Optimal taxation: the Mirrlees model

Paulo B. Brito
Advanced Macroeconomics
PhD in Economics: 2018-2019
ISEG
Universidade de Lisboa
pbrito@iseg.ulisboa.pt

November 13, 2018
revised, December 19, 2018

Contents

1 Introduction **3**

2 The economy **4**

3 The optimal taxation with complete information **8**
3.1 The primal approach 8
3.1.1 Example 1 10
3.1.2 Example 2 12
3.1.3 Example 3 15
3.2 The dual approach 15

4 The Mirrlees model: optimal distributive tax policy with information frictions **17**
4.1 Incentive compatibility 17
4.2 The primal optimal tax problem 18
4.3 Diamond (1998) simplified version 19

4.4 Implementing the optimal tax 21

A General problem **24**

A.1 Simple calculus of variations problem free initial and terminal indexes and states 25

A.2 Isoperimetric problem 27

A.3 Optimal control problem 28

1 Introduction

In this note we address the redistribution function of tax policy in a static, i.e., time-independent, framework. Redistribution can only take place when the economy is populated by heterogeneous agents. The most obvious dimension of heterogeneity is related to differences in income. However, differences in income can be rooted to different types of primitives: differences in preferences (on time-preference, risk-aversion or other dimensions of attitude towards risk), differences in skills, differences in information, differences in luck, among many others.

In this note we assume that the heterogeneity in income has its source on heterogeneity of skills, which is associated with heterogeneity in productivity and in wages. Part of the analysis is the similar if we assume heterogeneity in luck, also called idiosyncratic heterogeneity.

Tax/transfer policy is a standard instrument to redistribute income, and therefore consumption attainment, within a population with heterogeneity of skills. When the population is heterogeneous, tax policies, even when they are used as instruments of macroeconomic stabilization, always have effects on distributions of income, intended or not.

Optimal tax/transfer policy deals with the explicit design of a redistribution mechanism such that social welfare is the best which is attainable. In principle, redistribution can be achieved when the marginal tax on higher incomes is proportionally higher, i.e., the income tax schedule should be progressive.

However, redistribution creates an incentive problem: if higher incomes are associated to higher ability or higher willingness to develop better or higher efforts, and not with higher rents, then heavier taxes will generate a negative incentive to higher skilled or industrious people to work, which will negatively effect the aggregate output and therefore, the total amount of resources to redistribute.

Information play one important role here. If government can observe income and clearly distinguish its skill or effort component, it can design the tax schedule such that a social optimum may be attained, by balancing the redistributing and incentive features of the tax/transfer structure.

However, if the information on the skill abilities or work effort is private, then a particular tax schedule may have unintended consequences on incentives throughout the economy. One un-properly designed tax policy may induce the most skilled part of the population to reduce effort this reducing the total tax collected and the aggregate income of the economy. This

problem is more generally faced by any contract between a principal and an agent in which the agent has private information with a bearing in the contract. There is adverse selection if the private information concerns the type of the agent and moral hazard if the type of information concerns the actions of the agent [1].

The seminal paper dealing with this problem is Mirrlees (1971), and Mirrleesian taxation has been identified with the distributional and incentive role of taxation. Mirrlees (1971) introduces the incentive considerations by adding an incentive compatibility constraint to the optimal tax problem. Therefore, the optimal taxation problem not only involves dealing with redistribution of income and changes in incentives but also with one informational friction.

There is a huge literature on Mirrlesian taxation. For a simpler and clearer version of the model see Diamond (1998), and for a thorough discussion of their properties see Saez (2001). A textbook presentation is (Tuomala, 2016, ch. 4). There are some extensions of the model to dynamic contexts: see Golosov et al. (2011), and Werning (2007) and Farhi and Werning (2013) for dynamic Mirrleesian economies, andSargent et al. (2017) for a recent contribution.

In this note, in order to separate the distributional from the information problem we present the an optimal taxation model with perfect information in section 3. In this section we specify the two methods for presenting and solving optimal taxation problems: the primal and dual approaches. In section 4 we present the model with imperfect information, in which the incentive effects of taxation are incorporated via an incentive compatibility constraint. In the appendix A auxiliary results on dynamic optimization, needed to solve optimal taxation problems, are presented.

2 The economy

In this section we present a general equilibrium allocation, of consumption and hours worked, depending upon on an income tax structure such that the government constraint holds. We also introduce the approaches to allows us to find an optimal tax structure.

Assume an economy populated by agents with heterogeneous skills or earnings ability. The skill levels, θ, are ordered in a continuum, from the minimum $\underline{\theta} \geq 0$ to the maximum $\bar{\theta}$ that can be finite or infinite: formally, $\theta \in \Theta \equiv (\underline{\theta}, \bar{\theta}) \subseteq (0, \infty)$. The population is heteroge-

[1] This is now standard in the mechanism design literature (see Bolton and Dewatripont (2005).

nous and is distributed according to the skill levels. The proportion of population with skill θ is given by the density function $f(\theta) = F^{'}(\theta)$, where the cumulative skill distribution, $F(\theta)$ satisfies the following properties: $\int_\Theta dF(\theta) = \int_\Theta f(\theta)d\theta = 1$, and $f(\theta) > 0$ for all $\theta \in \Theta$.

We assume that gross income for an agent with skill level θ is a linear in the hours worked, $\ell(\theta)$, is

$$y(\theta) = \theta\, \ell(\theta)$$

where $w(\theta) = \theta$ is the wage rate for agents with skill level θ. In this simple model θ is equal to the productivity of agents with skill level θ. We also assume that the labor effort is measured in hours worked with introduces the following constraint: $0 \leq \ell(\theta) \leq 1$.

The tax faced by agents of type θ, $T(\theta)$, is set by the tax authority, and has a shape which is a-priori unknown to the agent. The functional form of the tax function $T(\theta)$ is a-priori unknown, and is not necessarily a linear function of agents' income. It can be non-linear, or can have a lump-sum structure to, and can have any sign. However, one would expect that it should be negative for low levels of income and positive for high levels of income.

The after-tax income for an agent with skill level θ is $y(\theta) - T(\theta)$ and is a function of both the skill level and the hours worked. If $T(\theta) > 0$ agents of type θ are taxed and if $T(\theta) < 0$ they receive a transfer.

As we are dealing with a static economy (or with a steady state of a dynamic economy), there are no savings, implying that consumption is equal to post-tax income

$$c(\theta) = y(\theta) - T(\theta) = \theta\, \ell(\theta) - T(\theta). \tag{1}$$

Considering both the constraint on hours worked, $0 < \ell(\theta) < 1$ and introducing the constraint that consumption should be positive, for every skill level, then the hours worked are assumed to belong to the set

$$\mathcal{L}(\theta) = \left\{ \ell(\theta) : \max\left\{0, \frac{T(\theta)}{\theta}\right\} < \ell(\theta) < 1 \right\},$$

for every $\theta \in \Theta$.

We assume agents derive utility from consumption and leisure, and that their preferences are homogeneous throughout the skill distribution. Thus their utility function is $U(c(\theta), \ell(\theta))$, for every $\theta \in \Theta$. Furthermore, we let the standard properties hold: utility is increasing in consumption and is decreasing on hours worked, $U_\ell(c, \ell) < 0 < U_c(c, \ell)$, and utility is a strictly concave function of (c, ℓ).

The utility of agent θ, $u(\theta)$, can be written as a function of the hours worked, $u(\theta) = U(\theta\,\ell(\theta) - T(\theta), \ell(\theta))$.

The optimal working time, $\ell^*(\theta)$, is skill-specific and is the solution to the problem

$$\max_{\ell(\theta)\in\mathcal{L}} u\left(\theta\,\ell(\theta) - T(\theta), \ell(\theta)\right).$$

The necessary first-order conditions for an interior maximum, $0 < \ell^*(\theta) < 1$ is is

$$u_c^*(\theta)\theta + u_\ell^*(\theta) = 0, \text{ for every } \theta \in \Theta \tag{2}$$

where $u_c^*(\theta) = U_c\left(\theta\,\ell^*(\theta) - T(\theta), \ell^*(\theta)\right)$ and $u_\ell^*(\theta) = U_\ell\left(\theta\,\ell^*(\theta) - T(\theta), \ell^*(\theta)\right)$.

