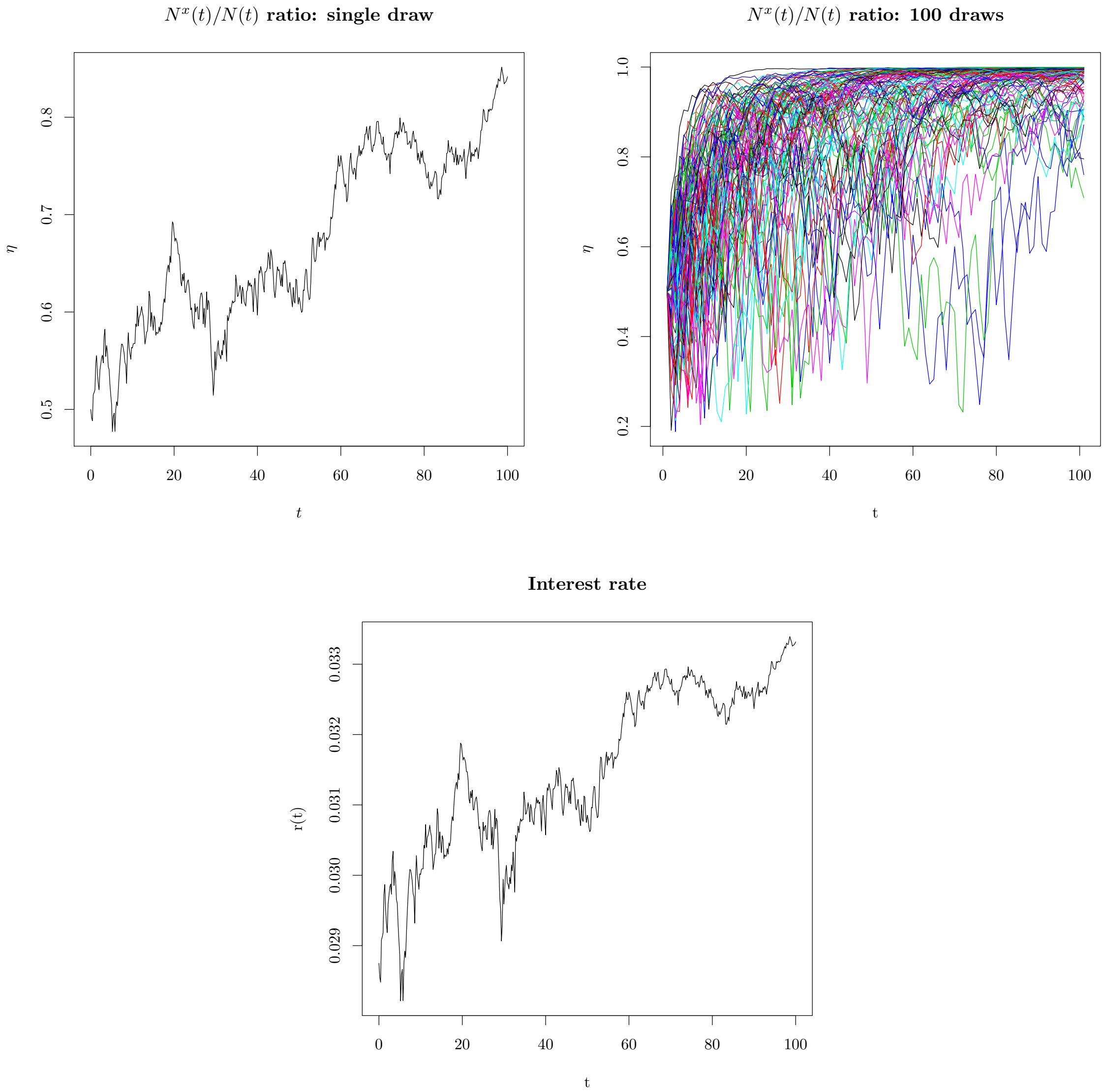

Figure 1: The limited participation model: logarithmic utility

The risk-averse case In a scenario with higher risk aversion, that is with $\gamma > 1$, we would have to solve system (34) in order to obtain the equilibrium expressions for $r(t)$ and $\sigma_s(t)$. It does not seem to have a closed form solution.

