Consumption, savings and asset accumulation

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20.10.2020
(revised 17.12.2020)

1 Introduction

This lecture note is dedicated to the consumer problem. It is thus a partial equilibrium exercise. The analysis is conducted in continuous time and within a deterministic setting. That is, it is assumed that the consumer has perfect information.

After studying the main concepts of the intertemporal choice in a continuous-time setting we apply them to both intertemporal additive and intertemporally dependent preferences.

This is applied to the consumer problem in several environments: without resource constraints, with a initial finite resource (cake eating), with borrowing constraints, with financial and non-financial income, and with a stochastic horizon.

This allows us to provide background to the benchmark consumer problem in macroeconomics: the discounted infinite horizon problem in a complete market setting in which the rate pf return is equal to the rate of time preference but in which the consumer has a solvability (or non-Ponzi game) constraint.

Although this model has a closed form (explicit) solution, we present the method of comparative statics applied to this case. This has an interest per se but also is the only tool available to characterize the solution to the intertemporally dependent consumer problem.

This model provides a relatively simple theory for one of the stylized facts in macroeconomics: the relatively more persistent behavior of consumption relative to aggregate income.

In the additive

In section 2 we present the benchmark intertemporal additive utility case. In section 3 we present the consumer problem in the continuous time setting. In section 4 the method of continuous-time

comparative dynamics is applied to the additive model. In section 5 the habit formation is presented and its comparative dynamics is studied, for an increase in non-financial income.

2 Intertemporaly additive utility functional

In this section we provide some definitions regarding the intertemporal preference properties which are implicit in intertemporal utility functionals. We start with the case of additive utility functionals both in discrete and continuous time, in subsections 2.1 and 2.2. In section **??** we study, with the same concepts, a non-additive intertemporal utility functional arising in the habit formation model.

2.1 Discrete time

Let measurements be taken at discrete time intervals $\mathrm{T} = \{0, 1, \ldots, t, \ldots, T\}$. In this discrete time setting, the sequence of consumption $\{c\} = \{c_0, c_1, \ldots, c_t, \ldots, c_T\}$ is valued by the utility functional

$$\mathsf{U}[c] = \mathsf{U}[\{c_0, c_1, \ldots, c_t, \ldots, c_T\}].$$

The functional $\mathsf{U}[c]$ maps every consumption sequence into a number. The higher $\mathsf{U}[c]$ is the higher is the value of a consumption sequence.

Given some assumptions from choice theory, we represent an order relationship between consumption sequences by a cardinal relationship between their intertemporal utilities. That is, given two consumption sequences $\{c'\} = \{c'_0, c'_1, \ldots, c'_t, \ldots, c'_T\}$ and $\{c''\} = \{c''_0, c''_1, \ldots, c''_t, \ldots, c''_T\}$ the consumer is indifferent between the two if and only if $\mathsf{U}[c'] = \mathsf{U}[c'']$ and the consumer prefers c' to c'' if and only if $\mathsf{U}[c'] > \mathsf{U}[c'']$.

In order to characterize the implicit properties introduced by the utility functional $\mathsf{U}[c]$ consider a given consumption sequence $\{c\}$ and introduce a change in consumption at any point in time t, i.e, consider the change from $\{c\} = \{c_0, c_1, \ldots, c_t, \ldots, c_T\}$ to $\{c+dc\} = \{c_0, c_1, \ldots, c_t + dc_t, \ldots, c_T\}$, where $\{dc\} = \{0, \ldots, 0, dc_t, 0, \ldots 0\}$. The value of the consumtion sequence changes from $\mathsf{U}[c]$ to $\mathsf{U}[c + dc] = \mathsf{U}[c] + d\mathsf{U}[c; c_t]$ where

$$d\mathsf{U}[c; c_t] = \frac{\partial \mathsf{U}[c]}{\partial c_t}\, dc_t,$$

where $\dfrac{\partial \mathsf{U}[c]}{\partial c_t}$ as the partial simple derivative for consumption at time t. We call it the **marginal utility for a change consumption at any time** t and denote it as

$$\mathsf{U}_t[c] \equiv \frac{\partial \mathsf{U}[c]}{\partial c_t}, \text{ for any } t \in \mathrm{T}.$$

Observe that $\mathsf{U}_t[c]$ is also a functional. Therefore we can define the second partial derivatives derivatives

$$U_{t,t'}[c] \equiv \frac{\partial^2 \mathsf{U}[c]}{\partial c_t \, \partial c_{t'}} \, c_{t'}, \text{for any } t, t' \in \mathrm{T},$$

which yields the change in the marginal utility of consumption at time c_t for a change in consumption at time t', $dc_{t'}$, where $t' = t$, in which case we have own derivative, or $t' \neq t$, in which case we have cross derivative.

Now consider the following case: consider the initial sequence $c = \{c_0, c_1, \ldots, c_{t_0}, \ldots, c_{t_0+\tau}, \ldots, c_T\}$ and perturb it by changing consumption in two periods t_0 and $t_1 = t_0+\tau$, to $c+dc = \{c_0, c_1, \ldots, c_{t_0} + dc_{t_0}, \ldots, c_{t_0+\tau} + dc_{t_0+\tau}, \ldots, c_T\}$. The initial sequence has value with value $\mathsf{U}[c]$ and the perturbed sequence has value

$$\mathsf{U}[c + dc] = \mathsf{U}[c] + d\mathsf{U}[c] = \mathsf{U}[c]\mathsf{U}_{t_0} \, dc_{t_0} + \mathsf{U}_{t_1} \, dc_{t_1}.$$

