Optimal taxation:
Chamley-Judd-Ramsey taxation model

Paulo B. Brito
Advanced Macroeconomics (PhD in Economics): 2018-2019
ISEG
Universidade de Lisboa
pbrito@iseg.ulisboa.pt

November, 13, 2018
(revised December 17, 2018)

Contents

1 Introduction

In this note we present the Chamley (1986) model. This paper, jointly with Judd (1985), are still a benchmark regarding optimal taxation and, in particular, as regard optimal capital-tax policy. Their main result is that, in the long-run, capital taxes should be zero. This result, is still puzzling (for a recent discussion see Straub and Werning (2018)).

Here, we are more interested on the setting up and solution approaches rather than in the specific discussion of the long-run capital tax. Therefore, we present first a simpler version allowing for closed-form solutions in which the production function is linear (the so-called Ak economy). Later on, we present the original Chamley (1986) model in which there are both labor and capital taxes.

There are two approaches for solving the optimal tax problem: the **dual approach** (Chamley's original one) and the **primal approach** (following a method used in Lucas and Stokey (1983)). Although the dual approach is more intuitive, there is a considerable simplification if we use the primal approach. The primal approach has dominated the literature recently.

Also we present the models in continuous time, as in the original papers. A big part of the macro literature uses discrete time (see Ljungqvist and Sargent (2012)). Although the results are similar, from a qualitative point of view, there are some differences in some of the key formulas. This is due to the fact that some constraints in the first period are important and the fact that the first period of a discrete time model is an integral of the analogous solution for a continuous time case.

2 A simple economy without labor taxes

We start with a simple model in which we can derive closed form solutions. Time is the independent variable, $t \in [0, \infty)$, there is perfect information, and private agents are homogeneous. The representative consumer has total financial wealth a and his only source of income is capital income ra, where r is the interest rate. The financial wealth is composed of private equity and government bonds, b. Equity finances physical capital, which is the only input in producing output y. The output is used in households consumption, c, government consumption, g, and capital accumulation $\dot{k}$. The government finances its expenditures, on goods and interest payments, by issuing new debt, $\dot{b}$ and capital taxes. Capital taxes are levied at the same rate on every component of financial wealth, private issued equity or government bonds. Because of arbitrage in the financial markets the rates of return for investment on bonds and equity are the same.

A **competitive equilibrium** is an allocation, $(c(t), k(t), b(t))_{t \in \mathbb{R}_+}$, defined below, which is parameterized by the government tax policy and government expenditures $(\tau(t), g(t))_{t \in \mathbb{R}_+}$.

A **Ramsey equilibrium** is a competitive equilibrium such that the government chooses its tax policy in an efficient way, i.e, such that the competive equilibrium allocations are optimal, in the sense of maximizing a social welfare function. In the present note we assume the social utility function is equal to the utility function of the representative agent.

The primitives of the model refer to preferences and technology. We assume the following:

- Preferences:
  - intertemporal preferences are characterized by additive separability and time-discounting: $U(c(.)) = \int_0^\infty u(c(t))e^{-\rho t}dt$ where $\rho > 0$
  - the instantaneous utility function is increasing, concave and it has the Inada properties, as a function of consumption: $u^{'}(c) > 0 > u^{''}(c)$ and $\lim_{c\to 0^+} u^{'}(c) = \infty$ and $\lim_{c\to+\infty} u^{'}(c) = 0$;
  - in particular, we assume an utility function that satisfies those properties

$$u(c) = \begin{cases} \dfrac{c^{1-\sigma}}{1-\sigma} & , \ \sigma > 0, \ \sigma \neq 1 \\ \ln{(c)}, & \sigma = 1 \end{cases}$$

- Technology: capital is the only input and there is constant returns to scale: $y(t) = Ak(t)$

2.1 The competitive equilibrium

Definition 1. ***A dynamic general equilibrium*** *(DGE) is an allocation path* $(c^{eq}(t), k^{eq}(t), b^{eq}(t))_{t\in\mathbb{R}_+}$ *and a price system* $(r(t))_{t\in\mathbb{R}_+}$*, such that, given* $(g(t), \tau(t))_{t\in\mathbb{R}_+}$*: (1) the representative household solves its problem; (2) firms optimize; (3) the government budget constraints is satisfied; (4) there is equilibrium in the goods market; and (5) the balance sheet constraint holds.*

The firms optimization problem is

$$\max_{k}\{Ak - r(t)k\}, \text{for every } t.$$

Because we assume that capital taxes are paid by the owners of capital, i.e., households, we obtain, from the firms' first-order condition, before-tax rate of return of capital

$$r(t) = A, \ \forall t \in [0, \infty). \tag{1}$$

The **household problem** is to find $(c(t), a(t))_{t\in\mathbb{R}_+}$, where $a(t) = k(t) + b(t)$, that solve

$$\max_{c(.)} \int_0^\infty \frac{c(t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt$$

subject to the instantaneous budget constraint, non-Ponzi-game condition, and the initial wealth:

$$\dot{a} = \bar{r}(t)a - c,\ t \geq 0 \tag{2a}$$

$$\lim_{t\to\infty} e^{-\int_0^t (\bar{r}(u)du} a(t) = 0 \tag{2b}$$

$$a(0) = a_0,\ t = 0 \tag{2c}$$

given $(r(t), \tau(t))_{t\in\mathbb{R}_+}$. The after-tax rate of return of capital, which is

$$\bar{r}(t) = A(1 - \tau(t)),$$

is time-varying because capital taxes can be time-varying. We are assuming linear taxes.