However, we can determine the volatility of the risky asset, σ_s, from the second equation

$$\frac{\sigma - \sigma_s}{q} = -\left(\frac{\gamma - 1}{2}\right)(1-\eta)(\sigma_s)^3. \tag{38}$$

We readily see that its is higher than the fundamental volatility, that is $\sigma_s \geq \sigma$, for $0 < \eta \leq 1$, meaning that when agents are highly risk averse the existence of limited participation generates endogenous volatility and that this volatility is dependent upon the asset distribution. In equation (38) we readily see that the asset volatility is positively related to leverage (if we set $\sigma_s = \sigma_s(\eta)$ we have $\sigma_s'(\eta) < 0$). This means that,

- if the leverage ratio is low, i.e., η is closed to one, the endogenous component of volatility will be low and the volatility of the risky asset will approach the fundamental volatility;
- if the leverage ratio is high, i.e., η is closed to zero, the endogenous component of volatility is high and the volatility of the risky asset will be higher than the fundamental volatility

4 Technological illiquidity frictions

In this section we present a version of a model presented in Brunnermeier and Sannikov (2016) in which there are, in the words of those authors, **technological illiquidity**. In this model the dividends process and the capital accumulation processes are endogeneized (following a model by Bernanke et al. (1999)) with two assumptions: (1) the production function displays constant returns to scale; and (2) there adjustments costs in investment which have a convex deterministic component and a linear stochastic component. The last assumption is the source of technological illiquidity.

We first consider the frictionless financial markets version and then present the limited market participation version. In order to isolate the source of market volatility generated by technologic liquidity we assume that preferences are homogeneous and the utility function is logarithmic.

4.1 Unrestricted market participation

There are two dimensions of the technology in this economy: first, the production function displays constant returns to scale, $Y(t) = AK(t)$; second, the investment technology involves adjustment costs taking the form convex costs for installing capital. In particular, we assume that investment expenditures, $I(t) = \iota(t)K(t)$, generate a deterministic increase in gross capital $\Phi(\iota)K(t)$, where

$\Phi(0) = 0$, $\Phi''(\iota) < 0 < \Phi'(\iota)$, and have a random component, capturing fundamental uncertainties in the investment process.

The capital accumulation equation follows the diffusion equation

$$\frac{dK(t)}{K(t)} = (\Phi(\iota(t)) - \delta)\, dt + \sigma dW(t)$$

where δ is the depreciation rate. We denote $\mu_k(t) = \Phi(\iota(t)) - \delta$ and $\sigma_k = \sigma$.

Investment in physical generates a dividend, which is equal to output subtracted by the investment expenditures,

$$D(t) = (A - \iota(t))\, K(t).$$

Because the asset value of the firm is $S(t) = q(t)K(t)$, where q is Tobin's q, the return for holding the capital is

$$dr_s(t) = \frac{d\,(q(t)K(t))}{q(t)K(t)} + \frac{D(t)}{q(t)K(t)}dt.$$

We conjecture that the relative price of capital $q(t)$ is driven by the process

$$\frac{dq(t)}{q(t)} = \mu_q(t)dt + \sigma_q(t)dW(t)$$

where $\mu_q(t)$ and $\sigma_q(t)$ are to be determined in equilibrium. Therefore, the capital gains dynamics is driven by the equation, after applying Itô's formula,

$$\begin{aligned}
\frac{d\,(q(t)K(t))}{q(t)K(t)} &= (\mu_q(t) + \mu_k(t) + \sigma_q(t)\sigma_k(t))\, dt + (\sigma_q(t) + \sigma_k(t))\, dW(t) \\
&= (\mu_q(t) + \Phi(\iota(t)) - \delta + \sigma_q(t)\sigma)\, dt + (\sigma_q(t) + \sigma)\, dW(t)
\end{aligned}$$

where σ represents fundamental, exogenous, volatility and σ_q is the endogenous volatility component.

At last, we obtain the process followed by the rate of return for investment in the risky asset

$$dr_s(t) = (d(q(t), \iota(t)) + \Phi(\iota(t)) - \delta + \sigma_q(t)\sigma)\, dt + \mu_q(t) + (\sigma_q(t) + \sigma)\, dW(t). \tag{39}$$

where the dividend-price ratio

$$d(q(t), \iota(t)) \equiv \frac{A - \iota(t)}{q(t)}$$

is a negative function of both the relative price of capital and the investment rate.

Using our previous notation, this is equivalent to $dr_s(t) = \tilde{\mu}_s(t)dt + \sigma_s(t)dW(t)$, where

$$\begin{aligned}
\tilde{\mu}_s(t) &= d(q(t), \iota(t)) + \Phi(\iota(t)) - \delta + \mu_q(t) + \sigma_q(t)\sigma \\
\sigma_s(t) &= \sigma_q(t) + \sigma
\end{aligned}$$

If we assume that there is no heterogeneity and no financial frictions, this return is received by the representative household which own the firms, and has the balance sheet constraint

$$B(t) + q(t)K(t) = N(t),$$

where $B(t)$ is the household stock of risk-free assets and $N(t)$ is the household net wealth.