If there are no singularities, $\ell^*(\theta)$ will be unique, which would allows us to find the aggregate labor input, income and consumption (all this data is in per-capita terms) $L = \int_\Theta \ell^*(\theta)f(\theta)d\theta$, $Y = \int_\Theta \theta\,\ell^*(\theta)f(\theta)d\theta$ and $C = \int_\Theta c^*(\theta)f(\theta)d\theta$.

The government sets taxes/transfers within a balanced budget policy. In per-capita terms the government budget constraint is

$$\int_\Theta T[y(\theta)]f(\theta)d\theta \geq G \tag{3}$$

where G are exogenous per capita government net expenditures.

Definition 1. *A **general equilibrium in this economy** is an allocation $(c^{eq}(\theta), \ell^{eq}(\theta))_{\theta\in\Theta}$ such that, for a given tax/transfer policy $(T(\theta))_{\theta\in\Theta}$: first, the hours worked for agents of skill level θ, equation* (2) *holds; second, consumption of agents os skill level θ satisfies equation* (1)*; third, the government budget constraint* (3) *is satisfied; and, fourth, the goods' market clears $Y = C + G$.*

Up to this point, we have assume the tax/transfer policy is arbitrary. The **optimal taxation problem** is to find the tax schedule such that the tax authority optimizes a welfare function.

We will assume next that the social welfare functional is

$$\mathcal{W} = \int_\Theta W[u(\theta)]f(\theta)d\theta. \tag{4}$$

The social welfare function is the **average of the social value** of the private utility distribution for the population with all the skill levels. Here considerations over social justice enter into the model [2].

If $W(u) = u$, the social welfare function is called utilitarian because $\mathcal{W}$ is just a simple average of the utility levels for people with different skills, weighted by their proportion in the total population. If, in general, $W(u)$ is increasing in u then the social welfare function will not involve a change in the order relationship which exists in the distribution of private utilities. However, its concavity properties may entail a redistribution such that the the social differences in utility will be smoother than the private ones.

Next we determine optimal allocations for two different information environments: first, we assume that the tax authority has perfect information both on consumption, income, and hours worked, $c(\theta)$, $y(\theta)$, and $\ell(\theta)$; second, we assume the tax authority faces an **information friction** because it observes consumption and income, $c(\theta)$, $y(\theta)$, but does not observe works worked, $\ell(\theta)$. In particular, in the second case, it does not know whether a given income comes from the agents type (or luck), θ, or on the agents' effort, $\ell(\theta)$

Definition 2. *An **optimal allocation** $(c^*(\theta), \ell^*(\theta))_{\theta\in\Theta}$ is an allocation that maximizes the welfare functional* (4)*.*

Definition 3. *A tax/transfer policy $T^*(\theta)$ **implements** the optimal allocation if is a tax/transfer function that makes an equilibrium allocation optimal.*

In the literature, particularly in the literature relative to the Chamley-Judd capital income taxation, a distinction can be made between two methods for finding an optimal taxation: the primal and the dual approaches. The **primal approach** consists in solving a centralized optimization problem for finding the optimal allocation with the tax policy implicit. After this step the tax policy that implements the optimal allocation can be found. The **dual approach** solves the problem in two steps: in the first step the problem for the agents is solved, given the tax policy, and in second step the tax authority determines the optimal taxation taking the equilibrium allocations as a constraint. Although the second approach is intuitive, because taxation is used directly as an instrument, in the second step, the first is simpler to apply.

Under some circumstances, that we will discuss later, the two methods yield the same optimal tax schedule. While the first method uses a local approximation to the optimal

[2]For the economics of the social welfare function see any textbook in public economics, v.g. Atkinson and Stiglitz (1980).

taxation problem, the second method has a global nature. Therefore, the difference between two has analogies with the general difference between local and global approximations to optima.

3 The optimal taxation with complete information

In this section, we derive the properties of the optimal tax structure $(T^*(\theta))_{\theta\in\Theta}$ when the government has perfect information. In subsection 3.1 we solve the problem using the primal approach, and in subsection 3.2 we use the dual approach. Some examples allow for more explicit characterizations.

3.1 The primal approach

The primal approach consists in finding the optimal allocation, of consumption and hours worked, subject to government budget constraint, by taking the tax structure implicitly, that it by observing that $T(\theta) = y(\theta) - c(\theta)$. In order to compare to the Mirrlees model (see section 4) we will consider an equivalent problem in which the optimization is done by using the distribution of utility, $u(\theta)$, instead of the distribution of consumption, $c(\theta)$.

If the source of heterogeneity was luck, or idiosyncratic uncertainty, this problem would be similar to an optimal insurance problem.

Central planner problem We assume that the utility function, $U(c,\ell)$ is monotonic as a function of consumption, c, that is $U_c(c,\ell) > 0$ for any $(c,\ell) \in \mathbb{R}_{++} \times (0,1)$. Therefore, if the utility of an agent of skill level θ is $u(\theta) = U\left(c(\theta), \ell(\theta)\right)$, then, by the implicit function theorem we can write consumption as a function of the level of utility and hours worked

$$c(\theta) = C\left(u(\theta), \ell(\theta)\right),$$

that has the following first derivatives,

$$\frac{\partial C(u,\ell)}{\partial u} = \frac{\partial U(c,\ell)}{\partial c} > 0,\ \frac{\partial C(u,\ell)}{\partial \ell} = -\frac{\dfrac{\partial U(c,\ell)}{\partial \ell}}{\dfrac{\partial U(c,\ell)}{\partial c}} > 0.$$

Therefore, consumption increases with utility and there is complementarity between consumption and hours worked.

The set of admissible values for consumption and hours worked is

$$\mathscr{U} = \{(u(\theta), \ell(\theta)) : \ C\left(u(\theta), \ell(\theta)\right) > 0, \ 0 < \ell(\theta) < 1, \ \forall \theta \in \Theta\}.$$

Definition 4. *An optimal allocation* $(c^*(\theta), \ell(\theta))_{\theta\in\Theta}$ *is an allocation such that* $c^*(\theta) = C\left(u^*(\theta), \ell^*(\theta)\right)$ *and* $(u^*(\theta), \ell(\theta))_{\theta\in\Theta}$ *solve the tax-planner problem:*

$$\max_{(\ell(.),u(.))\in\mathscr{U}} \int_\Theta W[u(\theta)] f(\theta) d\theta \tag{5}$$

subject to the (per capita) government budget constraint

$$\int_\Theta \left(\theta\, \ell(\theta) - C\left(u(\theta), \ell(\theta)\right)\right) f(\theta) d\theta \geq G \tag{6}$$

The planner problem is to find an optimal distribution of utilities and hours worked, such that the government budget constraint is satisfied, in order to maximize a social welfare function which is the average of the social value of private utilities for agents of all skills.

Although it is infinite dimensional, this is a static redistribution problem. Next, we assume that the conditions for an interior solution are satisfied. If there is an optimal allocation $(u^*(\theta), \ell^*(\theta))_{\theta\in\Theta}$ will satisfy, jointly with the Lagrange multiplier λ, the necessary first order conditions for an interior maximum

$$C_\ell\left(u^*(\theta), \ell^*(\theta)\right) = \theta, \text{ for every } \theta \in \Theta \tag{7a}$$

$$\lambda\, C_u\left(u^*(\theta), \ell^*(\theta)\right) = W^{'}[u^*(\theta)], \text{ for every } \theta \in \Theta \tag{7b}$$

$$\int_\Theta \left(\theta\, \ell^*(\theta) - C\left(u^*(\theta), \ell^*(\theta)\right)\right) f(\theta) d\theta = G. \tag{7c}$$

Equation (7a) expresses an efficiency condition: the marginal increase in consumption of agent of skill θ should be equal to its productivity (which in this case is equal to its wage). Equation (7b) says that the marginal social value of the utility of agents with skill θ should be equal to their cost, measured by the value of the marginal effect on aggregate consumption which is generated by an increase in their utility. Also, from equation (7b) we can see that

$$\lambda = \frac{W^{'}[u^*(\theta)]}{C_u\left(u^*(\theta), \ell^*(\theta)\right)} \text{ for any } \theta \in \Theta$$

which means that the optimal policy would equalize the social and private value of utility across the continuum of skills.

Now, we need to determine which tax policy would generate an optimal allocation, that is would satisfy conditions (7a)-(7c).