From the first order optimality condition, for the representative household, we obtain the dynamics for consumption

$$\dot{c}(t) = c(t)\bar{\gamma}(t), t \geq 0, \tag{3}$$

where the rate of growth of consumption is proportional to the difference between the after-tax rate of return of capital and the rate of time preference,

$$\bar{\gamma}(t) = \frac{\bar{r}(t) - \rho}{\sigma} = \begin{cases} \dfrac{A(1 - \tau(t)) - \rho}{\sigma}. & \text{if } \sigma \neq 1 \\ A(1 - \tau(t)) - \rho & \text{if } \sigma = 1. \end{cases}$$

Solving equation (3) we obtain

$$c(t) = c(0)e^{\int_0^t \bar{\gamma}(s)ds}$$

where $c(0)$ is determined endogenously. Solving equations (2a), with the admissibility constraint (2c), we obtain the present value of households's wealth using the after-tax rate of return as the discount factor,

$$a(t)e^{-\int_0^t \bar{r}(s)ds} = a_0 - \int_0^t e^{-\int_0^s \bar{r}(u)du} c(s)ds.$$

Substituting optimal consumption yields

$$a(t)e^{-\int_0^t \bar{r}(s)ds} = a_0 - c(0)\int_0^t e^{-\int_0^s \bar{\beta}(u)du} ds,$$

where

$$\bar{\beta}(t) = \bar{r}(t) - \bar{\gamma}(t) = \frac{A(1 - \tau(t))(\sigma - 1) + \rho}{\sigma}.$$

If the NPG condition, equation (2b), holds then consumption at time $t = 0$ is proportional to initial wealth

$$c(0) = \frac{a_0}{\bar{\psi}(\sigma)} \tag{4}$$

where the wealth-consumption ratio is

$$\bar{\psi}(\sigma) = \begin{cases} \rho^{-1}, & \text{if } \sigma = 1 \\ \int_0^\infty e^{-\int_0^s \bar{\beta}(u)du} ds, & \text{if } \sigma \neq 1 \end{cases}$$

Observations:

- while in the logarithmic case optimal consumption is independent of taxes, because $\bar{\beta} = \rho$, in the general isoelastic case it depends on the future path of taxes. This provides a further policy instrument for the general iso-elastic case (see Lansing (1999) for a discussion)

- we require that $0 < \bar{\psi}(\sigma) < \infty$. A sufficient condition for this is that $\bar{r}(t) > \bar{\gamma}(t)$ or equivalently $\rho + (\sigma - 1)A(1 - \tau(t)) > 0$ for all t.

Therefore, the demand for consumption at any time $t \geq 0$, which we can take as the equilibrium demand for consumption is

$$c^{eq}(t) = \frac{a_0}{\bar{\psi}(\sigma)} e^{\int_0^t \bar{\gamma}(s)ds}, \; t \geq 0, \tag{5}$$

where households wealth at time $t = 0$ satisfies the balance-sheet condition $a(0) = a_0 = k_0 + b_0$.

Substituting in the solution to the budget constraint yields

$$\begin{aligned} a(t) &= a_0 e^{\int_0^t \bar{r}(s)ds} \left(\bar{\psi}(\sigma) - \int_0^t e^{-\int_0^s (\bar{r}(u) - \bar{\gamma}(u))du} ds \right) \bar{\psi}(\sigma)^{-1} \\ &= a_0 e^{\int_0^t \bar{r}(s)ds} \left(\bar{\psi}(\sigma) - \int_0^\infty e^{-\int_0^s \bar{\beta}(u)du} ds + e^{-\int_0^t \bar{\beta}(s)ds} \int_t^\infty e^{-\int_t^s \bar{\beta}(u)du} ds \right) \bar{\psi}(\sigma)^{-1} \\ &= a_0 e^{\int_0^t \bar{\gamma}(s)ds} \int_t^\infty e^{-\int_t^s \bar{\beta}(u)du} ds \bar{\psi}(\sigma)^{-1}. \end{aligned}$$

Therefore

$$a^{eq}(t) = a_0 e^{\int_0^t \bar{\gamma}(s)ds} \mu(t), \tag{6}$$

where

$$\mu(t) \equiv \frac{\int_t^\infty e^{-\int_t^s \bar{\beta}(u)du} ds}{\int_0^\infty e^{-\int_0^s \bar{\beta}(u)du}}.$$

This function has the following properties: First, $\mu(0) = 1$. Second, If $\bar{r}(t) - \bar{\gamma}(t)$ is constant then $\mu(t) = 1$ for any $t \in [0, \infty)$. This is the case if $\sigma = 1$ because $\bar{r}(t) - \bar{\gamma}(t) = \rho$ which implies

$$a(t) = a_0 e^{\int_0^t \bar{\gamma}(s)ds} \text{ for } \bar{\gamma}(t) = A(1 - \tau(t)) - \rho$$

This is also the case if capital taxes are set to zero $\tau(t) = 0$ permanently. Third, if $\bar{r}(t) - \bar{\gamma}(t)$ is not constant but converges to a positive constant then asymptoticaly $\lim_{t\to\infty} \mu(t) \to 1$. Fourth, if this is not the case it may happen that $\lim_{t\to\infty} \mu(t) \to 0$.