Definition 4. ***General equilibrium*** *Is defined by the allocations* $(C^{eq}(t), i^{eq}(t), B^{eq}(t), K^{eq}(t))_{t\in\mathbb{R}_+}$ *and the returns* $(r^{eq}(t), r^{s,eq}(t))_{t\in\mathbb{R}_+}$ *such that, given the aggregate risk process* $(W(t))_{t\in\mathbb{R}_+}$*:*

1. *the representative household solves its problem, that is, it maximizes the utility functional* (2) *subject to the instantaneous budget constraint* (13)*, given the asset income processes;*
2. *the representative firm maximizes its profits* $\pi(\iota) = q\Phi(\iota) - \iota$*, at every point in time;*
3. *markets clear:* $C^{eq}(t) + I^{eq}(t) = Y(t)$*,* $B^{eq}(t) = 0$*, and* $S^{eq}(t) = N^{eq}(t)$*, at every time and state of nature.*

If we assume a logarithmic utility function, the optimality condition for the household is proportional to net wealth

$$C(t) = \rho N(t), \tag{40}$$

and the demand for the two assets is $B(t) = (1 - w(t))N(t)$ and $S(t) = w(t)N(t)$, where

$$w(t) = \left(\frac{\tilde{\mu}_s(t) - r(t)}{(\sigma_s(t))^2}\right) = \frac{X(t)}{\sigma_s(t)}. \tag{41}$$

As in the previous model consumption and net wealth have their rates of growth perfectly correlated

$$\frac{dC(t)}{C(t)} = \frac{dN(t)}{N(t)} = \mu_n(t)dt + \sigma_n(t)dW(t) \tag{42}$$

where

$$\begin{aligned}\mu_n &= r - \rho + \left(\frac{\tilde{\mu}_s - r}{\sigma_s}\right)^2\\ \sigma_n &= \frac{\tilde{\mu}_s - r}{\sigma_s}.\end{aligned}$$

Because the firm's profit, per unit of capital, is

$$\pi(\iota) = q\Phi(\iota) - \iota$$

then the optimum investment condition is

$$q\Phi^{'}(i) = 1.$$

From the implicit function theorem, investment is an increasing function of the relative price of capital,

$$\iota(t) = \Psi(q(t)),\ \Psi^{'}(q) > 0.$$

This implies that the dividend-price ratio is decreasing with q, $d = \hat{d}(q) = d(q, \Psi(q))$, for $\hat{d}^{'}(q) < 0$.

The aggregation and market equilibrium for the risk free asset, $B^{eq}(t) = 0$, implies

$$w^{eq} = \frac{q(t)K(t)}{N(t)} = 1.$$

There are two implications: first

$$q(t)K(t) = N(t) \tag{43}$$

and

$$\tilde{\mu}_s(t) = r(t) + (\sigma_s(t))^2. \tag{44}$$

The goods' market equilibrium condition, is

$$AK(t) = C(t) + \iota(t)K(t).$$

As at the equilibrium $C(t) = \rho q(t)K(t)$, then $A = \rho q(t) + \iota(t)$. Therefore, the equilibrium dividend per unit of capital is equal to the rate of time preference,

$$\hat{d}(q) = \rho.$$

This condition together with the optimality condition for firms yield a system of two equations in $(q(t), \iota(t))$,

$$\begin{cases} \rho q + \iota = A \\ q\Phi^{'}(\iota) = 1 \end{cases} \tag{45}$$

which has an unique solution for q and ι, $\bar{q} = \bar{q}(A, \rho)$ and $\bar{\iota} = \bar{\iota}(A, \rho)$, which are both positively related to productivity and negatively related to the rate of time preference [3].

[3] To see this take the total differential to the second equation: $dq\Phi^{'}(\iota) + q\Phi^{''}(\iota)d\iota = 0$. Because $\Phi(.)$ is increasing and concave then this equation features a positive relation between q and ι. As the other equation features a negative relation, therefore the system has one unique and positive solution.

As the price of capital, and the investment rate, is stationary at the equilibrium we obtain $\mu_q^{eq} = \sigma_q^{eq} = 0$. Therefore

$$\frac{d\left(q(t)K(t)\right)}{q(t)K(t)} = \left(\Phi\left(\bar{\iota}(A,\rho)\right) - \delta\right)dt + \sigma dW(t).$$

The dynamics for the net wealth, from the solution of the household problem is

$$\frac{dN(t)}{N(t)} = \left(r(t) - \rho + (\sigma_s(t))^2\right)dt + \sigma_s(t)dW(t).$$

From the market equilibrium condition (43) we should have

$$\frac{d\left(q(t)K(t)\right)}{q(t)K(t)} = \frac{dN(t)}{N(t)}.$$

Matching the two equations allows us to find the equilibrium asset return volatility, $\sigma_s(t) = \sigma$, and equilibrium risk-free interest rate

$$r^{eq}(t) = \rho + R(A,\rho) - \sigma^2. \tag{46}$$

where the net rate of return on capital is

$$R(A,\rho) \equiv \Phi\left(\bar{\iota}(A,\rho)\right) - \delta.$$

Because $R_A = \dfrac{\partial R}{\partial A} > 0$ this means that an increase in productivity will increase the risk-free interest rate. Substituting in equation (44) we determine the equilibrium process for the rate of return for capital

$$dr_s^{eq}(t) = \tilde{\mu}_s^{eq}dt + \sigma_s^{eq}dW(t) = \left(\rho + R(A,\rho)\right)dt + \sigma dW(t), \tag{47}$$

Furthermore, the equilibrium rate of growth of aggregate net wealth is $N(t) = \bar{q}(A,\rho)K(t)$,

$$\frac{dN^{eq}(t)}{N^{eq}(t)} = \left(R(A,\rho) - \rho\right)dt + \sigma dW(t)$$

The rate of growth increases with the productivity, and, again the uncertainty is driven by the fundamentals.