Implementing the optimal plan If we find, explictly or implicitly, an optimal allocation, $u^*(.)$ and $\ell^*(.)$, we can determine the associated income $y^*(\theta)$ and consumption $c^*(\theta) = C\left(u^*(\theta), \ell^*(\theta)\right)$ and by substituting in

$$T(\theta) = y^*(\theta) - c^*(\theta) = \theta ell^*(\theta) - C\left(u^*(\theta), \ell^*(\theta)\right)$$

we find the tax policy that **implements the optimal allocation**. However, this formula will gives the tax schedule as a function of the skill distribution.

In order to compare with the results for the model with imperfect information (and with actual tax codes) we need to determine the dependence of the tax function on income, and, in particular the marginal tax function. As

$$\begin{aligned} T(\theta) &= \theta\ell(\theta) - c(\theta) = \\ &= \theta\ell(\theta) - C\left(u(\theta), \ell(\theta)\right) = \\ &= y(\theta) - C\left(u(\theta), \frac{y(\theta)}{\theta}\right) \end{aligned}$$

then the marginal tax rate, as a function of income, is

$$T^{'}(y(\theta)) = 1 - \frac{C_\ell\left(u(\theta), \ell(\theta)\right)}{\theta},$$

where $\ell(\theta) = y(\theta)/\theta$.

We say we have a linear tax stucture if $T^{'}(y)$ is constant.

Optimal allocations, and the fiscal policy that implements them, will depend on the agents' utility function, $u(.)$ via function $C(.)$, on the social utility function $W(.)$, on the distribution of skills $f(.)$ and on the level of government income $R = G$. Next we derive them for particular cases considered in the literature.

3.1.1 Example 1

Assume that the utility function is $U(c, \ell) = \log(c) + \alpha \log(1 - \ell)$, where α is the weight of leisure relative to consumption, and that the social utility function is $W[u] = u$. This means that we are assuming a utilitarian social welfare function, weighting the distribution of utility just by their weight in total population. The private utility function also means that there are both income and substitution effects from the private choice between consumption and leisure.

In this case, we have

$$c = C(u, \ell) = e^u (1 - \ell)^{-\alpha}$$

and the optimality conditions (7a) and (7b) take the form

$$\theta \ell(\theta) = \theta - \frac{\alpha}{\lambda}, \text{ and } c(\theta) = \frac{1}{\lambda}.$$

Substituting in the resource constraint (7c) we obtain the Lagrange multiplier as proportional to the difference between the per capita wage and the per-capita government expenditure

$$\frac{1}{\lambda} = \frac{\alpha}{1+\alpha} (\mathbb{E}[\theta] - G),$$

because the average wage is

$$\mathbb{E}[\theta] = \int_\Theta \theta f(\theta) d\theta.$$

Then a candidate allocation satisfies $c(\theta) = \frac{1}{1+\alpha} (\mathbb{E}[\theta] - G)$ and $y(\theta) = \theta \ell(\theta) = \theta - \frac{\alpha}{1+\alpha} (\mathbb{E}[\theta] - G)$. This is an only if the admissibility conditions hold: $c(\theta) > 0$ and $y(\theta) \in (0, \theta)$, that is if and only if $\mathbb{E}[\theta] - G > 0$ and $\theta > \underline{\theta} = \frac{\alpha}{1+\alpha} (\mathbb{E}[\theta] - G)$.

Therefore, if $\left(\frac{1+\alpha}{\alpha}\right) \theta > \mathbb{E}[\theta] - G > 0$ then an optimal allocation is characterized by

$$c^*(\theta) = \frac{1}{1+\alpha} (\mathbb{E}[\theta] - G)$$

and

$$y^*(\theta) = \theta \ell^*(\theta) = \theta - \frac{\alpha}{1+\alpha} (\mathbb{E}[\theta] - G).$$

Observe that there is complete insurance or complete redistribution (i.e, consumption is skill-independent) and labor income is increasing in skill.

The tax schedule that implements this optimum is

$$T^*(\theta) = \theta - (\mathbb{E}[\theta] - G).$$

If we define a critical skill threshold by $\theta_c = \mathbb{E}[\theta] - G$, the optimal tax schedule can be written as a piecewise function of the skill level,

$$T(\theta) = \theta - \theta_c \begin{cases} < 0 & \text{if } \underline{\theta} < \theta < \theta_c \\ = 0 & \text{if } \theta = \theta_c \\ > 0 & \text{if } \theta > \theta_c. \end{cases}$$

We see that there is a lump-sum subsidy, equal to $\mathbb{E}[\theta] - G$ and a linear tax schedule such that agents with skill bellow θ_c receive a net subsidy and agents with skill above θ_c pay net taxes.

However, the marginal tax rate, relative to income, $y(\theta)$ is

$$\begin{aligned} T'(y) &= 1 - \alpha \frac{c^*(\theta)}{1 - \ell^*(\theta)} = \\ &= \frac{\frac{\alpha c^*(\theta)}{\theta} - \alpha c^*(\theta)}{\frac{\alpha c^*(\theta)}{\theta}} = \\ &= 1 - \theta. \end{aligned}$$

Then $T'(y) \lesseqgtr 0$ if and only if $\theta \lesseqgtr 1$. This means that the income tax schedule has the typical Laffer form: an inverted U-shape with a maximum at $\theta = 1$, or $y^* = \dfrac{1 + \alpha(1 - \theta_c)}{1 + \alpha}$.

3.1.2 Example 2

This is the case considered in Diamond (1998). Assume that the utility function is $u(c, \ell) = c + (1 - \ell)^\xi$ and that the social utility function is concave $W[u] = -\dfrac{e^{-\beta\, u}}{\beta}$, where both ξ and β are positive. In this case, the social welfare function is not utilitarian, although it still displays an increasing social welfare function, but at decreasing rate: if $\lim_{u\to\infty} W'(u) = 0$. The private utility function is linear in consumption, which means that there are only substituting effects associated with labor supply.

In this case we have

$$c = C(u, \ell) = u - (1 - \ell)^\xi.$$

which implies $C_u = 1$ and $C_\ell = \xi(1 - \ell)^{\xi - 1}$.

The optimality conditions (7a) and (7b) take the form $\xi(1 - \ell(\theta))^{\xi - 1} = \theta$ and $\lambda = e^{-\beta u(\theta)}$. Therefore, the leisure time is

$$1 - \ell(\theta) = \left(\frac{\theta}{\xi}\right)^{\frac{1}{\xi - 1}}.$$

and the utility is skill-independent $u(\theta) = u$. If we substiitute in equation (7c), we can determine the utility level as

$$u^* = Y + V - G$$

where Y is the average per-capita income

$$\begin{aligned} Y &= \mathbb{E}[y] = \\ &= \int_\Theta \theta y(\theta) f(\theta) d\theta = \\ &= \int_\Theta \left(\theta - \xi \left(\frac{\theta}{\xi} \right)^{\frac{\xi}{\xi-1}} \right) f(\theta) d\theta \end{aligned}$$

and V is the average utility of leisure

$$\begin{aligned} V &= \mathbb{E}[(1-\ell)^\xi] = \\ &= \int_\Theta (1-\ell(\theta))^\xi f(\theta) d\theta = \\ &= \int_\Theta \left(\frac{\theta}{\xi} \right)^{\frac{\xi}{\xi-1}} f(\theta) d\theta. \end{aligned}$$

As optimal consumption for agents of skill level θ is [3]

$$c^* = u^* - v(\ell^*)$$

where the utility of leisure at the optimum

$$v(\ell^*) = \left(\frac{\theta}{\xi} \right)^{\frac{\xi}{\xi-1}}$$

is increasing (decreasing) with skill if $\xi > 1$ ($\xi < 1$), that is, if the elasticity of labor supply is positive or negative. Defining the elasticity of labor supply

$$\epsilon(\ell(\theta)) \equiv -\frac{v''(\ell(\theta))\ell(\theta)}{v'(\ell(\theta)} \tag{8}$$

that result is obtained if we observe that

$$\epsilon(\ell(\theta)) = \frac{(\xi-1)\ell^*(\theta)}{1-\ell^*(\theta)}.$$

Therefore, consumption is increasing (decreasing) with the skill level if agents have inelastic (elastic) labor supply.

[3] We leave the determination of the admissibility conditions for an interior optimum $c^\theta > 0$ and $0 < \ell^*(\theta) < 1$.