There are two takehomes from this:

- in the presence of capital taxes, and if the utility function displays intertemporal elasticities lower than one, the time behavior of households' wealth depends on the fiscal policy;
- there are two effects of capital taxes: a wealth effect which operates through $\bar{\psi}$ and a substitution effect between consumption in different points in time (see equation (5)). In this sense, capital tax can be seen as a tax on future consumption.

The government's budget constraint is

$$\dot{b} = g(t) + r(t)b - \tau(t)r(t)\left(b(t) + k(t)\right) = g(t) + \bar{r}(t)\left(b(t) + k(t)\right) - r(t)k(t). \tag{7}$$

and we assume that the intertemporal budget constraint, $\lim_{t\to\infty} e^{-\int_0^t \bar{r}(s)ds} b(t) = 0$, holds, and $b(0) = b_0$ is given.

The goods's market clearing condition is,

$$y(t) = Ak(t) = c(t) + \dot{k}(t) + g. \tag{8}$$

From the Walras Law one, one of three equations, among the two budget constraints, (2a) and (7), and the goods market clearing condition (8) is redundant.

The DGE path for $k(t)$ is

$$k^{eq}(t) = e^{rt}\left(k_0 - \int_0^t e^{-rs}\left(c^{eq}(s) + g(s)\right) ds\right)$$

and we obtain the government stock of bonds residually

$$b^{eq}(t) = a^{eq}(t) - k^{eq}(t).$$

2.2 The optimal-tax policy problem

An optimal tax-policy is defined by the flow of taxes $(\tau(t))_{t\in\mathbb{R}_+}$ such that we have a Ramsey-equilibrium.

Definition 2. *A **Ramsey equilibrium** is an allocation $(c^*(t), k^*(t))_{t\in\mathbb{R}_+}$ such that, given g, this is an equilibrium allocation that maximizes consumer utility.*

Definition 3. *An **optimal tax policy** is a tax path $(\tau(t))_{t\in\mathbb{R}_+}$ that implements a Ramsey equilibrium.*

There are two approaches to obtaining a Ramsey equilibrium:

- **primal approach** (attributed to Lucas and Stokey (1983) see Chari and Kehoe (1999)) : find equilibrium paths for c and k that are optimal and find indirectly the optimal tax policy from the first order conditions of that problem (this is the preferred approach in the recent literature)

- the **dual approach** (this was the original Chamley (1986) approach): find directly the optimal tax policy subject to the conditions characterizing an equilibrium .

2.3 The optimal-tax policy problem: the primal approach

The primal approach identifies the optimal allocation $(c^*(t), k^*(t))_{t\in\mathbb{R}_+}$ with the solution to the **Ramsey primal problem** $(c^{pr}(t), k^{pr}(t))_{t\in\mathbb{R}_+}$. There are several different approaches to modelling the Ramsey problem, next we present the simplest one.

In order to obtain closed form solutions, we assume that government expenditure is a constant proportion of private consumption

$$\eta \equiv \frac{c+g}{c} > 1.$$

The Ramsey primal-problem is:

$$\max_{c(.)} \int_0^\infty u(c(t))e^{-\rho t}dt$$

subject to two constraints: the **implementability** and **resource** constraints

$$c(0) = (k_0 + b_0)\bar{\psi}(\sigma)^{-1},\ t = 0 \tag{9a}$$

$$\dot{k} = Ak - c - g,\ t \in [0, \infty) \tag{9b}$$

given $k(0) = k_0$ and $b(0) = b_0$ and $\lim_{t\to\infty} k(t) \geq 0$.

The Hamiltonian is

$$H^{pr}(t) = u(c(t)) + q^{pr}(t)\left(Ak(t) - c(t) - g\right),\ \text{for } t \geq 0$$

where q^{pr} is the co-state variable associated to the state variable k. As the implementability constraint referred to time $t = 0$, (9a), is a static constraint, we write the Lagrangean, referring to time $t = 0$

$$\mathcal{L}(0) = H^{pr}(0) + \lambda\left((k_0 + b_0)\psi(\sigma)^{-1} - c(0)\right),\ \text{for } t = 0$$

There are, at $t = 0$, two first order conditions for consumption: an optimality condition $c(0)^{-\sigma} = q^{pr}(0)+\lambda$, and a complementary slackness condition introduced by the implementability constraint, $c(0) = (k_0 + b_0)\bar{\psi}(\sigma)^{-1}$. Therefore, differently from the consumer problem in the CE economy, consumption at time $t = 0$ is pre-determined

$$c^{pr}(0) = (k_0 + b_0)\bar{\psi}(\sigma)^{-1} \tag{10}$$

and the Lagrange multiplier is $\lambda = (k_0 + b_0)^{-\sigma}\bar{\psi}(\sigma)^{\sigma} - q^{pr}(0)$.