We call **intertemporal marginal rate of substitution**, between periods t_0 and $t_1 = t_0 + \tau$, the change in consumption at $t_1 = t + \tau$, where $\tau > 0$, for one unit increase in consumption at t_0 such that it leaves intertemporal utility constant:

$$IMRS_{t_0,t_1} = -\frac{dc_{t_1}}{dc_{t_0}}\Big|_{\mathsf{U}=\text{constant}}.$$

Therefore, the $IMRS_{t_0,t_1}$ is the ratio of changes between dc_{t_0} and dc_{t_1} such that

$$d\mathsf{U}[c] = \mathsf{U}_{t_0} \, dc_{t_0} + \mathsf{U}_{t_1} \, dc_{t_1} = 0.$$

Therefore, we have an equivalent definition of the intertemporal marginal rate of substitution as the ratio between marginal utilities

$$IMRS_{t_0,t_1}(c) = \frac{\mathsf{U}_{t_0}}{\mathsf{U}_{t_1}}.$$

We say the utility functional displays **impatience** if, for a stationary consumption path, such that $c_t = \bar{c}$ for any t, the intertemporal marginal rate of substitution is greater than one: $IMRS_{t_0,t_1}(\bar{c}) > 1$.

This means that, given a time-independent stationary consumption sequence, the consumer attaches a higher value for consumption closer in time that far away in time. Equivalently, the consumer is willing to sacrifice a given amount of consumption in the present if a higher level of consumption will be given in the future, and that level increases with the distance in the future in which it is made available.

The change in the marginal utility also provides insight on the implicit consumption behavior in $\mathsf{U}[c]$.

The Uzawa-Allen elasticity of intertemporal substitution, between consumption in period t_0 and t_1 is defined as

$$\varepsilon_{t_0,t_1} = -\frac{\mathsf{U}_{t_0,t_1}[c]\, c_{t_0}}{U_{t_0}[c]}, \text{for any } t_0, t_1 \in \mathrm{T},$$

which measures the relative change in the marginal utility of consumption at time t_0 for a change in consumption at time t_1.

Considering a stationary consumption path, the utility functional displays one of the three following properties [1]:

- there is **intertemporal complementarity** if the marginal utility of consumption at time t_0 increases with consumption at time any future date t_1. In this case $\varepsilon_{t_0,t_1}(\bar{c})$ is negative;
- there is **intertemporal independence** if the marginal utility of consumption at time t_0 does not change with consumption at any future time t_1. In this case $\varepsilon_{t_0,t_1}(\bar{c})$ is equal to zero;
- or there is **intertemporal substitution** if the marginal utility of consumption at time t_0 decreases with consumption at time any future date t_1. In this case $\varepsilon_{t_0,t_1}(\bar{c})$ is positive.

In order to measure intertemporal substitution/complementarity the definition of **intertemporal elasticity of substitution** is introduced

$$EIS_{t_0,t_1} = \frac{\partial\big(c_{t_1}/c_{t_0}\big)}{\partial IMRS_{t_0,t_1}} \frac{IMRS_{t_0,t_1}}{c_{t_1}/c_{t_0}}.$$

which measures the elasticity of the consumption ratio as regards the elasticity of the $IMRS$. Expanding the definition, we have the equivalent formula

$$EIS_{t_0,t_1} = \frac{c_{t_0}\,\mathsf{U}_{t_0} + c_{t_1}\,\mathsf{U}_{t_1}}{c_{t_0}\,\mathsf{U}_{t_0}\,\varepsilon_{t_1,t_1} - 2\,c_{t_0}\,\mathsf{U}_{t_0}\,\varepsilon_{t_0,t_1} + c_{t_1}\,\mathsf{U}_{t_1}\,\varepsilon_{t_0,t_0}}.$$

The **benchmark utility functional** is the additive utility functional

$$\mathsf{U}[c] = \sum_{t=0}^{T} \beta^t u(c_t) \tag{1}$$

[1] Next we refer to complementarity and substitutability in the Edgeworth-Pareto, or uncompensated, sense not in the Hicks-Allen, or compensated, sense. The issue of complementarity and substitutability should be dealt with care. This is discussed in the next subsection.

where $\beta \equiv \dfrac{1}{1+\rho}$ is the psychological discount factor and ρ is the rate of time preference. It means that the value of a consumption sequence c is equal to the present value of the sequence of period utilities of consumption. It is therefore linear on the utilities of consumption for every period.

Exercise: Prove that the utility functional (1) displays impatience, intertemporal independence, and, if we define the elasticity of marginal substitution by

$$\sigma(c) = -\frac{u''(c)\,c}{u'(c)}$$

then the elasticity of intertemporal substitution is equal to the inverse of the elasticity of marginal utility $EIS_{t_0,t_1}(\bar{c}) = \dfrac{1}{\sigma(\bar{c})}$

2.2 Continuous time

Let time be a real non-negative variable, $t \in \mathrm{T} \subseteq \mathbb{R}_+$. Consumption at time t is $c(t)$ and the flow of consumption is $c = \big(c(t)\big)_{t\in\mathrm{T}}$.

While the definition of marginal utility regarding intertemporal choice in discrete time is analogous to the static choice among several goods, the definition of marginal utility in a continuous time setting requires some background in integral derivatives, because the change in consumption at a point in time takes place within an infinitesimal time interval.

In this section we introduce the concepts of marginal utility, intertemporal marginal rate of substitution, Uzawa-Allen elasticities and elasticity of intertemporal substitution for the benchmark time additive utility functional. In section 5 we apply them to a particular type of intertemporally dependent preferences.