The tax function that implements the optimum is

$$T^*(\theta) = \psi(\theta) - u^* = \psi(\theta) - \Psi + G.$$

where

$$\psi(\theta) \equiv y^*(\theta) + v(\ell^*) = \theta + (1-\xi)\left(\frac{\theta}{\xi}\right)^{\frac{\xi}{\xi-1}}$$

and

$$\Psi \equiv \mathbb{E}[\psi] = \int_\Theta \psi(\theta) f(\theta) d\theta.$$

The tax structure, as regards the skill level, depends on the elasticity of labor supply. As

$$\psi'(\theta) = 1 - \left(\frac{\theta}{\xi}\right)^{\frac{1}{1-\xi}}, \text{ and } \psi''(\theta) = \frac{1}{\xi - 1}\left(\frac{\theta}{\xi}\right)^{\frac{\xi}{1-\xi}},$$

then: if $\xi < 1$ then $\psi(\theta)$ is concave with a maximum at $\theta = \xi$, and if $\xi > 1$ is convex with a minimum at $\theta = \xi$.

This means that we have now two critical levels for skills $\underline{\theta}_c < \xi < \bar{\theta}_c$ but the distributional properties of the tax structure are symmetrical, depending on the level of expenditures to finance G: if labor supply is inelastic ($\xi < 1$) then the two extremes of the skill distribution will pay lower taxes (or may be subsidized) and the middle level would pay higher taxes; but if labor supply is elastic ($\xi > 1$) the opposite shape is optimal with the two extremes of the skill distribution will pay higher taxes and the middle level would pay lower taxes (or may be subsidized).

In this case, the marginal tax distribution, related to income, yields a surprising result

$$T'(y) = 1 - \frac{C_\ell\left(u^*(\theta), \ell^*(\theta)\right)}{\theta} = 0$$

that is the tax schedule is a lump-sum tax as a function of income.

We can readily see that this result holds for any utility function which is additive in consumption and leisure and linear in consumption, of the form $U(c, \ell) = c + v(1-\ell)$, where $v(.)$ is an increasing function of leisure. For this utility function we have $C_u = 1$ and $C_\ell = v'(1-\ell)$ and the optimality condition (7a) is $v'(1-\ell^*(\theta)) = \theta$. Therefore, the optimal marginal tax, as a function of income, tax implements the optimum is always zero,

$$T'(y) = 1 - \frac{C_\ell\left(u^*(\theta), \ell^*(\theta)\right)}{\theta} = 1 - \frac{v'(1-\ell^*(\theta))}{\theta} = 0.$$

3.1.3 Example 3

The case that is most common in the recent macro literature, $U(c,\ell) = (1-\sigma)^{-1}c^{1-\sigma} - (1+\xi)^{-1}\ell^{1+\xi}$ and $W = u$, is left as an exercise.

3.2 The dual approach

The dual approach involves two steps. In the first step, we find equilibrium allocations for arbitrary fiscal policies, and, in the second step, we find optimal (or second-best) allocations by solving a problem for the tax policy maker using the tax function as a control variable.

Again, an **equilibrium allocation** is an allocation of consumption and hours worked $(c(\theta))_{\theta\in\Theta}$ and $(\ell(\theta))_{\theta\in\Theta}$ such that households solve their problem, markets clear and the government budget constraint holds, given the tax policy $(T(\theta))_{\theta\in\Theta}$ and the level of government expenditures G.

The equilibrium allocations, satisfy the first order condition for household (2), together with the budget constraint, equation (3), and the market clearing condition: $Y = C + G$, or

$$\int_{\theta\in\Theta} \theta\,\ell(\theta)\,f(\theta)d\theta = \int_{\theta\in\Theta} c(\theta)\,f(\theta)d\theta + G$$

From the Walras law, and because the budget constraint of the consumer is $c(\theta) = \theta\,\ell(\theta) - T(\theta)$, and because the market equilibrium condition is equivalent to a macroeconomic resource constraint, the general equilibrium is characterized only by equations (2) and (3).

In order to formulate the policy-maker's problem of finding the optimal tax we have two possibilities: first, if we have an explicit functional form for the equilibrium hours worked as a function of taxes, then we can write the social utility function as the tax schedule and solve the problem taking the tax function as a control variable; second, if we do not have an explicit functional form for the equilibrium hours worked we introduce equation (2) as a constraint to the optimization problem and solve it using both hours worked and taxes as control variables.

Next, we follow the second approach.

The optimal tax policy is the following perimetric problem (see Appendix subsection A.2 for the optimality condition):

$$\max_{(\ell(\theta),T(\theta))\in\mathscr{U}^*} \int_\Theta W\left[u\left(\theta\,\ell(\theta) - T(\theta), \ell(\theta)\right)\right]\,f(\theta)\,d\theta \tag{9}$$

for $\mathscr{U}^* = \{(\ell(\theta), T(\theta))_{\theta\in\Theta} : 0 < \ell(\theta) < 1,\ \theta\,\ell(\theta) > T(\theta)\}$ subject to

$$u_c\left(\theta\,\ell(\theta) - T(\theta), \ell(\theta)\right)\,\theta + u_\ell\left(\theta\,\ell(\theta) - T(\theta), \ell(\theta)\right) = 0,\ \theta \in \Theta \tag{10a}$$

$$\int_\Theta T(\theta)\,f(\theta)\,d\theta = G. \tag{10b}$$

We define the Lagrangean associated to every skill-level θ

$$\begin{aligned}\mathcal{L}(\theta) = \{W\left[u\left(\theta\,\ell(\theta) - T(\theta), \ell(\theta)\right)\right] + \lambda T(\theta)\}\ f(\theta) + \\ h(\theta)\left[u_c\left(\theta\,\ell(\theta) - T(\theta), \ell(\theta)\right)\,\theta + u_\ell\left(\theta\,\ell(\theta) - T(\theta), \ell(\theta)\right)\right]\end{aligned}$$

where λ is the Lagrange multiplier associated to constraint (10b) and $h(\theta)$ are the Lagrange multipliers associated to every constraint (10a). The first order conditions for an interior solution are, simplifying the notation,

$$W^{'}[u]\left(u_c(\theta)\theta + u_\ell(\theta)\right) + h(\theta)\left(u_{cc}(\theta)\theta^2 + 2u_{c\ell}(\theta)\theta + u_{\ell\ell}(\theta)\right) = 0 \tag{11a}$$

$$\left(-W^{'}[u]u_c(\theta) + \lambda\right) f(\theta) + h(\theta)\left(u_{cc}(\theta)\theta + u_{\ell c}(\theta)\right) = 0 \tag{11b}$$

together with constraints (10a) and (10b). Constraint (10a) together with the assumption that the agents' utility function is strictly concave (therefore the Hessian of $u(.)$ is positive definite) implies that restriction (11a) is equivalent to $h(\theta) = 0$ for every $\theta \in \Theta$. Therefore condition (11b) becomes $W^{'}[u]u_c(\theta) = \lambda$.

Going back to the primal problem, we have solved $u(\theta) = u(c(\theta), \ell(\theta))$ for c as $c = C(u, \ell)$. As, locally, $du = u_c dc + u_\ell d\ell$ if there are no singularities we find $C_u = 1/u_c$ and $C_\ell = -u_\ell/u_c$.

Therefore, with the restrictions that the utility function $u(.)$ is strictly concave and has no singularities (i.e, u_c and u_ℓ are different from zero in all their domains), the solution to the dual problem is equivalent to the solution of the primal problem (see equations (7a)-(7b)).

The literature uses (explicitly or implicitly) this equivalence result to deal with more complicated problems of optimal tax policy by using the primal approach which leads to more straightforward results. But, again, this equivalence only works if the local and global properties of the problem are similar.

4 The Mirrlees model: optimal distributive tax policy with information frictions

In Mirrlees (1971) the optimal tax policy problem is addressed when the tax authority has **imperfect information**: it **observes** again both the consumption and the income distributions, $c(\theta)$ and $y(\theta)$, but it **does not observe** the individual productivity, θ, and the effort level of agents $\ell(\theta)$. This creates a problem for policy: a more productive agent may have an interest in reducing the income it reports by reducing its effort. If this is the case, the social welfare will be reduced because the total resources of the economy will be reduced, because, again the resource constraint

$$\int_{\Theta} \theta\ \tilde{\ell}(\theta)d\theta = \int_{\Theta} c(\theta)d\theta + G$$

should be satisfied, where $\tilde{\ell}(\theta)$ has a distortion generated by the tax policy relative to the perfect information case. This problem creates an **information friction** in the derivation of the optimal tax policy.