4.2 Limited asset market participation

Now the balance sheet constraints are: for households $B^h(t) = N^h(t)$ and for specialists $S(t) + B^x(t) = N^x(t)$, where $S(t) = q(t)K(t)$. The aggregate net wealth is $N(t) = N^h(t) + N^x(t)$.

Again we denote the weight of specialists on the aggregate wealth by $\eta(t) = N^x(t)/N(t)$.

Assuming again that the risk-free asset is in zero net supply the market equilibrium condition is $B^h(t) + B^x(t) = 0$. Consolidating the accounts and using this market equilibrium condition we obtain again $S(t) = q(t)K(t) = N(t)$.

The good's market equilibrium condition is

$$AK(t) = C(t) + \iota(t)K(t)$$

where aggregate consumption is

$$C(t) = C^h(t) + C^x(t) = \rho(N^h(t) + N^x(t)) = \rho N(t) = \rho q(t)K(t).$$

assuming again a logarithmic utility function.

Therefore, we have again $A - \iota(t) = \rho q(t)$. This equation together with the optimality condition for investment by the firm, $q(t)\Phi'(\iota(t)) = 1$, implies that both q and ι are constant as in the previous section, and functions of (A, ρ), $q = \bar{q}(A, \rho)$ and $\iota = \bar{\iota}(A, \rho)$.

This implies that the equilibrium value of capital follows the process

$$\frac{d\left(q(t)K(t)\right)}{q(t)K(t)} = \left(\Phi\left(\bar{\iota}(A, \rho)\right) - \delta\right)dt + \sigma dW(t) = R(A, \rho)dt + \sigma dW(t).$$

In order to derive the equilibrium dynamics for the aggregate demand of risky assets, which satisfies $S(t) = N(t) = N^h(t) + N^x(t)$, we have

$$\begin{aligned}
dN(t) &= dN^h(t) + dN^x(t) = \\
&= N^h(t)(r(t) - \rho)dt + N^x(t)\left(r(t) - \rho + \left(\frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)}\right)^2 + \frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)}dW(t)\right) = \\
&= N(t)\left[\left(r(t) - \rho + \eta(t)\left(\frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)}\right)^2\right)dt + \eta(t)\left(\frac{\tilde{\mu}_s(t) - r(t)}{\sigma_s(t)}\right)dW(t)\right] = \\
&= N(t)\left[\left(r(t) - \rho + \frac{(\sigma_s(t))^2}{\eta(t)}\right)dt + \sigma_s(t)dW(t)\right]
\end{aligned}$$

Matching again the two diffusion processes, because at the equilibrium $N(t) = q(t)K(t)$, we find $\sigma_s^{eq}(t) = \sigma$ and

$$r^{eq}(t) = \rho + R(A, \rho) - \frac{\sigma^2}{\eta(t)}$$

the process for the return of the risky asset is the same as in equation (47), and is independent from the distribution of wealth, η

The risk-free interest rate is, as in the model in section 3. We also find that the dynamics for the wealth distribution is driven by the equation

$$\frac{d\eta(t)}{\eta(t)} = \left(\frac{1-\eta(t)}{\eta(t)}\sigma\right)^2 dt + \left(\frac{1-\eta(t)}{\eta(t)}\sigma\right) dW(t).$$

5 Other frictions

The models we have presented are very simple. Most of the models do not display endogenous risk, with the exception of the model in which there is limited participation and non logarithmic preferences.

In order to obtain richer dynamics other type of frictions can be introduced. Most of the frictions are generated by some sort of discontinuity. But the number of frictions considered in the literature is huge.

Brunnermeier and Sannikov (2014) consider a model in which both households and specialists invest in two assets, risky and riskless, and there is some sort of heterogeneity affecting the rate of return of the risky capital ($\tilde{\mu}_s$ and/or σ_s). If there are no more requirements on structure of the balance sheet of the agents, the model will display results similar to those in section 4. However, if there is a requirement that the specialists keep a higher share of investment in equity and the non-specialists are constrained to lend in the risk-free markets, this introduces another mechanism for the share of wealth η affecting the volatility σ_s away from the fundamentals.