The benchmark **intertemporal utility functional** in continuous time is

$$\mathsf{U}[c] = \int_0^T u(c(t))e^{-\rho t}dt,\ \rho > 0 \tag{2}$$

where $e^{-\rho t}$ is the psychological discount factor and ρ is the rate of time preference, where the **utility function** $u(\cdot)$ is assumed to be continuous, differentiable (at least second-order differentiable), increasing and concave.

As we saw in last subsection, a crucial feature of the allocation of consumption through time is the intertemporal substitutability (or complementarity) of consumption. In a continuous time setting the utility functional $\mathsf{U}[c]$ is infinite-dimensional and consumption substitutability is related to the change in consumption in two points in time, say t and $t+\tau$, by $dc(t_0)$ and $dc(t_0+\tau)$, such

that the utility functional $\mathsf{U}[c]$ remains constant. That is, we want to compare the flow $\big(c(t)\big)_{t\in \mathrm{T}}$ with the flow $\big(\hat{c}(t)\big)_{t\in \mathrm{T}}$ such that $\hat{c}(t) = c(t)$ if $t \neq \{t_0, t_0+\tau\}$ and $\hat{c}(t_0) = c(t_0) + dc(t_0)$ and $\hat{c}(t_0+\tau) = c(t_0+\tau) + dc(t_0+\tau)$.

There are two difficulties in dealing with intertemporal substitution/complementarity in continuous time.

The first difficulty is related to the economic definition of complementarity/substitutability. The concept of complementarity and substitutability are not unequivocal. In a static microeconomic setting, let preferences among different consumption bundles $\mathbf{c} = (c_1, \dots c_n)$ be represented by the utility function $u(\mathbf{c})$, and the consumer's problem be $\max_{\mathbf{c}}\{u(\mathbf{c}) : \mathbf{p}\cdot\mathbf{c} \leq y\}$, where $\mathbf{p}$ is the related price vector and y is income. In this setting we say two goods, indexed for instance by i and j, are substitutability in the Edgeworth-Pareto sense if $\dfrac{\partial^2 U(\mathbf{c})}{\partial c_i\, c_j} < 0$ and they are substitutable in the Hicks-Allen sense if, at the optimum, the demand function is $\mathbf{c} = C(\mathbf{p}, y)$ and $\dfrac{\partial c_i}{\partial p_j} > 0$. The first type can be called uncompensated substitutability and the second compensated substitutability because it takes into account a normalizing effect of the budget constraint. Given the difficulty of defining an extension to the Hicksian sense of substitutability in continuous time [2] , the definition we present next is an extension of the Edgeworth-Pareto concept.[3]

The second difficulty is mathematical. Although continuous time makes analytical derivations of results easier, one has to clarity the mathematical nature of the definition of the intertemporal marginal rate of substitution between times t_0 and $t_0+\tau$, because the time variations become infinitesimal, i.e., variations have a measure zero.

As regards the utility functional (2), we can use the concept of a Gâteaux derivative for a "spyke" variation of consumption at time t, introduced in the appendix A. Applying the definition of the integral derivative, provided there, to equation (14), we obtain the **marginal utility of consumption** at any time t is given by

$$\mathsf{U}_t \equiv \delta\mathsf{U}[c;t] = u'\big(c(t)\big)\, e^{-\rho t}, \text{ for any } t_i \in \mathrm{T},$$

where $\delta\mathsf{U}[c;t]$ is the integral derivative of the functional $\mathsf{U}[c]$ for a "spyke" variation in consumption at time $t \in [0,T]$.

[2] For a discussion see Biswas, 1976, and Ryder and Heal, 1973 and Heal and Ryder, 1976.

[3] There are issues here that are more profound than they look at first. There is a long tradition between ordinalists and cardinalists related comparative statics. The most recent approach to microeconomics, the monotone comparative statics approach, initiated by Milgrom and Shannon, 1994, associates substitutability defined in an ordinal way with the property of supermodularity. If a utility function is continuous and differentiable, the Edgeworth-Pareto substitutability criteriUm is equivalent to supermodularity.

Therefore, the **intertemporal marginal rate of substitution** between consumption at time t_0 and $t_1 = t_0 + \tau$, for $\tau \geq 0$ for the intertemporal utility functional (2) is

$$\text{IMRS}_{t_0,t_1} = \frac{\mathsf{U}_{t_0}}{\mathsf{U}_{t_1}} = \frac{u'\big(c(t_0)\big)}{u'\big(c(t_1)\big)} e^{\rho\,\tau}.$$

Using the second order Gâteaux derivatives for a "spyke" variation at the same time and at a different time, we obtain

$$\mathsf{U}_{t_i t_i} \equiv \delta^2 \mathsf{U}[c; t_i, t_i] = u''\big(c(t_i)\big)\, e^{-\rho t_i}, \text{ for any } t \in \mathrm{T}.$$

and $\mathsf{U}_{t_i t_j} \equiv \delta^2 \mathsf{U}[c; t_i, t_j] = 0$ for $t_i \neq t_j$ both in T.