The Mirrlees (1971) paper was one of the first papers in the mechanism design literature that adresses principal-agent problems in contexts of imperfect information.

4.1 Incentive compatibility

The solution put forward by Mirrlees (1971) is to make the policy **incentive compatible**, in the sense that there should exist a truth revealing mechanism: that is agents of type $\theta > \theta^{'}$ should work at least a fraction $\frac{\theta^{'}}{\theta}$ of the time worked by agents of type $\theta^{'}$. This is possible if and only if

$$u\left(c(\theta), \ell(\theta)\right) \geq u\left(c(\theta^{'}), \frac{\theta^{'}}{\theta}\ell(\theta^{'})\right).$$

As there is no savings, consumption is equal to after-tax income, and after-tax income is $y(\theta) = \theta\ell(\theta) - T(\theta)$. As the utility of agent of type θ is $u(\theta) \equiv u\left(c(\theta), \ell(\theta)\right) = u\left(\theta\ell(\theta) - T(\theta), \ell(\theta)\right)$ it maximizes utility if condition (2) holds. Given its type θ, the incentive compatibility condition holds if the marginal increase in its skill level induces a marginal change in its income and therefore to an increase in utility [4]

$$\frac{du}{d\theta} = u_c(\theta)\ell(\theta) = -\frac{\ell(\theta)u_\ell(\theta)}{\theta}$$

[4]If r is the report of an agent of type θ then the income reported by agent of type θ is $r\ell(\theta)$. Therefore the change in utility obtained by a small increase in reporting if $\frac{du}{dr} = \frac{d}{dr}u(c(r)) = \frac{d}{dr}u\left(r\ell(.) - T(.), \ell(.)\right)$.

where we introduced the optimality condition (2): $u_c(\theta)\theta + u_\ell(\theta) = 0$.

4.2 The primal optimal tax problem

Using the same primal approach as for the perfect information case, the policy problem is to find

$$\max_{l(\theta)\in(0,1)} \int_{\underline{\theta}}^{\bar{\theta}} W[u(\theta)]f(\theta)d\theta \tag{12}$$

where the skill domain is $\Theta = [\underline{\theta}, \bar{\theta}]$, subject to the following constraints

$$\int_{\underline{\theta}}^{\bar{\theta}} \left[\theta\ell(\theta) - C(u(\theta), \ell(\theta)]\ f(\theta)d\theta \geq G \right. \tag{13a}$$

$$\frac{du}{d\theta} = -\frac{\ell(\theta)u_\ell(\theta)}{\theta} \tag{13b}$$

$$\underline{\theta}, \bar{\theta} \text{ free} \tag{13c}$$

$$u(\underline{\theta}), u(\bar{\theta}) \text{ free.} \tag{13d}$$

Equation (13a) is the **resource constraint**, equation (13b) is the **incentive compatibility constraint**. The constraints (13c) and (13d) are introduced to account for the fact that the tax authority limits and levels of taxes at both ends of the skill distribution should be optimally derived. This means that there can be upper or lower extremes of the skill distribution that are not taxed.

This is a control problem with state variable $u(\theta)$ and control variable $\ell(\theta)$, whose optimality conditions are derived in the Appendix A. We have to introduce two types of adjoint variables: λ is skill-independent and is associated to constraint (13a), and $h(\theta)$ is skill-dependent and is associated to state variable $u(\theta)$. The Hamiltonian is

$$\begin{aligned} H(\theta) &= H(\theta, \lambda, y(\theta), u(\theta), h(\theta)) \equiv \\ &\equiv \{W[u(\theta)] - \lambda\left(C\left(u(\theta), \ell(\theta)\right) - \theta\,\ell(\theta)\right)\} f(\theta) - h(\theta)\frac{\ell(\theta)}{\theta} u_\ell\left(C\left(u(\theta), \ell(\theta)\right), \ell(\theta)\right) \end{aligned}$$

Next we present the conditions for an interior solution, i.e., for $0 < \ell^*(\theta) < 1$. The static optimality condition $H^*_\ell(\theta) = 0$ (see equation (23a)) yields the optimal distribution of income

$$\lambda\left(C^*_\ell(\theta) - \theta\right)\ f(\theta) = \frac{h(\theta)}{\theta}\left[u^*_\ell(\theta) + \ell(\theta)\left(u^*_{c\ell}(\theta) + u^*_{\ell\ell}(\theta)\right)\right],\ \theta \in [\underline{\theta}^*, \bar{\theta}^*]. \tag{14}$$

Again, we denote $C^*_j(\theta) \equiv C_j\left(u^*(\theta), \ell^*(\theta)\right)$, for $j = u, \ell$, $u^*_\ell(\theta) \equiv u_\ell\left(C\left(u^*(\theta), \ell^*(\theta)\right), \ell^*(\theta)\right)$ and analogously for the higher order derivatives of the utility function $u(.)$.

The Euler equation $h^{'}(\theta)+H_u^*(\theta)=0$ (see equation (23b)) yields the change in the value of the utility along the skill distribution

$$\frac{h(\theta)}{d\theta}=\left(\lambda C_u^*(\theta)-W^{'}[u^*(\theta)]\right)f(\theta)+\left(\frac{\ell^*(\theta)}{\theta}u_{\ell c}^*(\theta)C_u^*(\theta)\right)h(\theta),\ \theta\in[\underline{\theta}^*,\bar{\theta}^*]. \tag{15}$$

The optimal conditions associated to the limit values for households' utility in the two limits of the skill distribution, $u^*(\underline{\theta})$ and $u^*(\bar{\theta})$ (see equation (23c)), satisfy

$$h(\bar{\theta})=h(\underline{\theta})=0 \tag{16}$$

and the optimal cutoff-values for skill distribution which is taxable, $\underline{\theta}^*$ and $\bar{\theta}^*$, (see equation (23d)) are

$$H^*(\theta^*)=h(\theta^*)u^{'}(\theta^*),\ \text{for}\ \theta^*=\underline{\theta}^*,\bar{\theta}^* \tag{17}$$

The admissibility conditions (13a) and (13b) should also hold for $\ell(\theta)=\ell^*(\theta)$ and $u(\theta)=u^*(\theta)$.

We see that the information friction introduces a skill-varying change when we compare to the analogous first-order conditions for the perfect information problem (compare with equations (7a) and (7b)):

$$C_\ell^*(\theta)-\theta=\frac{h(\theta)}{\lambda\theta f(\theta)}\left[u_\ell^*(\theta)+\ell(\theta)\left(u_{c\ell}^*(\theta)+u_{\ell\ell}^*(\theta)\right)\right]$$

$$\lambda C_u^*(\theta)-W^{'}[u^*(\theta)]=\frac{1}{f(\theta)}\left(\frac{h(\theta)}{d\theta}-\left(\frac{\ell^*(\theta)}{\theta}u_{\ell c}^*(\theta)C_u^*(\theta)\right)h(\theta)\right)$$

In addition, optimality conditions (16) and (17) constrain the range of taxable income and the level of taxes at the two extremes of the skill distribution.

4.3 Diamond (1998) simplified version

A little more intuition on the characterization of the optimal redistribution problem is gained by using the utility function assumed by Diamond (1998): $u(c,\ell)=c+v(1-\ell)$ where $v^{'}(.)>0$ and $v^{''}<0$. This utility function simplifies calculations by assuming there are no income effects associated to changes in taxes [5]. With this utility function the elasticity of labor supply, for skill-level θ is

$$\epsilon(\theta)=-\frac{v^{''}(1-\ell(\theta))\,\ell(\theta)}{v^{'}(1-\ell(\theta))}.$$

[5] Saez (2001) proves that introducing income effects do not change qualitatively the results.