Kiyotaki and Moore (1997) consider a case in which loans to firms, because of asymmetries of information, require a collateral. This means that loans cannot be higher than a given proportion of net wealth, v.g $B^h(t) \leq \kappa N^x(t)$. This generates an amplification mechanism contracting investment when asset prices are depressed.

He and Krishnamurty (2012) and He and Krishnamurty (2013) assume that households do not invest directly in the risky asset but invest through an intermediary. This generates an agency problem because households and intermediaries have asymmetric on the market conditions for risky assets. In order to secure financing from households, via risk-free assets, the intermediary has to consider incentive compatible contracts with the household. In depressed markets for the risky asset, the intermediary may be constrained from investing in the risky asset because it should satisfy the incentive compatibility constraint for securing financing from the households.

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Appendix

A Arrow-Debreu equilibrium

Given the assumptions on boundedness we can form the Lagrangean

$$\mathcal{L} = E_0 \left[\int_0^\infty \left(u(C(t)) e^{-\rho t} + \lambda M(t)(Y(t) - C(t)) \right) dt \right]$$

where λ is a constant. The f.o.c are

$$u^{'}(c(t,\omega)) e^{-\rho t} = \lambda m(t,\omega) \;\; (t, \omega(t)) \in \mathbb{R}_+ \times \mathcal{F}_t.$$

As, at time $t = 0$ we have $u^{'}(c(0)) = \lambda$, where consumption is deterministic, which allows us to determine λ. Then we can write

$$\frac{u^{'}(c(t,\omega))}{u^{'}(c(0))} = e^{\rho t} m(t,\omega), \;\; (t, \omega(t)) \in \mathbb{R}_+ \times \mathcal{F}_t.$$

By the market clearing condition, we get **equilibrium stochastic discount factor** for any state of nature

$$m^*(t,\omega) = e^{-\rho t} \frac{u^{'}(y(t,\omega))}{u^{'}(y(0))}, \;\; (t,\omega) \in \mathbb{R}_+ \times \mathcal{F}_t.$$

Then $M^*(t) = e^{-\rho t} \dfrac{u^{'}(Y(t))}{u^{'}(y(0))}$ for any $t \in \mathbb{R}_+$.

B Solving the representative household problem in a finance economy

Here we consider the representative household problem in a finance economy, when it participates in all the financial markets and chooses consumption and its financial portfolio to maximize an intertemporal utility functional,

$$\max_{c,w} E_0 \left[\int_0^\infty u(C(t)) e^{-\rho t} dt \right] \tag{48}$$

subject to the instantaneous budget constraint, in its differential representation,

$$dN(t) = \{ [r(t)(1 - w(t)) + \mu^s(t) w(t)] N(t) + y(t) - c(t) \} dt + w(t) \sigma^s(t) N(t) dW(t). \tag{49}$$

where $N(0) = n_0$ is given and $N(.)$ is bounded, v.g., $\lim_{t\to\infty} N(t) \geq 0$. This is a stochastic optimal control problem with infinite horizon, and two control variables. There are three different methods to solve this problem: (1) dynamic programming; (2) stochastic control; and (3) martingale methods.

B.1 Solution by the principle of dynamic programming

Next we solve it by using the principle of dynamic programming (see Fleming and Rishel (1975) or Seierstad (2009)). This is the most common method of solution.

Let the realization at an arbitrary time t of the stochastic processes for consumption, non-financial income, portfolio weights, and net financial wealth be $C(t) = c$, $Y(t) = y$, $w(t) = w$, and $N(t) = n$. We use the same notation for the rate of return processes: $r(t) = r$, $\mu_s(t) = \mu_s$ and $\sigma_s(t) = \sigma_s$.

The Hamilton-Jacobi-Bellman equation is a second order ODE in implicit form, for the value function $V(n)$,

$$\rho V(n) = \max_{c,w} \left\{ u(c) + V^{'}(n)[(r(1-w) + \mu^s w)n + y - c] + \frac{1}{2} w^2 n^2 (\sigma^s)^2 V^{''}(n) \right\}. \tag{50}$$

The policy functions for consumption and portfolio composition, c^* and w^*, are obtained from the equations

$$u^{'}(c^*) = V^{'}(n), \tag{51}$$

$$w^* = \frac{1}{r_r(v)} \left(\frac{\mu^s - r}{\sigma^s} \right) \tag{52}$$

where

$$r_r(V(n)) = -\frac{v^{''}(n)n}{v^{'}(n)},\ p_r(v) = -\frac{v^{'''}(n)n}{v^{''}(n)}$$

are the coefficients of relative risk aversion and prudence for the value function, and $\frac{\mu^s - r}{\sigma^s}$ is the Sharpe index.

We obtain an explicit solution to the problem, if the utility function is isoelastic.