For $t \geq 0$, the other optimality conditions are

$$c^{pr}(t)^{-\sigma} = q^{pr}(t) \tag{11a}$$

$$\dot{q}^{pr} = q^{pr}(\rho - A) \tag{11b}$$

$$\lim_{t\to\infty} v(t)e^{-\rho t} = 0 \tag{11c}$$

where is the optimal value of capital $v(t) \equiv q^{pr}(t)k(t)$. Then

$$\begin{aligned}\frac{\dot{v}}{v} &= \frac{\dot{q}^{pr}}{q^{pr}} + \frac{\dot{k}}{k}\\ &= \rho - \eta c\end{aligned}$$

using the assumption that $c + g = \eta c$, and the constraint (9b). Then

$$\dot{v} = \rho v - \eta (q^{pr})^{\frac{\sigma-1}{\sigma}}.$$

Integrating (11b),

$$q^{pr}(t) = q^{pr}(0)e^{(\rho - A)t},$$

and substituting in the solution for v yields

$$\begin{aligned}v(t)e^{-\rho t} &= v(0) - \eta \int_0^t e^{-\rho s}(q^{pr}(s))^{\frac{\sigma-1}{\sigma}} ds\\ &= v(0) - \eta (q^{pr}(0))^{\frac{\sigma-1}{\sigma}} \int_0^t e^{-\beta s} ds\\ &= v(0) + \frac{\eta}{\beta}(q^{pr}(0))^{\frac{\sigma-1}{\sigma}} \left(e^{-\beta t} - 1\right).\end{aligned}$$

where

$$\beta \equiv \frac{A(\sigma - 1) + \rho}{\sigma}.$$

Imposing the transversality condition we obtain

$$v(0) = q^{pr}(0)k_0 = \frac{\eta}{\beta}(q^{pr}(0))^{\frac{\sigma-1}{\sigma}} \Leftrightarrow \frac{\eta}{\beta}(q^{pr}(0))^{-\frac{1}{\sigma}} = k_0$$

and, therefore

$$c^{pr}(0) = \frac{\beta}{\eta}k_0. \tag{12}$$

Because, from equations (11a) and (**??**), $\dot{c}^{pr} = \gamma c^{pr}(t)$, where

$$\gamma = \frac{A - \rho}{\sigma},$$

then the optimal consumption behavior

$$c^{pr}(t) = c^{pr}(0)e^{\gamma t} = \frac{\beta}{\eta}k_0 e^{\gamma t},\ t \geq 0 \tag{13}$$

This implies that the optimum capital stock is

$$k^{pr}(t) = k(0)e^{\gamma t}.$$

We say that the tax policy implements the Ramsey equilibrium if the tax policy which is chosen, in a competitive context, generates a consumption path which is equal to the optimum.

Therefore, the **optimal tax policy** that implements the Ramsey equilibrium should have the following features:

- for $t > 0$ (compare equations (5) and (13)): it should converge asymptotically to zero. This is because, $c^{eq}(t)$ can only converge asymptotically to $c^{pr}(t)$ if $\bar{\gamma}(t) \to \gamma$, which requires that $\tau(t) \to 0$ for $t \to \infty$.
- at time $t = 0$ the following condition should hold (compare equations (10) and (12))

$$\frac{\beta\bar{\psi}(\sigma)}{\eta}k_0 = (k_0 + b_0)$$

  that is

$$\beta \int_0^\infty e^{-\int_0^t \bar{\beta}(s)ds}dt = \eta\left(\frac{k_0 + b_0}{k_0}\right).$$

- In general, the literature assumes that $\tau(0) = \tau_0$ is pre-determined, such that $0 < \tau_0 < 1$. We could conceive two tax regimes: one initial regime in the period $[0, t^{'})$, for $t^{'} > 0$, in which the tax rate τ_0 is kept constant, and a second regime, for period $[t^{'}, \infty)$, in which the tax is reduced to zero. In this case, we can compute $t^{'} > 0$ solving

$$e^{-\beta t^{'}} + \frac{\beta}{\bar{\beta}(\tau_0)}\left(1 - e^{-\bar{\beta}(\tau_0)t^{'}}\right) = \eta\left(\frac{k_0 + b_0}{k_0}\right), \text{ for } 0 < t^{'} < \infty$$

  where $\bar{\beta}(\tau_0) \equiv \dfrac{A(\sigma - 1)(1 - \tau_0) + \rho}{\sigma}$.

- A policy of permanently zero taxes, except for time $t = 0$, would not be feasible if $\eta \neq \dfrac{k_0}{k_0 + b_0}$.

2.4 The optimal-tax policy problem: the dual approach

The primal approach identifies the optimal allocation $(c^*(t), k^*(t))_{t\in\mathbb{R}_+}$ with the solution to the **Ramsey dual problem** $\left(c^d(t), k^d(t)\right)_{t\in\mathbb{R}_+}$

The Ramsey dual problem takes the equations characterizing the competitive equilibrium allocation

$$\dot{c} = \bar{\gamma}(t)c, \text{with } \bar{\gamma} = \frac{\bar{r} - \rho}{\sigma} \tag{14a}$$

$$\dot{k} = Ak - c - g \tag{14b}$$

$$\dot{b} = g - Ak + \bar{r}(k + b) \tag{14c}$$

where c, k and b are state variables, and maximizes an intertemporal social welfare function by using the capital tax as a control variable and the constraint that capital tax should be bellow 1. If we assume that the social welfare function is equal to the consumer utility function and if the control variable is $\bar{r} = A(1-\tau(t))$, the dual Ramsey problem is

$$\max_{\bar{r}(.)} \int_0^\infty \frac{c(t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt$$

subject to equations (14a), (14b), (14c) and

$$\bar{r}(t) \geq 0 \tag{15}$$

where $k(0) = k_0$ and $b(0) = b_0$ are given, $c(0)$ is **free**, and asymptotic boundary conditions over k and b are also introduced.