With this utility function, the first order condition (14) becomes

$$\lambda\left(v^{'}(1-\ell^{*}(\theta))-\theta\right)f(\theta)=\frac{h(\theta)}{\theta}\left(v^{'}(1-\ell^{*}(\theta))-\ell^{*}(\theta)v^{''}(1-\ell^{*}(\theta))\right),\ \text{for}\ \theta\in[\underline{\theta}^{*},\bar{\theta}^{*}], \tag{18}$$

and condition (15) becomes

$$\frac{h(\theta)}{d\theta}=-\left(W^{'}[u^{*}(\theta)]-\lambda\right)f(\theta),\ \text{for}\ \theta\in[\underline{\theta}^{*},\bar{\theta}^{*}]. \tag{19}$$

This is an ordinary differential equation, which can be solved together with the terminal optimality conditions (16). Then, [6], (16),

$$h(\theta)=\int_{\theta}^{\bar{\theta}}\left(W^{'}[u^{*}(s)]-\lambda\right)f(s)ds=\int_{\theta}^{\bar{\theta}}\left(W^{'}[u^{*}(s)]-\lambda\right)dF(s),$$

is a balance equation between the utility of agents of type θ and the net benefit of reducing utility for agents with skill higher than θ.

Substituting in equation (18) yields

$$\lambda\left(\theta-v^{'}(1-\ell(\theta))\right)f(\theta)=\left(\frac{v^{'}(1-\ell(\theta))-\ell(\theta)v^{''}(1-\ell(\theta))}{\theta}\right)\int_{\theta}^{\bar{\theta}}(\lambda-W^{'}(s))dF(s).$$

Using the definition of the elasticity of labor supply, as in equation (8), and rearranging terms we get the well known expression (see Diamond (1998) and (Tuomala, 2016, ch. 4))

$$\frac{\theta-v^{'}(1-\ell(\theta))}{v^{'}(1-\ell(\theta))}=A(\theta)\,B(\theta)\,C(\theta) \tag{20}$$

where

$$A(\theta)\equiv 1+\frac{1}{\epsilon(\theta)}$$

$$B(\theta)\equiv\frac{\int_{\theta}^{\bar{\theta}}\left(\lambda-W^{'}[u(s)]\right)dF(s)}{\lambda\left(1-F(\theta)\right)}$$

$$C(\theta)\equiv\frac{1-F(\theta)}{\theta f(\theta)}$$

Equation (20) basically says that the ratio of the optimal tax policy should equate the marginal rate of substitution between consumption and labor supply, for an agent of skill θ

[6] From now on we delete the $*$ symbol in functions $\ell^{*}(\theta)$ and $u^{*}(\theta)$ and in numbers $\underline{\theta}^{*}$ and $\bar{\theta}^{*}$.

to the product of three terms: the deadweight burden generated by the income tax to people of skill θ ($A(\theta)$), the relative transfer of income from people with higher skills than θ ($B(\theta)$), and the weight of people with higher skills relative to the average skills of people with skill θ ($C(\theta)$).

4.4 Implementing the optimal tax

In order to find the conditions for an optimal tax policy we need to find which tax implements the optimal redistribution. Because the tax authority has imperfect information, as it only observes income $y(\theta)$, and not θ and $\ell(\theta)$, we need to find the tax policy that implements the optimal allocation as a function of the agents' income.

If we use the Diamond (1998) utility function the tax schedule becomes

$$\begin{aligned} T(\theta) &= \theta\ell(\theta) - c(\theta) = \\ &= \theta\ell(\theta) - u(\theta) + v\left(1-\ell(\theta)\right) \\ &= y(\theta) - u(\theta) + v\left(1 - \frac{y(\theta)}{\theta}\right) \end{aligned}$$

then the marginal tax rate that implements optimality condition (20) is $T^{'}(y(\theta)) = 1 - \frac{v^{'}(1-\ell(\theta))}{\theta}$ yielding

$$\frac{T^{'}(y(\theta))}{1-T^{'}(y(\theta))} = \frac{\theta - v^{'}(1-\ell(\theta))}{v^{'}(1-\ell(\theta))}.$$

Therefore, the optimal tax policy that allows for the optimal redistribution of income within an imperfect information environment is

$$\frac{T^{'}(y(\theta))}{1-T^{'}(y(\theta))} = A(\theta)\,B(\theta)\,C(\theta), \text{ for } \theta \in [\underline{\theta}^{*}, \bar{\theta}^{*}] \tag{21}$$

In the perfect information case, we saw that $T^{'}(y(\theta)) = 0$ because $\theta = v^{'}(1-\ell(\theta)$. In this imperfect information case the result is not so clear cut.

The literature has discussed the shape of the tax function $T(y(\theta))$, the marginal tax rates at the two extremes of the skill and income distribution, and the values of the cutoffs (see Diamond (1998), Saez (2001) and (Tuomala, 2016, ch 4 and 5)). All those features of the optimal tax policy depend on the nature of the utility function, $u(.)$, the welfare function, $W(.)$ and the distribution of skills, $F(.)$. Most of the results tend to generate non-linear

tax schedules with the marginal tax rates at the boundaries of the distribution close to zero. This result is not surprising because it is a consequence of the boundary optimality condition (16).

A detailed analysis of the Diamond model is provided in Dahan and Strawczynski (2000).

A survey on theory and policy implications of Mirrleesian taxation can be found in Diamond and Saez (2011).

References

Atkinson, A. B. and Stiglitz, J. E. (1980). *Lectures on Public Economics*. McGraw-Hill.

Bolton, P. and Dewatripont, M. (2005). *Contract Theory*. The MIT Press.

Dahan, M. and Strawczynski, M. (2000). Optimal income taxation: An example with a u-shaped pattern of optimal marginal tax rates: Comment. *American Economic Review*, 90(3):681–686.

Diamond, P. and Saez, E. (2011). The Case for a Progressive Tax: From Basic Research to Policy Recommendations. *Journal of Economic Perspectives*, 25(4):165–190.

Diamond, P. A. (1998). Optimal income taxation: An example with a u-shaped pattern of optimal marginal tax rates. *American Economic Review*, 88.

Farhi, E. and Werning, I. (2013). Insurance and Taxation over the Life Cycle. *Review of Economic Studies*, 80(2):596–635.

Gelfand, I. M. and Fomin, S. V. (1963). *Calculus of Variations*. Dover.

Golosov, M., Troshkin, M., and Tsyvinski, A. (2011). Optimal taxation: merging micro and macro approaches. *Journal of Money, Credit and Banking*, 43(5):147–174.

Mirrlees, J. A. (1971). An exploration in the theory of optimum income taxation. *Review of Economic Studies*, 38:175–208.

Saez, E. (2001). Using elasticities to derive optimal income tax rates. *Review of Economic Studies*, 68:205–229.

Sargent, T., Golosov, M., Evans, D., and Bhandari, A. (2017). Optimal fiscal-monetary policy with redistribution. 2017 Meeting Papers 1245, Society for Economic Dynamics.

Tuomala, M. (2016). *Optimal Redistributive Taxation*. Oxford University Press, 1 edition.

Werning, I. (2007). Optimal fiscal policy with redistribution. *Quarterly Journal of Economics*, 122(3):925 – 967.

A General problem

Independent variable, or index, $x \in \mathcal{X} \subseteq \mathbb{R}_+$, where $\mathcal{X} \equiv [x_0, x_1]$, state variable $y : \mathcal{X} \to \mathbb{R}$ and control variable $u : \mathcal{X} \to \mathbb{R}$.

The problem

$$\max_{x_0, x_1, y(x_0), y(x_1), (u(x))_{x \in [x_0, x_1]}} \int_{x_0}^{x_1} F\left(x, y(x), y'(x)\right) dx, \text{ subject to (22a), (22b), and (A)} \quad \text{(P1)}$$

$$\int_{x_0}^{x_1} G_0\left(x, y(x), u(x)\right) dx \leq \bar{G} \tag{22a}$$

$$\frac{dy(x)}{dx} = G_1\left(x, y(x), u(x)\right) \; x \in \mathcal{X} \tag{22b}$$

$$x_0, x_1, y(x_0), y(x_1) \text{ free} \tag{22c}$$

This problem optimal control problem has one functional constraint of the isoperimetric type, (22a), one ordinary differential equation constraint, (22b), and has free initial and terminal indices and free initial and terminal values for the state variable . There are several versions of it. For instance: (1) the simplest problem is the one in which x_0, x_1, $y(x_0)$ and $y(x_1)$ are fixed; (2) the free terminal problem which is common in optimal control problems in which the index variable is time in which x_0 and $y(x_0)$ are known and x_1 and $y(x_1)$ are free; (3) a problem in which the limit values of the indices, x_0 and x_1, are fixed and the state values, $y(x_0)$ and $y(x_1)$, are free; or (4) a problem in which the limit values of the indices, x_0 and x_1, are free and the state values, $y(x_0)$ and $y(x_1)$, are fixed.