This implies that the **Uzawa-Allen elasticities** are [4] are

$$\epsilon_{t_i,t_i} = \sigma\big(c(t_i)\big) \equiv -\frac{u''\big(c(t_i)\big)\, c(t_i)}{u'\big(c(t_i)\big)}$$

where $\sigma\big(c(t_i)\big)$ is the elasticity of the marginal utility $u'(\cdot)$, and

$$\epsilon_{t_i,t_j} = 0 \text{ if } t_i \neq t_j.$$

Therefore the **elasticity of intertemporal substitution**[5]

$$\begin{aligned} IES_{t_0,t_1} &= \frac{d \log\big(c(t_0)/c(t_1)\big)}{d \log \text{IMRS}_{t_0,t_1}} = \\ &= \frac{c_{t_0}\, \mathsf{U}_{t_0} + c_{t_1}\, \mathsf{U}_{t_1}}{c_{t_0}\, \mathsf{U}_{t_0}\, \epsilon_{t_1,t_1} - 2\, c_{t_0}\, \mathsf{U}_{t_0}\, \epsilon_{t_0,t_1} + c_{t_1}\, \mathsf{U}_{t_1}\, \epsilon_{t_0,t_0}}. \end{aligned} \tag{3}$$

Applying to the intertemporal utility functional (2) yields

$$IES_{t_0,t_1} = \frac{c(t_0)\, u'(c(t_0))\, e^{-\rho t_0} + c(t_1)\, u'(c(t_1))\, e^{-\rho t_1}}{c(t_1)\, u'(c(t_1))\, e^{-\rho t_1}\, \sigma(c(t_0)) + c(t_1)\, u'(c(t_0))\, e^{-\rho t_0}\, \sigma(c(t_1))}.$$

[4] They are deined as

$$\epsilon_{t_i,t_j} = -\frac{\mathsf{U}_{t_i t_j}\, c(t_i)}{\mathsf{U}_{t_i}} \equiv -\frac{\delta^2 \mathsf{U}[c; t_i, t_j]\, c(t_i)}{\delta \mathsf{U}[c; t_i]} \text{ for any } t_i, t_j \in \mathrm{T}.$$

[5] Expanding the expression we obtain

$$\frac{d \log\big(c(t_i)/c(t_j)\big)}{d \log \text{IMRS}_{t_i,t_j}} = \frac{c(t_i)\, \mathsf{U}_{t_i} + c(t_j)\, \mathsf{U}_{t_j}}{c(t_j)\, \mathsf{U}_{t_j}\, \mathsf{U}_{t_i t_i} - 2\, c(t_i)\, \mathsf{U}_{t_i}\, \mathsf{U}_{t_i t_j} + c(t_i)\, \mathsf{U}_{t_i}\, \mathsf{U}_{t_j t_j}}.$$

Taking a stationary flow of consumption, such that $c(t) = \bar{c}$ for any $t \in \mathrm{T}$, we obtain $IMRS_{t_0,t_1} = e^{\rho\,\tau} > 1$, $\epsilon_{t_0,t_1} = 0$ and $IES_{t_0,t_1} = \dfrac{1}{\sigma(\bar{c})}$, which we interpret as establishing that it displays **impatience**, **intertemporal independence** and an **intertemporal elasticity of substitution which is the inverse of the elasticity of the marginal utility** .

For the case in which the utility function is isoelastic (indeed a generalized logarithm),

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta},$$

we obtain $IES_{t_0,t_1} = \dfrac{1}{\theta}$, which led to the literature calling θ the **inverse of the elasticity of intertemporal substitution**.

3 The consumer problem in the continuous time setting

In this section we apply the previous concepts, in particular the continuous-time definition of marginal utility, to the consumer problem with several types of constraints. We assume a deterministic setting in which the consumer has perfect information.

We start by dealing with the meaning of maximizing utility in an unconstrained problem in subsection 3.1. In subsection 3.2 we consider the problem for a rentier, i.e., for an agent which consumes out of a an initial resource from which the only activity is to consume it through time. Next, in subsection 3.3 we consider the problem for a rentier which is also an investor. Section 3.4 considers the case in which the consumer has also non-financial income. All those models consider a finite and known horizon.

In subsection 3.5 we present a justification for considering an infinite-horizon and solve the household problem. This model is the benchmark partial equilibrium model in intertemporal macroeconomics. It can also be seen as a simple model for a small open economy facing a perfect international capital market [6]

3.1 Maximizing utility without constraints

We address the first problem: finding the maximum consumption path in an unconstrained setting. That is, which path $\big(c(t)\big)_{t\in\mathrm{T}}$, where $\mathrm{T} = [0, T]$, will the consumer choose in the case she/he has no constraints.

[6] For a discrete time version see Uribe and Schmitt-Grohé, 2017, ch 2.

The problem is to find the maximum value

$$\mathsf{U}[c^*] = \max_{c} \int_0^T u(c(t))\, e^{-\rho t} dt \tag{P1}$$

where we denote $c^* = \big(c^*(t)\big)_{t\in \mathrm{T}}$ the maximum utility consumption path.

If c^* is optimum, then any arbitrary perturbation φ will not change intertemporal utility, that is $\mathsf{U}[c^* + \varphi] = \mathsf{U}[c^*]$. Therefore, at the optimum

$$\int_0^T u'(c^*(t))\, e^{-\rho t}\, \varphi(t)\, dt = 0$$

This is a linear functional which can be satisfies, because $\varphi t \neq 0$ for every $t \in \mathrm{T}$ [7] if and only if

$$u'(c^*(t))\, e^{-\rho t} = 0, \text{ for every } t \in [0, T]. \tag{P1:foc}$$

The optimum level of consumption, because $e^{-\rho t} \in (0, 1)$, depends on the properties of the utility function $u(\cdot)$:

1. if the utility function has the Inada property, i.e, if $\lim_{c\to 0} u'(c) = \infty$, then the optimum will be reached for $c^*(t) = \infty$ for any t;

2. however, if the utility function does not have the Inada property, then the optimum can be reached for a finite level of consumption. If there is a satiation point, that is a point $c^s \in \mathbb{R}_+$ such that $u'(c^s) = 0$, then the optimum will consist in consuming at that satiation point at every moment time $c^*(t) = c^s$ for any t.