Assumption 1. *We assume that capital productivity is higher than the rate of time preference* $A > \rho$.

The Hamiltonian is

$$H^d\left(\bar{r}, c, k, b, q_c^d, q_k^d, q_b^d\right) = \frac{c^{1-\sigma}}{1-\sigma} + q_c^d\left(\frac{\bar{r}-\rho}{\sigma}c\right) + q_k^d\left(Ak - c - g\right) + q_b^d\left(g - Ak + \bar{r}(k+b)\right)$$

where q_j^d, for $j = c, k, b$ are the co-state variables associated to the three state variables. The co-state variable $-q_b^d$ can be interpreted as the marginal burden introduced by government expenditures.

Because we have an inequality constraint, we have to consider the Lagrangean function

$$\mathcal{L}^d\left(\bar{r}, c, k, b, q_c^d, q_k^d, q_b^d, \nu\right) = H^d\left(\bar{r}, c, k, b, q_c^d, q_k^d, q_b^d\right) + \nu\bar{r},$$

for every $t \in [0, \infty)$.

The first order conditions for the control variable, $\bar{r}$, are

$$\frac{q_c^d(t)c(t)}{\sigma} + q_b^d(t)(k(t)+b(t)) + \nu(t) = 0, \text{for } t \in [0,\infty) \tag{16}$$
$$\nu(t)\bar{r}(t) = 0,\ \nu(t) \geq 0,\ \bar{r} \geq 0, \text{for } t \in [0,\infty). \tag{17}$$

The KKT (Kharush-Kuhn-Tucker) conditions are satisfied in two cases: $\nu(t) = 0$, $\bar{r}(t) > 0$ and $0 < \tau(t) < 1$, or $\nu(t) > 0$, $\bar{r}(t) = 0$ and $\tau(t) = 1$. Therefore, the biding condition implies that there are two tax regimes:

1. a regime in which the constraint is biding, with $\bar{r}(t) = 0$, meaning that $\tau(t) = 1$, and $\nu(t) > 0$; or
2. a regime in which the constraint is not biding, with $\bar{r}(t) > 0$, meaning that $\tau(t) < 1$, and $\nu(t) = 0$.

Because we have three state variables, we should have three Euler equations for the associated co-state variables

$$\begin{aligned}
\dot{q}_c^d &= q_c^d(t)\left(\frac{(1+\sigma)\rho-\bar{r}(t)}{\sigma}\right) - c(t)^{-\sigma} + q_k^d(t),\ t\geq 0, && (18)\\
\dot{q}_k^d &= q_k^d(t)\left(\rho-A\right) - q_b^d(t)\left(\bar{r}(t)-A\right)\right),\ t\geq 0, && (19)\\
\dot{q}_b^d &= q_b^d(t)\left(\rho-\bar{r}(t)\right),\ t\geq 0, && (20)
\end{aligned}$$

Because $c(0)$ is free and $k(0)$ and $b(0)$ are pre-determined we have

$$q_c^d(0) = 0,\ q_k^d(0) \text{ free } q_b^d(0) \text{ free.}$$

Evaluating equation (16) at time $t=0$, and because $q_c(0) = 0$ [1] we have $q_b^d(0)a_0 = -\nu(0) < 0$ which should be non-zero. Therefore, the **constraint** (17) **is biding at time** $t = 0$**, i.e,** $\tau(0) = 1$**.**

In order to see if this regime can be asymptotically optimal, we have from equation (16)

$$\nu(t) = -\left(\frac{q_c^d(t)c(t)}{\sigma} + q_b^d(t)(k(t)+b(t))\right).$$

Taking time derivatives, and using equations (14a) to (14c), and (16) and (18), we obtain

$$\dot{\nu} = \rho\nu + \frac{c}{\sigma}Z$$

where $Z \equiv c^{-\sigma} - q_k + \sigma q_b$. Taking time derivatives we get

$$\dot{Z} = Z(\rho - \bar{r}) + x(r - \bar{r})$$

where $r = A$ and $x \equiv q_k - q_b$. Taking time derivatives, and using equations (17) and (18) yields $\dot{x} = x(\rho - A)$, which has solution

$$x(t) = q_k^d(t) - q_b^d(t) = (q_k^d(0) - q_b^d(0))e^{-(A-\rho)t}.$$

Because we assumed $A > \rho$, then $x(\infty) = 0$ which means that q_k and q_b converge asymptotically to the same value.

Then transversality conditions to this problem

$$\lim_{t\to\infty} q_c^d(t)c(t)e^{-\rho t} = \lim_{t\to\infty} q_k^d(t)k(t)e^{-\rho t} = \lim_{t\to\infty} q_b^d(t)b(t)e^{-\rho t} = 0$$

[1]This is related to the fact the initial value of the state variable c is free at time $t = 0$. This is an uncommon assumption in optimal control problems applied to macroeconomics, but it is common in the mechanism design literature. Apparently the proof in Turnovsky (1995) missed this point.

imply that $\nu(\infty) = 0$. Therefore, **constraint** (17) **is not biding asymptotically implying , i.e,** $\tau(\infty) < 1$.