Defining

$$\begin{aligned} H^*(x) &= H\left(x, y^*(x), u^*(x), \lambda_0, \lambda_0, \lambda_1(x)\right) = \\ &= F\left(x, y^*(x), u^*(x)\right) - \lambda_0 G_0\left(x, y^*(x), u^*(x)\right) + \lambda_1(x) G_1\left(x, y^*(x), u^*(x)\right), \; x \in [x_0^*, x_1^*] \end{aligned}$$

The first-order necessary conditions for optimality are

$$H_u^*(x) = 0, \text{ for } x \in [x_0^*, x_1^*] \tag{23a}$$

$$\lambda'(x) + H_y^*(x) = 0, \text{ for } x \in [x_0^*, x_1^*] \tag{23b}$$

$$\lambda_1(x)\delta y_t = 0, \text{ for } x = x_t^*, \; t = 0, 1 \tag{23c}$$

$$\left(H^*(x) - \lambda_t(x)(y^*)'(x)\right)\delta x_t = 0, x = x_t^*, \text{ for } t = 0, 1 \tag{23d}$$

for admissible solutions, i.e., satisfying

$$\int_{x_0^*}^{x_1^*} G_0\left(x, y^*(x), u^*(x)\right) dx = \bar{G} \tag{24a}$$

$$\left(y^*\right)'(x) = G_1\left(x, y^*(x), u^*(x)\right) \; x \in (x_0^*, x_1^*) \tag{24b}$$

In order to simplify the derivation of the necessary conditions we consider two simpler problems: problem (P2) in which we address the free limits problem and problem (P3) in which we deal with the functional constraint.

A.1 Simple calculus of variations problem free initial and terminal indexes and states

The problem is (see (Gelfand and Fomin, 1963, ch. 3))

$$\max_{x_0, x_1, (y(x))_{x\in[x_0,x_1]}} \int_{x_0}^{x_1} F\left(x, y(x), y'(x)\right) dx, \text{ subject to (A)} \tag{P2}$$

We define the value functional

$$V[y] = \int_{x_0}^{x_1} F\left(x, y(x), y'(x)\right) dx.$$

As we assume that the the initial and terminal indices and va ues of the state variable are free, we write x_0^* and x_1^* the optimal initial and terminal indices and the solution for the state variable as the path $y^* = \left(y^*(x)\right)_{x\in[x_0^*,x_1^*]}$. In particular optimal initial and terminal values for the state variable are $y_j^* = y^*(x_j^*)$ for $j = 0, 1$. The optimal value is

$$V[y^*] = \int_{x_0^*}^{x_1^*} F\left(x, y^*(x), \left(y^*\right)'(x)\right) dx. \tag{25}$$

We introducing a continuous perturbation $y(x) = y^*(x) + h(x)$. Because of the nature of the optimization problem, the initial and the terminal points of the perturbation are endogenous. We denote by $P_j^* \equiv (x_j^*, y_j^*)$ for $j = 0, 1$ the values of the indexes and of the states at the two boundaries at the optimum. The related terminal points for the perturbed solution are written as $P_j = (x_j^* + \delta x_j, y_j^* + \delta y_j$ for $j = 0, 1$.

Therefore the variation in the functional, $\delta V = V[y^*+h]-V[y^*]$ is (omitting the functional dependence when possible)

$$\begin{aligned}\delta V &= \int_{x_0^*+\delta x_0}^{x_1^*+\delta x_1} F\left(x, y^*(x)+h(x), (y^*)'(x)+h'(x)\right)dx - \int_{x_0^*}^{x_1^*} F\left(x, y^*(x), (y^*)'(x)\right)dx \\ &= \int_{x_0^*}^{x_1^*} F\left(x, y^*(x)+h(x), (y^*)'(x)+h'(x)\right) - F\left(x, y^*(x), (y^*)'(x)\right)dx + \\ &+ \int_{x_1^*}^{x_1^*+\delta x_1} F\left(x, y^*(x)+h(x), (y^*)'(x)+h'(x)\right)dx - \int_{x_0^*+\delta x_0}^{x_0^*} F\left(x, y^*(x)+h(x), (y^*)'(x)+h'(x)\right)dx\end{aligned}$$

Using a first-order Taylor approximation and integration by parts yields, if we denote $F^*(x) = F\left(x, y^*(x), (y^*)'(x)\right)$ and an analogous notation for the derivatives,

$$\begin{aligned}\delta V &= \int_{x_0^*}^{x_1^*} F_y(x)\,h(x) + F_{y'}(x)\,h'(x)dx + F(x)|_{x=x_1^*}\,\delta x_1 - F(x)|_{x=x_0^*}\,\delta x_0 \\ &= \int_{x_0^*}^{x_1^*}\left(F_y(x) - \frac{d}{dx}F_{y'}(x)\right)h(x)dx + \\ &+ F_{y'}(x)h(x)\big|_{x=x_1^*} - F_{y'}(x)h(x)\big|_{x=x_0^*} + F(x)|_{x=x_1^*}\,\delta x_1 - F(x)|_{x=x_0^*}\,\delta x_0\end{aligned}$$

If we approximate

$$h(x_t^*) \approx \delta y_j - y'(x_t^*)\delta x_t, \text{ for } t = 0, 1$$

we obtain

$$\begin{aligned}\delta V &= \int_{x_0^*}^{x_1^*}\left(F_y(x) - \frac{d}{dx}F_{y'}(x)\right)h(x)dx + F_{y'}(x)\big|_{x=x_1^*} - F_{y'}(x)h(x)\big|_{x=x_0^*} + \\ &+ \left(F(x) - F_{y'}(x)y'(x)\right)\Big|_{x=x_1^*}\delta x_1 - \left(F(x) - F_{y'}(x)y'(x)\right)\Big|_{x=x_0^*}\delta x_0\end{aligned}$$

Therefore the necessary conditions for optimality are

$$F_y\left(x, y^*(x), (y^*)'(x)\right) = \frac{d}{dx}F_{y'}\left(x, y^*(x), (y^*)'(x)\right),\ x \in [x_0^*, x_1^*] \tag{26a}$$

$$F_{y'}(x_0^*, y^*(x_0^*), (y^*)'(x_0^*))\,\delta y_0 = 0,\ (y(x_0^*)) \tag{26b}$$

$$F_{y'}(x_1^*, y^*(x_1^*), y'(x_1^*))\,\delta y_1 = 0,\ (y(x_1^*)) \tag{26c}$$

$$\left(F(x_0^*, y^*(x_0^*), (y^*)'(x_0)) - F_{y'}(x_0^*, y^*(x_0^*), (y^*)'(x_0^*))(y^*)'(x_0^*)\right)\,\delta x_0 = 0\ (x_0^*) \tag{26d}$$

$$\left(F(x_1^*, y^*(x_1^*), (y^*)'(x_1^*)) - F_{y'}(x_1^*, y^*(x_1^*), (y^*)'(x_1^*))(y^*)'(x_1^*)\right)\,\delta x_1 = 0, (x_1^*) \tag{26e}$$

To apply the limit conditions (26b) to (26e), observe that

- if the value of the index-j variable, x_j, is known we set $\delta x_j = 0$ in equation (26d) for $j = 0$ or in equation (26e) for $j = 1$;
- if the value of the index-j variable is free to find x_j^* we use

$$F(x_j^*, y(x_j^*), y'(x_j^*)) - F_{y'}(x_j^*, y(x_j^*), y'(x_j^*))y'(x_j^*) = 0$$

  in equation (26d) for $j = 0$ or in equation (26e) for $j = 1$, to free optimal index-variable limit;
- if the value of the state variable associated to index-j variable, $y(x_j) = y_j$ or $y(x_j^*) = y_j$, is known we set $\delta y_j = 0$ in equation (26b) for $j = 0$ or in equation (26c) for $j = 1$;
- if the value of the state variable associated to index-j variable, $y^*(x_j)$ or $y^*(x_j^*)$, is free we set

$$F_{y'}(x_j^*, y^*(x_j^*), (y^*)'(x_j^*)) = 0$$

  in equation (26b) for $j = 0$ or in equation (26c) for $j = 1$.