Exercise: Prove that the utility function of the isoelastic type $u(c) = \dfrac{c^{1-\theta} - 1}{1 - \theta}$, for $\theta > 0$ has the Inada property, and therefore the optimum consumption is $c^*(t) = \infty$ for any t.

Exercise: Prove that the quadratic utility function $u(c) = c - \dfrac{\beta}{2}\, c^2$, for $\beta > 0$ has a satiation point.

The message from this is clear: if there are no constrains on consumption and the utility function displays no satiation, then optimal consumption would be infinite for every point in time, as in the Cockaigne land.

[7] See Gel'fand and Fomin, 1963, p.9

3.2 Maximizing utility for a rentier

Now assume the problem of a lazy rentier: it has an initial stock of net wealth, $a_0 > 0$ which is used to finance the purchase of consumption goods throughout lifetime. Assume prices are always equal to one.

Consider, again, any two moments t and $t + \tau$. Assuming that the rentier still possesses the stock level $a(t)$ at time t, and that she/he consumes a constant quantity $c(t)$ during any period of length τ, what will be the stock of wealth remaining at time $t + \tau$? It is easy to see that $a(t+\tau) = a(t) - c(t)\,\tau$. If the time interval shrinks to zero, the infinitesimal change in his wealth is

$$\lim_{t \to 0} \frac{a(t+\tau) - a(t)}{\tau} = \dot{a}(t) = -c(t).$$

Solving the budget constraint $\dot{a}(t) = -c(t)$ we find that the level of net wealth at any point in time is

$$a(t) = a_0 - \int_0^t c(s)\, ds.$$

From now on we assume that the initial value of wealth is finite, positive and known, that is $a(0) = a_0 > 0$.

Does the optimal consumption path changes as regards problem (P1), i.e., does existence of an initial finite resource constrains consumption ? The answer is, not necessarily, because it depends on the assumptions we made regarding the domain of a.

Consider a first case: the future net wealth level a can take any real value, that is the agent can borrow without limit. As a denotes the stock oh net wealth we assume that $-\infty < a(t) < \infty$ for any time $t > 0$.

The problem of the consumer becomes

$$\begin{aligned} &\max_c \int_0^T u(c(t))\, e^{-\rho t} dt \\ &\text{subject to} \\ &\dot{a}(t) = -c(t), \text{ for } t \in \mathrm{T} \\ &a(0) = a_0 > 0 \text{ given} \\ &a(t) \in \mathbb{R} \cup \infty \end{aligned} \tag{P2}$$

We show in the appendix B that the optimality condition for this problem is the same as the solution of problem (P1), in equation (P1:foc):

$$u'(c^*(t))\, e^{-\rho t} = 0, \text{ for every } t \in [0, T]. \tag{P2:foc}$$

Again, if the utility has the Inada property (v.g, if it has a satiation point) then the consumer will want to consume an infinite amount at every point in time. This means that a becomes negative and unbounded just immediately after $t = 0$ and the initial constraint of the resource is not active.

A second, more realistic case is the one in which the consumer faces a lower bound on a, that we denote by $\underline{a}$. If he cannot borrow then $\underline{a} \geq 0$. If he can become a net debtor until some prescribed limit, then he faces a borrowing constraint $\underline{a} < 0$ which is finite. This is a source in which financial frictions can be introduced in the model.

We assume this constraint is active at every point in time, and the consumer has an initial net wealth level which is positive. The problem is now:

$$\begin{aligned} &\max_{c} \int_0^T u(c(t))\, e^{-\rho t} dt \\ &\text{subject to} \\ &\dot{a}(t) = -c(t), \text{ for } t \in \mathrm{T} \\ &a(t) \in [\underline{a}, \infty), \text{ for every } t \in [0, T] \\ &a(0) = a_0 > \max\{0, \underline{a}\} \text{ given} \end{aligned} \tag{P3}$$

Let us assume that the utility function $u(c)$ has the Inada property: $u''(c) < 0 < u'(c)$ and $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. In order to make the analysis clear let us assume that the utility function is $u(c) = \dfrac{c^{1-\theta} - 1}{1 - \theta}$, for $\theta > 0$. This yields $u'(c) = c^{-\theta}$

Two important observations are raised by introduction of the borrowing constraint: first, we assume that initial level of wealth is above the lower limit for a, $\underline{a}$ (which can be positive or negative); second, as the utility function would induce an unconstrained consumer to consume an infinite amount at every point in time, then the borrowing constraint will be active at some point in time, say $\underline{t} \in (0, T]$. The time of activation of the constraint will be higher than zero because the initial level of wealth is by assumption bigger than the lower limit, $a_0 > \underline{a}$ but it is **endogenous**. This raises a natural question: is it optimal to hit the constraint earlier or at the horizon T ?

The optimality conditions are

$$\begin{cases} u'(c(t))e^{-\rho t} = \lambda(t) \\ \dot{a} = -c(t) \\ \dot{\lambda} = -\eta(t) \\ \eta(t)(a(t) - \underline{a}) = 0, \quad \eta(t) \geq 0,\ a(t) \geq \underline{a} \text{ for } t \in (0, T] \\ \lambda(T^-) = \eta(T). \end{cases}$$

Therefore, we can divide the solution in two periods: the period $[0, \underline{t})$ in which the constraint is not active, and we have $\eta(t) = 0$ which implies $\lambda(t) = \lambda_0$ is a constant, consumption is positive and the net wealth diminishes through time, because we still have $a_0 > a(t) > \underline{a}$; and, the period $[\underline{t}, T]$ in which the constraint is active, and we have $\eta(t) > 0$, $\dot{\lambda} < 0$, consumption is equal to zero and net wealth is at the borrowing limit.