As $\nu(\infty) = 0$ we can avoid consumption to be zero asymptotically only if $Z(\infty) = 0$.

This, and the fact that $q_k(\infty) = q_b(\infty)$, requires that $\bar{r}(\infty) = A$ (from equations (19) and (20)). This can only occur if constraint (15) is non-binding.

The fact that $Z(\infty) = 0$, which is equivalent to having $c(\infty)$ unbounded, is also consistent with the solution of (14a) when $\bar{r}$ converges to A, that is

$$c^d(t) = c(0)e^{\int_0^t \bar{\gamma}(s)ds} \to c^*(0)e^{\gamma t}.$$

which is unbounded because $\gamma > 0$.

Because the capital tax at $t = 0$ should be $\tau(0) = 1$, the fact that it converges to $\tau(\infty) = 0$ means that there should be a downward, continuous or discrete, adjustment of the tax rate .

This was the result in Chamley (1986) proposition 3. He provided the famous result that it would be optimal to **tax capital at a zero marginal rate in the steady state**. This is a strong result and has been widely discussed (see Straub and Werning (2018) for a recent survey of the problem).

As we saw in this simple model, we didn't prove that this was the unique solution.

3 An economy with labor taxes

In order to introduce other taxes, beyond capital taxes, we now assume that labor supply is endogenous. Now labor is one input in production but also a source of disutility, because people value leisure. We start with a simple Ramsey problem undistorted by taxes. Next we consider a competitive economy with taxes and at last consider the (Chamley-Judd) problem of finding the optimal tax policy. We use the primal approach.

In this section we derive expressions characterizing the optimal policies for consumption and labor taxes. We show that: first, the previous results for capital taxes continue to hold; second, one of the other two taxes are redundant; and third, the second instrument chosen, in this case labor taxes, should be kept close to constant.

3.1 Competitive equilibrium without government

We start with a competitive equilibrium without government, that is, without taxes, government expenditures and government debt. As equilibrium is Pareto efficient, it is equivalent to the solution for centralized Ramsey model with endogenous labor.

The assumptions regarding the primitives, preferences and technology, are the following: First, the utility function satisfies $u_\ell(c, \ell) < 0 < u_c(c, \ell)$ and $u(c, \ell)$ is concave in (c, ℓ), and, second the production function $F(k, \ell)$ is increasing in both inputs it is concave and it is homogeneous of first degree. This last assumption implies that the Euler theorem holds as $y = F(k, \ell) = F_k(k, \ell)k + F_\ell(k, \ell)\ell$.

The household problem is

$$\max_{c,\ell} \int_0^\infty u(c(t), \ell(t)) e^{-\rho t} dt$$

subject to

$$\dot{k} = F(k, \ell) - c - \delta k$$

where $k(0) = k_0$ given and $\lim_{t\to\infty} k(t) \geq 0$.

The first order conditions are

$$\begin{aligned}
u_c(c(t), \ell(t)) &= q(t),\ t \in [0, \infty) \\
u_\ell(c(t), \ell(t)) &= -q(t) F_\ell(k(t), \ell(t)),\ t \in [0, \infty) \\
\dot{q} &= q(t)\left(\rho - r(t)\right), t \in [0, \infty),\ \lim_{t\to\infty} q(t)k(t)e^{-\rho t} = 0 \\
\dot{k} &= F(k(t), \ell(t)) - c(t) - \delta k(t), t \in [0, \infty),\ k(0) = k_0.
\end{aligned}$$

The intratemporal arbitrage condition between consumption and leisure (or labor) is

$$w(t) = -\frac{u_\ell(c(t), \ell(t))}{u_c(c(t), \ell(t))}. \tag{21}$$

where, in a competitive economy, the wage is equal to the marginal productivity of labor

$$w(t) = F_\ell(k(t), \ell(t)).$$

The intertemporal arbitrage condition, between savings and future consumption is

$$\frac{d}{dt}\left(u_c(c(t), \ell(t))\right) = u_c(c(t), \ell(t))(\rho - r(t)) \tag{22}$$

where the net rate of return is, in a competitive economy, equal to the net marginal productivity of capital

$$r(t) = F_k(k(t), \ell(t)) - \delta.$$

The competitive equilibrium path $(c^{eq}(t), \ell^{eq}(t), k^{eq}(t))_{t\in\mathbb{R}_+}$, given the concavity of preferences and technology, and the transversality condition, differently from the previous AK model, converges to a steady state. This implies that the asymptotic rate of growth of marginal utility

$$g_{u_c}(t) = \frac{\frac{d}{dt}\left(u_c(c(t), \ell(t))\right)}{u_c(c(t), \ell(t))}$$

tends to zero asymptotically.

3.2 A competitive economy with labor taxes

Now, we introduce government and government policy. We consider an economy similar to the one presented in section 2 but in which there are labor, τ^l, and consumption, τ^c, taxes in addition to capital taxes, τ^k. Again assume that the government expenditures are g and can be financed by taxes and/or government bonds, b.