In this case we conjecture that the solution for equation (50) is of the form

$$V(n) = x \left(\frac{A(n)^{1-\gamma}}{1-\gamma} \right)$$

where x is an unknown constant, and

$$A(n) = \frac{y}{r} + n.$$

If the agent does not receive a non-financial income then $A(n) = n$.

If the conjecture is correct, note that the $V^{'}(n) = x(A(n))^{-\gamma}$, $V^{''}(n) = -x\gamma(A(n))^{-(1+\gamma)}$, and $V^{'''}(n) = -x\gamma(1+\gamma)(A(n))^{-(2+\gamma)}$. As for the utility function, we can compute coefficients of relative risk aversion and prudence for the value function (which can be thought as an intertemporal indirect utility function over net wealth,

$$r_r(V(n)) = \gamma,\ p_r(V) = 1 + \gamma.$$

Therefore, the optimal policy functions for consumption is

$$c^* = x^{-\frac{1}{\gamma}} A(n)$$

and for the weight of the risky asset on net wealth is

$$w^* = \left(\frac{\mu^s - r}{(\sigma^s)^2}\right) \frac{A(n)}{\gamma n}. \tag{53}$$

Substituting the trial function for $V(n)$ and optimal policies in the HJB equation (50), and after some algebra, we obtain

$$x = \left\{ \frac{\rho}{\gamma} - \left(\frac{1-\gamma}{\gamma}\right) \left[r + \frac{1}{2\gamma} \left(\frac{\mu^s - r}{\sigma^s}\right)^2 \right] \right\}^{-\gamma}.$$

Therefore, the optimal policy for consumption is

$$C^* = \left\{ \frac{\rho}{\gamma} - \left(\frac{1-\gamma}{\gamma}\right) \left[r + \frac{1}{2\gamma} \left(\frac{\mu^s - r}{\sigma^s}\right)^2 \right] \right\} A(N). \tag{54}$$

Therefore, the SDE for the optimal new wealth becomes, after substituting optimal consumption and portfolio policies, equations (54) and (53), in the budget constraint (49)

$$dN^* = \frac{A(N)}{\gamma} \left\{ \left[r - \rho + \frac{(1+\gamma)}{2\gamma} \left(\frac{\mu^s - r}{\sigma^s}\right)^2 \right] dt + \left(\frac{\mu^s - r}{\sigma^s}\right) dW(t) \right\} \tag{55}$$

In the case of a logarithmic utility function we have:

$$w^* = \left(\frac{\mu^s - r}{(\sigma^s)^2}\right) \frac{A(N)}{N} \tag{56a}$$

$$C^* = \rho A(N(t)) \tag{56b}$$

$$dN^* = A(N) \left\{ \left[r - \rho + \left(\frac{\mu^s - r}{\sigma^s}\right)^2 \right] dt + \left(\frac{\mu^s - r}{\sigma^s}\right) dW(t) \right\} \tag{56c}$$

If the agent does not receive non-financial income and has a logarithmic utility function the solution simplifies to

$$w^* = \left(\frac{\mu^s - r}{(\sigma^s)^2}\right) \tag{57a}$$

$$C^* = \rho N(t) \tag{57b}$$

$$dN^* = N(t) \left\{ \left[r - \rho + \left(\frac{\mu^s - r}{\sigma^s}\right)^2 \right] dt + \left(\frac{\mu^s - r}{\sigma^s}\right) dW(t) \right\} \tag{57c}$$

References: Merton (1971), Merton (1990), Duffie (2001) and (Cvitanić and Zapatero, 2004, p. 395-398).

B.2 Solution by the stochastic Pontriyagin maximum principle

Next we consider again the problem of maximizing the intertemporal utility functional (48) subject to the stochastic differential equation (49) where we assume there is no non-financial income, that is $Y(t) = 0$.

In this case there are two control variables, C and w, but one control variable, w, affects the volatility term. Because of the last fact we have to introduce two dynamic adjoint functions p and P and two static ones, q and Q.

The adjoint equations are

$$\begin{cases} dp(t) = -\left[(r + (\mu - r)w(t))\, p(t) + \sigma w(t) q(t)\right] dt + q(t) dW(t) \\ \lim_{t\to\infty} p(t) = 0 \end{cases}$$

and

$$\begin{cases} dP(t) = -\left[2\left(r + (\mu - r)w(t)\right) P(t) + (\sigma w(t))^2 P(t) + 2\sigma w(t) Q(t)\right] dt + Q(t) dW(t) \\ \lim_{t\to\infty} P(t) = 0. \end{cases}$$

To find the optimal controls we write the generalized Hamiltonian

$$G(t, N, C, w, p, P) = e^{-\rho t}\frac{C^{1-\gamma}}{1-\gamma} + p\left[\,(r + (\mu - r)w)\, N - C\right] + \frac{1}{2}\sigma^2 w^2 N^2 P$$

and

$$\mathcal{H}(t, N, C, w) = G(t, N, C, w, p, P) + \sigma w N \left(q - P\sigma w^* N\right).$$