That is, we have the following solutions for consumption and net wealth

$$c(t) = \begin{cases} \lambda_0^{-\frac{1}{\theta}} e^{\gamma_c t} & \text{for } t \in [0, \underline{t}) \\ 0 & \text{for } t \in (\underline{t}, T] \end{cases}$$

here $\gamma_c = -\dfrac{\rho}{\theta} < 0$ is the (negative) rate of growth of consumption, and

$$a(t) = \begin{cases} a_0 + \dfrac{\lambda_0^{-\frac{1}{\theta}}}{\gamma_c}\left(1 - e^{\gamma_c t}\right) & \text{for } t \in [0, \underline{t}) \\ \underline{a} & \text{for } t \in (\underline{t}, T], \end{cases}$$

where λ_0 is unknown and can only be determined when $\underline{t}$ is found.

At time $\underline{t}$ we have $c(\underline{t}^+) = c(\underline{t}^-) = 0$ which is only possible if $\lambda = \infty$ which implies $a(\underline{t}) = \underline{a} = a_0$, which is not possible for $\underline{t} < T$, because we have assumed that $\underline{a} < a_0$.

If we set, $\underline{t} = T$ from $\underline{a} = a_0 + \dfrac{\lambda_0^{-\frac{1}{\theta}}}{\gamma_c}\left(1 - e^{\gamma_c T}\right)$ we can determine λ_0 from

$$\lambda_0^{-\frac{1}{\theta}} = \gamma_c \frac{\underline{a} - a_0}{1 - e^{\gamma_c T}}$$

and obtain the optimum consumption

$$c^*(t) = \gamma_c \frac{\underline{a} - a_0}{1 - e^{\gamma_c T}} e^{\gamma_c t}\ t \in [0, T).$$

At time $t = T$, because we assumed that $\underline{a} < a_0$, there should exist a discontinuous jump in consumption because $c(T^-) = \gamma_c \dfrac{\underline{a} - a_0}{1 - e^{\gamma_c T}} e^{\gamma_c T} > 0 \neq c(T) = 0$ (recall that $\gamma_c < 0$). The net optimum wealth stock

$$a^*(t) = a_0 + \gamma_c \frac{\underline{a} - a_0}{1 - e^{\gamma_c T}} \left(1 - e^{\gamma_c t}\right) \text{ for } t \in [0, T].$$

Therefore, if the consumer attributes an unlimited utility to consumption when it is close to zero, when the utility function has the Inada properties, if there is a financial constraint, and the consumer has perfect foresight, she/he will stay away from the borrowing constraint before the problem's horizon T. It finds optimal to exhaust his possibilities for borrowing only at the last moment.

Exercise If the utility function displays satiation what will be the solution to the problem ?

3.3 Maximizing utility for an investor

Next we assume that the household can extract a financial income, equal to $r(t)\, a$, from the ownership of the asset there is investment. However, if the household is a net debtor it would represent an extra expenditure, instead. We assume that the interest rate is given to the household. To simplify, we assume that the interest rate r is constant.

Again we assume that the utility function has the Inada property and that, justified by the results in the last subsection, there is a terminal constraint on the level of net wealth.

Assuming an isoelastic utility function, the problem is

$$\begin{aligned} &\max_c \int_0^T \frac{c(t)^{1-\theta} - 1}{1 - \theta} e^{-\rho t} dt \\ &\text{subject to} \\ &\dot{a}(t) = r\, a - c(t), \text{ for } t \in \mathrm{T} \\ &a(0) = a_0 > \underline{\mathrm{a}} \text{ given} \\ &a(T) \geq \underline{a} \end{aligned} \tag{P4}$$

In appendix D we prove that optimal consumption is generated by the function

$$c^*(t) = \left(\frac{(r - \gamma_c)\left(a_0 - \underline{a}\, e^{-r\, T}\right)}{1 - e^{(\gamma_c - r)T}} \right) e^{\gamma_c t}, \text{for every } t \in [0, T] \tag{4}$$

where the rate of growth of consumption is

$$\gamma_c \equiv \frac{r-\rho}{\theta}$$

and the optimal net wealth is generated by the function

$$a^*(t) = e^{r\,t}a_0 - \frac{a_0 - \underline{a}\,e^{-r\,T}}{1-e^{(\gamma_c - r)T}}\left(e^{r\,t} - e^{\gamma_c\,t}\right), \text{for every } t \in [0,T]. \tag{5}$$

The following observations can be made: first, as the sign of γ_c is the same as if $r-\rho$ then if $r > \rho$ consumption will grow across the lifetime, it will remain constant if $r = \rho$ and will diminish if $r < \rho$. Second, for any value of the parameters, the initial and the terminal values of net wealth are $a(0) = a_0$ and $a(T) = \underline{a}$, which means that the time profile of consumption (and savings) featuring more or less consumption (or less or more savings) in the beginning of the period or at the end of the period, depend on the relative value of the parameters, ρ and θ and on the market interest rate r. IN particular, if θ is higher then the absolute value of γ_c will be lower.

In this model the path of income is $y(t) = r\,a(t)$ because there is only financial income.

3.4 Maximizing utility with non-financial income

In this subsection we assume that the household is entitled to a non-financial stream of income $\big(w(t)\big)_{t\in \mathrm{T}}$. It can be labor income or any other type of non-financial income. In the case of labor income, we assume that the household supplies inelastically a constant flow of working hours (or effort) normalized to one.