The consumer problem is now

$$\max_{c,\ell} \int_0^\infty u(c(t), \ell(t))e^{-\rho t}dt$$

subject to

$$\dot{a} = \bar{w}(t)\ell(t) + \bar{r}(t)a(t) - (1 + \tau^c(t))c \tag{23a}$$

$$a(0) = a_0 \tag{23b}$$

$$\lim_{t\to\infty} e^{-\int_0^t \bar{r}(s)ds}a(t) \geq 0 \tag{23c}$$

where $\bar{w} = (1 - \tau^l(t))w(t)$ and $\bar{r} = (1 - \tau^k(t))r(t)$ and $a(t) = k(t) + b(t)$ is the state variable.

The arbitrage conditions, analogous to equations (22) and (21), are now distorted by the presence of taxes.

The intertemporal arbitrage condition is

$$\frac{d}{dt}\left(u_c(c(t), \ell(t))\right) = u_c(c(t), \ell(t))(\rho - \bar{r}(t)), \tag{24}$$

where $r(t) = (F_k(k(t), \ell(t)) - \delta)$. The intratemporal arbitrage condition, between consumption and labor, is

$$\frac{\bar{w}(t)}{1 + \tau^c(t)} = w(t)\left(\frac{1 - \tau^l(t)}{1 + \tau^c(t)}\right) = -\frac{u_\ell(c(t), \ell(t))}{u_c(c(t), \ell(t))} \tag{25}$$

where $w(t) = F_\ell(k(t), \ell(t))$.

From equation (25) we conclude that one of the taxes, on labor or on consumption, is redundant. Therefore, we set $\tau^c(t) = 0$. In this case the intratemporal arbitrage condition becomes

$$\bar{w}(t) = w(t)\left(1 - \tau^l(t)\right) = -\frac{u_\ell(c(t), \ell(t))}{u_c(c(t), \ell(t))}. \tag{26}$$

If the utility function is separable in consumption and labor $u(c, \ell) = u(c) - v(\ell)$ then the intratemporal arbitrage condition simplifies further to

$$\bar{w}(t) = w(t)\left(1 - \tau^l(t)\right) = -\frac{v^{'}(\ell(t))}{u_c(c(t))}.$$

Because $w > \bar{w}$ and $v^{''}(\ell) < 0$, then the labor tax induces a reduction in labor supply relative to the undistorted case.

The competitive equilibrium definition is supplemented by the government budget constraint, the balance equation and the good's market equilibrium condition (where the time dependence is ignored)

$$\dot{b} = g - \tau^l w\ell - \tau^k(r-\delta) + rb \tag{27a}$$

$$F(k,l) = c + g + \dot{k} + \delta k \tag{27b}$$

$$a = k + b. \tag{27c}$$

3.3 Optimal taxation policy

As in the exogenous labor case, the optimal tax problem, in its primal version, is defined by the maximization of the social welfare function dependent upon a resource constraint and an implementability constraint.

The **resource constraint** is again equivalent to the market equilibrium equation (27b) as a result of the Walras law.

In order to find the **implementability constraint** we need to remember that it condenses the optimality conditions for the consumer in a competitive equilibrium, given by equations (23a) to (23c).

If we integrate the consumer instantaneous budget constraint, equation (23a), and introduce the initial, equation (23b), and the non-Ponzi game condition, equation (23c), (and observe that it would not be optimal to satisfy it with an inequality) we obtain

$$\int_0^\infty e^{-\int_0^t \bar{r}(s)ds}\Big(c(t) - (1-\tau^l(t))w(t)\ell(t)\Big)\,dt = a_0. \tag{28}$$

Integrating the optimality condition (24), we obtain

$$e^{-\rho t} u_c\left(c(t),\ell(t)\right) = u_c\left(c(0),\ell(0)\right) e^{-\int_0^t \bar{r}(s)ds}$$

Multiplying both sides of equation (28) by $u_c\left(c(0),\ell(0)\right)$ and using this equation yields

$$\int_0^\infty e^{-\rho t} u_c\left(c(t),\ell(t)\right)\Big(c(t) - (1-\tau^l(t))w(t)\ell(t)\Big)\,dt = u_c\left(c(0),\ell(0)\right) a_0.$$

Using the the optimality condition (26) allows us to obtain the implementability condition for this case

$$\int_0^\infty e^{-\rho t}\left[u_c\left(c(t),\ell(t)\right)c(t) + u_\ell\left(c(t),\ell(t)\right)\ell(t)\right]dt = u_c\left(c(0),\ell(0)\right) a_0.$$

This condition should be compared to the analogous equation (9a) in the economy without endogenous labor. In the case of this model, we cannot come up with an explicit integration of the constraint because the utility and the production functions are given in implicit form. However, the meaning is similar.

The **primal problem** is

$$\max_{c,\ell} \int_0^\infty u(c(t), \ell(t)) e^{-\rho t} dt$$

subject to the implementability and the resource constraints

$$\int_0^\infty e^{-\rho t} \left[u_c\left(c(t), \ell(t)\right) c(t) + u_\ell\left(c(t), \ell(t)\right) \ell(t) \right] dt = u_c\left(c(0), \ell(0)\right)\left(k_0 + b_0\right) \tag{29a}$$

$$\dot{k} = F(k, \ell) - c - g - \delta k, \ t \in [0, \infty) \tag{29b}$$

with $k(0) = k_0$, $b(0) = b_0$ and $\lim_{t\to\infty} k(t) \geq 0$.