The optimal controls, C^* and w^* are found by maximizing function $\mathcal{H}(t, N, C, w)$ for C and w. Therefore, we find

$$C^*(t) = e^{-\frac{\rho}{\gamma}t} p(t)^{-\frac{1}{\gamma}} \tag{58}$$

and the condition

$$p(t)(\mu - r)N^*(t) + w^*(t)\sigma^2 N^*(t)^2 P(t) + \sigma N^*(t)\left(q(t) - \sigma w^*(t) N^*(t) P(t)\right) = 0$$

which is equivalent to $p(t)(\mu - r)N^*(t) + \sigma q(t) N^*(t) = 0$. Therefore we find

$$q(t) = -p(t)\left(\frac{\mu - r}{\sigma}\right),$$

and, substituting in the adjoint equation,

$$dp(t) = -p(t)\left(r dt + \left(\frac{\mu - r}{\sigma}\right) dW(t)\right).$$

Observe that the structure of the model is such that the shadow value of volatility functions P and Q have no effect in the shadow value functions associated with the drift component p and q, which simplifies the solution.

Applying the Itô's formula to consumption (58), and using this expression for the adjoint variable q, we find

$$\begin{aligned} dC(t) &= -\frac{\rho}{\gamma}C(t)dt - \frac{C(t)}{\gamma p(t)}dp(t) + \frac{(1+\gamma)}{\gamma}\frac{C(t)}{p^2(t)}(dp(t))^2 = \\ &= -\frac{\rho C(t)}{\gamma}dt + \frac{C(t)}{\gamma}\left(rdt + \left(\frac{\mu - r}{\sigma}\right)dW(t)\right) + C(t)\frac{(1+\gamma)}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2 dt = \\ &= \frac{C(t)}{\gamma}\left\{\left(r - \rho + \frac{(1+\gamma)}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)dt + \left(\frac{\mu - r}{\sigma}\right)dW(t)\right\}. \end{aligned}$$

Now, we **conjecture** that consumption is a linear function of net wealth $C = \theta N$. If this is the case this would allow us to obtain the optimal portfolio composition w^*. If the conjecture is right then we will also have

$$\begin{aligned} dC(t) &= \theta dN(t) \\ &= \theta N(t)\left[\,(r + (\mu - r)w - \xi)\,dt + \sigma w dW(t)\right] \\ &= C(t)\left[\,(r + (\mu - r)w - \xi)\,dt + \sigma w dW(t)\right] \end{aligned}$$

This can only be consistent with the previous derivation if

$$\begin{cases} \frac{1}{\gamma}\left[r - \rho + \frac{(1+\gamma)}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right] = r + (\mu - r)w - \theta \\ \frac{1}{\gamma}\left(\frac{\mu - r}{\sigma}\right) = \sigma w \end{cases}$$

Solving for θ and w we obtain the optimal controls

$$\begin{aligned} \theta &= \frac{1}{\gamma}\left[\rho + (\gamma - 1)\left(r + \frac{1}{2\gamma}\left(\frac{\tilde{\mu}_s - r}{\sigma_s}\right)^2\right)\right] \\ w^* &= \frac{1}{\gamma}\left(\frac{\mu - r}{\sigma^2}\right) \end{aligned}$$

Substituting in the budget constraint we have the optimal net wealth process

$$\frac{dN^*(t)}{N^*(t)} = \mu_n dt + \sigma_n dW(t)$$

where

$$\mu_n = \frac{1}{\gamma}\left[r - \rho + \frac{(1+\gamma)}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2\right) \tag{59}$$

$$\sigma_n = \frac{1}{\gamma}\left(\frac{\mu - r}{\sigma}\right) \tag{60}$$

which can be explicitly solved with the initial condition $N^*(0) = n_0$. We also find that

$$\frac{dC^*(t)}{C^*(t)} = \mu_n dt + \sigma_n dW(t)$$

the rates of return for consumption and wealth are perfectly correlated.

References (Yong and Zhou, 1999, chap. 3)

B.3 Solution by martingale representation methods

Suppose there is financial market with two assets, one risk-free asset with price B and one risky asset with price S following the processes $\{(B(t), S(t)), t \geq 0\}$ represented by

$$\begin{aligned} dB(t) &= r(t)B(t)dt, \quad B(0) = 1 \\ dS(t) &= \mu_s(t)S(t)dt + S(t)\sigma_s(t)dW(t), \quad S(0) = S_0. \end{aligned}$$

Then asset 1 is risk-free and asset 2 is risky.