We consider now the following problem

$$\begin{aligned}
&\max_c \int_0^T \frac{c(t)^{1-\theta}-1}{1-\theta}\,e^{-\rho t}dt \\
&\text{subject to} \\
&\dot{a}(t) = r\,a + w(t) - c(t), \text{ for } t \in \mathrm{T} \\
&a(0) = a_0 > \underline{\mathrm{a}} \text{ given} \\
&a(T) \geq \underline{a}
\end{aligned} \tag{P5}$$

Using the same approach as in the previous problem, we can prove that the solution is

$$c^*(t) = \left(\frac{(r-\gamma_c)\big(a_0 + h(0) - \underline{a}\,e^{-r\,T}\big)}{1-e^{(\gamma_c - r)T}}\right)e^{\gamma_c t}, \text{for every } t \in [0,T] \tag{6}$$

and

$$a^*(t) = e^{r\,t}a_0 + h(0) - \frac{a_0 + h(0) - \underline{a}\,e^{-r\,T}}{1 - e^{(\gamma_c - r)T}}\left(e^{r\,t} - e^{\gamma_c\,t}\right), \text{for every } t \in [0,T]. \tag{7}$$

where

$$h(0) = \int_0^T e^{-r\,t}\,w(t)\,dt$$

is the human capital of the household at time $t = 0$. Comparing with the previous problem, we see that while in the previous problem, consumption and savings essentially led net wealth to go from a_0 to $\underline{a}$, in this case, the initial total wealth is $a_0 + h(0)$ and the terminal wealth is still $\underline{a}$. This means that the present flow of consumption in this problem is much higher than in the former problem.

Furthermore, if human wealth can be seen as a collateral for net borrowing, it is possible that the lower limit on net financial wealth $\underline{a}$ might be reduced by the existence of human wealth which could be offered as a collateral.

3.5 Infinite horizons

We saw that the determination of the initial level of consumption $c(0)$, and therefore, the level of consumption is dependent on the horizon of the problem, T, and on the terminal constraints on wealth. In this section we present the infinite-horizon case.

There are two main justifications for considering the infinite horizon case: first, the existence of incomplete information on the terminal time, and second, the existence of concern by the household of the utility of its family beyond the life of those living at time $t = 0$. We call the second case the dynastic model.

The uncertain horizon In order to study this case we assume that the lifetime T is stochastic. Let the cumulative distribution of lifetime be $F(T) = \int_0^T f(t)dt \in (0,1)$ for $T \in (0,\infty)$ and the density function $f(t)$ follow a Poisson process, with the instantaneous probability of death μ, $f(t) = \mu\,e^{-\mu t}$. Then $F(T) = 1 - e^{-\mu T}$. Off course $F^{'}(t) = f(t)$, $F(0) = 0$ and $F(\infty) = 1$.

Let $\mathsf{U}(T) = \int_0^T u(c(t))\,e^{-\rho t}\,dt$ be the utility functional for an household having horizon T and let the intertemporal utility with a stochastic lifetime be given by the **expected utility functional**

$$\mathsf{U}[c] = \int_0^\infty f(T)\,\mathsf{U}(T)\,dT = \int_0^\infty f(T)\int_0^T u(c(t))\,e^{-\rho t}\,dt\,dT.$$

We can prove that

$$\mathsf{U}[c] = \int_0^\infty u(c(t))\, e^{-(\rho+\mu)t}\, dt. \tag{8}$$

To prove this observe that

$$\mathsf{U}[c] = \int_0^\infty F'(T)\, \mathsf{U}(T)\, dT.$$

Using integration by parts,[8] we have

$$\begin{aligned}
\mathsf{U}[c] &= F(T)\, U(T)\Big|_{T=0}^{\infty} - \int_0^\infty F(T)\, \mathsf{U}'(T)\, dT \\
&= F(\infty)\, U(\infty) - F(0)\, U(0) - \int_0^\infty F(T)\, e^{-\rho T} u(c(T))\, dT \quad \text{because } U'(T) = e^{-\rho T} u(c(T)) \\
&= U(\infty) - \int_0^\infty \left(1 - e^{-\mu T}\right) e^{-\rho T}\, u(c(T))\, dT \\
&= \int_0^\infty u(c(t))\, e^{-\rho t}\, dt - \int_0^\infty \left(1 - e^{-\mu T}\right) e^{-\rho T}\, u(c(T))\, dT \\
&= \int_0^\infty u(c(t))\, e^{-\rho t}\, dt - \int_0^\infty e^{-\rho t}\, u(c(t))\, dt + \int_0^\infty e^{-(\rho+\mu)\, t}\, u(c(t))\, dt \\
&= \int_0^\infty u(c(t))\, e^{-(\rho+\mu)t}\, dt.
\end{aligned}$$

Therefore, interpreting the discount factor as the sum of the rate of time preference plus the instantaneous mortality rate for an uncertain lifetime model with a Poisson distribution of the time of death, allows us to assume an utility functional for a single agent with an infinite horizon, as in equation (8).

The dynastic model Another interpretation for an infinite horizon is the dynastic interpretation is mathematically equivalent. In this case the economic agent can be seen as an household which cares for the utility not only of the present but for all future generations. In this case we have

$$\int_0^\infty u(c(t))\, e^{-\rho t}\, dt$$

We consider now the following problem, in which the rate of time preference can include the

[8] Assuming that some mathematical requirements are satisfied, implying the boundedness of $u(c)$.

mortality rate or not.

$$\begin{aligned}
&\max_{c} \int_0^{\infty} \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\, a + w(t) - c(t), \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&\lim_{t\to\infty} a(t)\, e^{-r\, t} \geq 0
\end{aligned} \tag{P6}$$

The terminal condition is called the non-Ponzi game condition, it means that in present value terms the household will not be a net debtor asymptotically.