Obtaining the first order conditions, and characterizing the solution to the problem, can be made easier if we define the **distorted utility function** by

$$W(c, \ell) \equiv u(c, \ell) + \lambda \left(u_c(c, \ell)c + u_\ell(c, \ell)\ell\right)$$

where λ is a Lagrange multiplier

The primal problem becomes similar to a distorted Ramsey problem

$$\max_{c,\ell} \int_0^\infty W(c(t), \ell(t)) e^{-\rho t} dt - \lambda u_c\left(c(0), \ell(0)\right)\left(k_0 + b_0\right)$$

subject to the resource constraint (29b).

If we denote the co-state variable by q^{pr}, the first order conditions are

$$\begin{aligned}
W_c(c(t), \ell(t)) &= q^{pr}(t), \ t \in [0, \infty) \\
W_\ell(c(t), \ell(t)) &= -q^{pr}(t) F_\ell(k(t), \ell(t)), \ t \in [0, \infty) \\
\dot{q}^{pr} &= q^{pr}(t)\left(\rho + \delta - F_k(k(t), \ell(t))\right), t \in [0, \infty) \\
\lim_{t\to\infty} q^{pr}(t) k(t) e^{-\rho t} &= 0,
\end{aligned}$$

for an admissible solution satisfying the resource and the implementability constraints. Therefore, we obtain the **distorted intratemporal condition**

$$F_\ell\left(k(t), \ell(t)\right) = -\frac{W_\ell(c(t), \ell(t))}{W_c(c(t), \ell(t))} \tag{30}$$

and **the distorted intertemporal arbitrage** condition

$$\frac{d}{dt}\left(W_c(c(t), \ell(t))\right) = W_c(c(t), \ell(t))\left(\rho + \delta - F_k\left(k(t), \ell(t)\right)\right). \tag{31}$$

If we compare with the arbitrage conditions for a competitive economy, in equations (26) and (24), we can obtain the short run values for taxes that **implement the optimum**. Comparing equation (30) with (26) we obtain the **optimal wedge** for labor

$$1 - \tau^{l*}(t) = \frac{u_\ell(c(t), \ell(t))}{u_c(c(t), \ell(t))} \frac{W_c(c(t), \ell(t))}{W_\ell(c(t), \ell(t))}. \tag{32}$$

Exercise For a benchmark utility function $u(c, l) = \dfrac{c^{1-\sigma}}{\sigma} - \dfrac{\ell^{1+\xi}}{1+\xi}$ where $\sigma > 0$ and $\xi > 0$, the ratio in equation (32) is constant. This means that the optimal labor tax is roughly constant throughout time.

Comparing equation (31) with (24) we obtain the **optimal wedge** for capital

$$\bar{r}(t) - r^*(t) = \frac{\frac{d}{dt} u_c(t)}{u_c(t)} - \frac{\frac{d}{dt} W_c(t)}{W_c(t)} = g_{W_c}(t) - g_{u_c}(t)$$

where $\bar{r}(t) - r^*(t) = (1 - \tau^k(t))(F_k(t) - \delta) - (F_k(t) - \delta)$, and g_x represents the rate of growth of variable x. This implies that that the optimal capital tax should verify

$$-\tau^{k*}(t)(F_k(t) - \delta) = g_{W_c}(t) - g_{u_c}(t). \tag{33}$$

Because of the concavity of the resource constraint regarding the state equation, there is an optimal steady state, satisfying $F_k(k, \ell) - \delta = \rho > 0$, and the solution to the problem, if it exists, will converge to it. Therefore, $\lim_{t\to\infty} \gamma_{W_c}(t) = \lim_{t\to\infty} \gamma_{u_c}(t) = 0$ implying $\lim_{t\to\infty} \tau^{k*}(t) = 0$

This implies that, again the optimal capital tax rate will converge to zero in the long run.

The optimal capital tax at time zero is, as in the AK economy, obtained from the implementability constraint (29a), and should be different from zero, with the exception of very particular cases.

4 Final remarks

There is a vast literature on optimal taxes. We have only presented the benchmark cases. If in the welfare function there are other objectives, like redistribution, the zero optimal capital tax in the long run result will not hold. See a survey of the recent discussion in Straub and Werning (2018).

References

Chamley, C. P. (1986). Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives. *Econometrica*, 54(3):607–22.

Chari, V. and Kehoe, P. J. (1999). Optimal fiscal and monetary policy. In Taylor, J. B. and Woodford, M., editors, *Handbook of Macroeconomics*, volume 1, chapter 26, pages 1671 – 1745. Elsevier.

Judd, K. L. (1985). Redistributive taxation in a simple perfect foresight model. *Journal of Public Economics*, 28.

Lansing, K. J. (1999). Optimal redistributive capital taxation in a neoclassical growth model. *Journal of Public Economics*, 73.

Ljungqvist, L. and Sargent, T. J. (2012). *Recursive Macroeconomic Theory*. MIT Press, Cambridge and London, 3rd edition.

Lucas, R. and Stokey, N. (1983). Optimal fiscal and monetary policy in an economy without capital. *Journal of Monetary Economics*, 12:55–93.

Straub, L. and Werning, I. (2018). Positive long-run capital taxation: Chamley-Judd revisited. `https://scholar.harvard.edu/files/straub/files/chamley-judd-revisited_june_2018.pdf`.

Turnovsky, S. (1995). *Methods of Macroeconomic Dynamics*. MIT Press.