We can show that $e^{-\int_0^t r(s)ds}S(t)$ **can be converted into a martingale**. Applying the Itô's lemma we get

$$\begin{aligned} d\left[e^{-\int_0^t r(s)ds}S(t)\right] &= S(t)d\left[e^{-\int_0^t r(s)ds}\right] + e^{-\int_0^t r(s)ds}dS(t) \\ &= e^{-\int_0^t r(s)ds}\left(-r(t)S(t)dt + \mu_s(t)S(t)dt + \sigma_s(t)S(t)dW(t)\right) \\ &= e^{-\int_0^t r(s)ds}\left((\mu_s(t) - r(t))S(t)dt + \sigma_s(t)S(t)dW(t)\right) \end{aligned}$$

If , in general $\mu_s(t) \neq r(t)$ then $e^{-\int_0^t r(s)ds}S(t)$ is not a martingale, with the initial probability distribution P. That is $\mathbb{E}^P\left[d\left[e^{-\int_0^t r(s)ds}S(t)\right]\right] \neq 0$

From the Girsanov theorem, we can find a $\mathcal{F}_t$-adapted process X_t and a new Wiener process $\tilde{W}(t)$ with probability measure Q and with density ξ_t relative to P, such that

$$dP(t) = \xi(t)dQ(t)$$

i.e., for which

$$\xi(t) = e^{\int_0^t X(s)dW(s) - \frac{1}{2}\int_0^t X(s)^2 ds}$$

with $\int_0^t X(s)^2 ds < \infty$, is a martingale. The new process is defined as

$$dW(t) = d\tilde{W}(t) - X(t)dt.$$

Applying that result we get

$$\begin{aligned} d[e^{-\int_0^t r(s)ds}S(t)] &= e^{-\int_0^t r(s)ds}\left((\mu_s(t) - r(t))S(t)dt + \sigma_s(t)S(t)dW(t)\right) \\ &= e^{-\int_0^t r(s)ds}\Big((\mu_s(t) - r(t))S(t)dt + \sigma_s(t)S(t)(d\tilde{W}(t) - X(t))\Big) \\ &= e^{-\int_0^t r(s)ds}\Big[((\mu_s(t) - r(t))S(t)dt - \sigma_s(t)S(t)X(t)dt) + \sigma_s(t)S(t)d\tilde{W}(t)\Big] \\ &= e^{-\int_0^t r(s)ds}\sigma_s(t)S(t)d\tilde{W}(t). \end{aligned}$$

Then $\mathbb{E}^Q\left[d\left[e^{-\int_0^t r(s)ds}S(t)\right]\right] = 0$ if and only if

$$X(t) = \frac{\mu_s(t) - r(t)}{\sigma_s(t)}.$$

This process is called **market price of risk**.

Considering the budget constraint For convenience, we rewrite the instantaneous budget constraint

$$dN(t) = \left[\left(r(t)(1 - w(t)) + \mu_s(t)w(t)\right)N(t) + Y(t) - C(t)\right]dt + w(t)\sigma_s(t)N(t)dW(t),$$

and $N(0) = N_0$ and assume that $Y(t)$ follows equation (1).

Consider the process

$$d\overline{M}(t) = -X(t)\overline{M}(t)dW(t)$$

also called in the literature the state density process, where $X(t)$ is the market price of risk. The process

$$M(t) = e^{-\int_0^t r(s)ds}\overline{M}(t)$$

is called the **state density deflator**. Using the Itô's lemma (prove this) it has the differential representation

$$dM(t) = -M(t)\left(r(t)dt + X(t)dW(t)\right).$$

Now consider the deflated value of wealth defined as

$$Z(t) = M(t)N(t)$$

which is a $\mathcal{F}_t$-measurable. By using the Itô's lemma its differential representation is [4]

$$dZ(t) = Z(t)[\sigma_s(t)w(t) - X(t)]dW(t) + M(t)(Y(t) - C(t))dt$$

then

$$Z(T) = Z(t) + \int_t^T M(s)(Y(s) - C(s))ds + \int_t^T Z(s)[\sigma_s(s)w(s) - X(s)]dW(s)$$

Under certain conditions, a **self-financing strategy** holds, i.e,

$$E_t[Z(T)] = Z(t) + E_t\left[\int_t^T \overline{M}(s)(Y(s) - C(s))ds\right]$$

[4] **Exercise:** prove this result by using the following version of the Itô's lemma: for $y = f(x_1, x_2)$

$$dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(dx_1)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}(dx_2)^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}dx_1dx_2.$$

If we set $t = 0$ and take $T = \infty$ and assume that there are no bubbles

$$E_0[\lim_{t \to \infty} \overline{M}(t)N(t)] = N_0$$

then we get the intertemporal budget constraint as

$$E_0 \left[\int_0^{\infty} \overline{M}(s)(Y(s) - C(s))ds \right] = 0$$

which is formally identical to the restriction for the consumer problem in the Arrow-Debreu economy. This means that the state price deflactor and the stochastic discount factor are equal.

An equivalent result would be obtained if the the household chose

$$w(t) = \frac{X(t)}{\sigma_s(t)},$$

which is the solution obtained by the other two methods.