A.2 Isoperimetric problem

Let us consider now the problem

$$\max_{x_0, x_1, (y(x))_{x\in[x_0,x_1]}} \int_{x_0}^{x_1} F\left(x, y(x), y'(x)\right) dx \text{subject to (A) and (27)} \tag{P3}$$

with the isoperimetric constant (observe the constraint is also a functional)

$$\int_{x_0}^{x_1} G\left(x, y(x), y'(x)\right) dx \leq \bar{G}. \tag{27}$$

The value of this program is (compare with in equation (25)) is

$$V[y^*] = \int_{x_0^*}^{x_1^*} F\left(x, y^*(x), (y^*)'(x)\right) dx + \lambda^* \left(\bar{G} - \int_{x_0^*}^{x_1^*} G\left(x, y^*(x), (y^*)'(x)\right) dx\right)$$

or, if we define the Lagrangean as

$$L(x, y(x), y'(x), \lambda) = F(x, y(x), y'(x)) - \lambda G(x, y(x), y'(x)).$$

it is

$$V[y^*] = \int_{x_0^*}^{x_1^*} L\left(x, y^*(x), (y^*)'(x), \lambda^*\right) dx + \lambda^* \bar{G} \tag{28}$$

Using the same method of proof we find the necessary conditions for optimality

$$L_y\left(x, y^*(x), (y^*)'(x), \lambda^*\right) = \frac{d}{dx} L_{y'}\left(x, y^*(x), (y^*)'(x), \lambda^*\right),\ x \in [x_0^*, x_1^*] \tag{29a}$$

$$L_{y'}\left(x_t^*, y^*(x_t^*), (y^*)'(x_t^*), \lambda^*\right)\ \delta y_t = 0,\ t = 0, 1 \tag{29b}$$

$$\left(L\left(x_t^*, y^*(x_t^*), (y^*)'(x_t), \lambda^*\right) - L_{y'}\left(x_t^*, y^*(x_t^*), (y^*)'(x_t^*))(y^*)'(x_t^*), \lambda^*\right)\right)\ \delta x_t = 0,\ t = 0, 1, \tag{29c}$$

$$\int_{x_0^*}^{x_1^*} G\left(x, y^*(x), (y^*)'(x)\right) dx \leq \bar{G} \tag{29d}$$

$$\lambda^* \left(\bar{G} - \int_{x_0^*}^{x_1^*} G\left(x, y^*(x), (y^*)'(x)\right) dx\right) = 0,\ \lambda^* \geq 0 \tag{29e}$$

where the derivatives of the $L(.)$ function are

$$L_y(x, y(x), y'(x), \lambda) = F_y(x, y(x), y'(x)) - \lambda G_y(x, y(x), y'(x))$$

$$L_{y'}(x, y(x), y'(x), \lambda) = F_{y'}(x, y(x), y'(x)) - \lambda G_{y'}(x, y(x), y'(x)).$$

For the problem with an equality constraint

$$\int_{x_0}^{x_1} G\left(x, y(x), y'(x)\right) dx = \bar{G}$$

the previous conditions are also value, but with constraint (29d) holding as

$$\int_{x_0^*}^{x_1^*} G\left(x, y^*(x), (y^*)'(x)\right) dx = \bar{G}$$

and the constraint (29e) holding with $\lambda^* > 0$.

A.3 Optimal control problem

Now we go back to problem (P1).

We define the Hamiltonian

$$H(x,y(x),u(x),\lambda_0,\lambda_1(x)) \equiv F\left(x,y(x),u(x)\right)-\lambda_0 G_0\left(x,y(x),u(x)\right)+\lambda_1(x)G_1\left(x,y(x),u(x)\right).$$

At the optimum the value function is

$$V[y^*,u^*]=\int_{x_0^*}^{x_1^*} F\left(x,y^*(x),u^*(x)\right)dx. \tag{30}$$

Equivalently

$$\begin{aligned}
V[y^*,u^*] &= \int_{x_0^*}^{x_1^*} \left(F\left(x,y^*(x),u^*(x)\right)-\lambda_0 G_0\left(x,y^*(x),u^*(x)\right)\right)dx+\lambda\bar{G}\\
&= \int_{x_0^*}^{x_1^*} \Big(F\left(x,y^*(x),u^*(x)\right)-\lambda_0 G_0\left(x,y^*(x),u^*(x)\right)+\lambda_1(x)\left(G_1\left(x,y^*(x),u^*(x)\right)\right)-(y^*)'(x)\Big)dx\\
&= \int_{x_0^*}^{x_1^*} \Big(H(x,y^*(x),u^*(x),\lambda_0,\lambda_1(x))+\lambda_1'(x)y^*(x)\Big)dx+\lambda_1(x_1^*)y^*(x_1^*)-\lambda_1(x_0^*)y^*(x_0^*)+\lambda_0\bar{G}
\end{aligned}$$

Now, we introduce the arbitrary (functional) perturbations $y^*(x)\to y(x)=y^*(x)+\varepsilon h_y(x)$, $u^*(x)\to u(x)=u^*(x)+\varepsilon h_u(x)$, and the (point) perturbations $x_t^*\to x_t=x_t^*+\varepsilon\delta x_t$, for $t=0,1$ and $y_t^*\to y_t=y_t^*+\varepsilon\delta y_t$, for $t=0,1$, such that

$$h_y(x_t^*)=\delta y_t-y'(x_t^*)\,\delta x_t,\ t=0,1 \tag{31}$$

At the optimum $\delta V[y^*,u^*]=0$ where the variational derivative is

$$\delta V[y^*,u^*]=\lim_{\epsilon\to 0}\frac{\Delta V}{\epsilon}$$

where $\Delta V=V[y^*+\varepsilon h_y,u^*+\varepsilon h_u]-V[y^*,u^*]$. Using derivations from the previous problems we find

$$\begin{aligned}
\Delta V[y,u] = \int_{x_0^*}^{x_1^*} &\big[H\left(x,y^*(x)+\varepsilon h_y(x),u^*(x)+\varepsilon h_u(x),\lambda_0,\lambda_1(x)\right)-H\left(x,y^*(x),u^*(x),\lambda_0,\lambda_1(x)\right)+\\
&+\lambda_1'(x)\left(y^*(x)+\varepsilon h_y(x)-y^*(x)\right)\big]dx+\\
&+\lambda_1(x_1^*)\left(y^*(x_1^*)+\varepsilon h_y(x_1^*)\right)-\lambda_1(x_0^*)\left(y^*(x_0^*)+\varepsilon h_y(x_0^*)\right)-\lambda_1(x_1^*)y^*(x_1^*)+\lambda_1(x_0^*)y^*(x_0^*)+\\
&+\Big(H\left(x,y^*(x),u^*(x),\lambda_0,\lambda_1(x)\right)|_{x=x_1^*}\Big)\delta x_1-\Big(H\left(x,y^*(x),u^*(x),\lambda_0,\lambda_1(x)\right)|_{x=x_0^*}\Big)\delta x_0
\end{aligned}$$

Using a first-order Taylor approximation and equation (31), collecting terms, factoring out and simplifying the notation we have,

$$\begin{aligned}\Delta V[y,u] = \varepsilon \Bigg\{ & \int_{x_0^*}^{x_1^*} \left[H_u^*(x) h_u(x) + \left(H_y^*(x) + \lambda_1'(x) \right) h_y(x) \right] dx + \\ & + \lambda_1(x_1^*) h_y(x_1^*) - \lambda_1(x_0^*) h_y(x_0^*) + H^*(x_1^*)\delta x_1 - H^*(x_0^*)\delta x_0 \Big\} = \\ = \varepsilon \Bigg\{ & \int_{x_0^*}^{x_1^*} \left[H_u^*(x) h_u(x) + \left(H_y^*(x) + \lambda_1'(x) \right) h_y(x) \right] dx + \\ & + \lambda_1(x_1^*)\delta y_1 - \lambda_1(x_0^*)\delta y_0 + \left(H^*(x_1^*) - \lambda_1(x_1^*)(y^*)'(x_1^*) \right) \delta x_1 - \left(H^*(x_0^*) - \lambda_1(x_0^*)(y^*)'(x_0^*) \right) \delta x_0 \Bigg\}\end{aligned}$$

at the optimum $\delta V[y^*, u^*] = 0$ from which we derive equations (23a)-(23d).