Another interpretation is that the three constraints imply that the household consumption path is sustainable, or that it is solvent. In order to see this, solving the household budget constraint, $\dot{a}(t) = r\, a + w(t) - c(t)$, together with the initial condition, $a(0) = a_0$, yields

$$a(t) = e^{r\, t}\Big(a_0 + \int_0^t e^{-r\, s}\, w(s) - c(s) ds\Big).$$

Multiplying both sides by the discount factor $e^{-r\, t}$ and taking the limit to infinity, yields

$$\lim_{t\to\infty} e^{-r\, t} a(t) = a_0 + h(0) - \int_0^{\infty} e^{-r\, t}\, c(t) dt$$

where we used the definition of human capital at time $t = 0$,

$$h(0) \equiv \int_0^{\infty} e^{-r\, t}\, w(t) dt.$$

If the non-Ponzi game condition holds then we obtain the following intertemporal budget constraint

$$\int_0^{\infty} e^{-r\, t}\, c(t)\, dt \leq a_0 + h(0)$$

which means that the present value of consumption is smaller than the initial total wealth which is equal to the sum of the financial and human wealth.

From now on, we assume that $r > 0$ and w is constant. Then $h(0) = \dfrac{w}{r}$.

By employing the methods already presented in the previous lecture[9] we can obtain explicitly the optimal policy function

$$c^* = C(a) = (r - \gamma_a)\Big(a + \frac{w}{r}\Big).$$

[9]See https://pmbbrito.github.io/cursos/phd/am/am2021_ramsey.pdf.

If $\theta \geq 1$. the rate of growth of net wealth is smaller than the interest

$$\gamma_a \equiv \frac{r-\rho}{\theta} < r,$$

which implies implies that optimal consumption is a positive function of total net wealth, financial and human, and the propensity to save is endogenously determined as a function of the difference $r - \gamma_a$.

If we substitute optimal consumption in the budget constraint we have

$$\begin{aligned}\dot{a} &= r\,a + w - (r-\gamma_a)\big(a + \frac{w}{r}\big) \\ &= \gamma_a \left(a + \frac{w}{r}\right) \\ &= \gamma_a \left(a - \bar{a}\right)\end{aligned}$$

where the steady state level of net financial wealth is the symmetric of human wealth

$$\bar{a} = -h(0) = -\frac{w}{r}.$$

Solving this differential equation, taking $a(0) = a_0$ yields

$$a^*(t) = \bar{a} + (a_0 - \bar{a})\, e^{\gamma_a\, t}, \text{for } t \in [0, \infty)$$

or, equivalently

$$a^*(t) = -h(0) + (a_0 + h(0))\, e^{\gamma_a\, t}, \text{for } t \in [0, \infty).$$

Therefore, the following dynamics are possible:

1. if $r > \rho$ then $\gamma_a > 0$ and $\lim_{t\to\infty} a(t) = +\infty$ if the initial wealth is positive $(a_0 + h(0) > 0)$ it is equal to $-h(0)$ if $a_0 + h(0) = 0$ and it is $-\infty$ if the initial wealth is negative;

2. if $r = \rho$ then $\gamma_a = 0$ and the financial wealth is stationary at the initial level $a(t) = a_0$ for any t;

3. if $r < \rho$ then $\gamma_a < 0$ and $\lim_{t\to\infty} a(t) = -h(0)$ the agent will be a net debtor if there are no borrowing constraints.

What is the role of the parameter θ in the solution ? As we saw the higher θ is the lower is the EIS, which means that the cost of transfering consumption between moments in time, in utility terms, is higher. We see that although it does not affect the sign of γ_a, it reduces its absolute value. For any point in time it also increases the propensity to consume out of total net wealth, this means

that savings will be smaller instantaneously and the transfer of consumption among periods will be smaller as well. Therefore, it tends to generate a smoother behavior of consumption.

The previous results contrast with those we obtained for the finite horizon case. In particular, it is not possible to have an unbounded net asset position. The reason for this is related to the fact that the terminal condition refers to the infinity, and given the fact that it involves discounting at a rate which is higher, in absolute value, than the rate of growth of the wealth position, it allows for an unbounded evolution of net financial wealth. On the other hand, if the agent is more impatient than the market, i.e, if $\rho > r$, the solution for consumption looks similar to the problem for an agent which depletes a given stock of wealth (i.e, to problem (P2)).

Another fundamental aspect of introducing an infinite horizon is related to the existence of stationary solutions, i.e., solutions which do not change in time, as we will see in the next section.

3.6 Conclusion

We discussed previously the effect of existence of satiation or no satiation in maximizing consumption utility, the effect of the constraints given by initial wealth or borrowing constraints, how constraints would have an effect on optimal solution when the consumer has no satiation, the justification and the effect of assuming an infinite horizon.

We found that when there are constraints, initial or otherwise, there is intertemporal substitution in consumption and the path of consumption is responsive to the elasticity of intertemporal substitution.

4 Comparative dynamics for the time additive model

In aggregate terms, in actual economies, one would expect that no agent would have an unbounded net wealth level, both as a net creditor or as a net debtor, because the individual agent interest rate would eventually be endogenous to its level of wealth. This means that it will become a large agent, if it is s creditor, or eventually it will face a borrowing constraint, if it is a debtor.

This means that if we consider problem (P6) as representative of an aggregate economy the natural case to take is the second, that is the case in which $r = \rho$. This is indeed the simplest model for a small open economy facing perfect international capital markets.

In addition to the those already presented, another reason for considering infinite horizons is related to the use of this model as (or within) a business cycle model in which we are interested in

studying deviations from a stationary trend. We can show that the infinite horizon version of the previous models produces stationary solutions, i.e. steady states and deviation from steady states.

In this section we study the comparative dynamics for a shock in the non-financial income w, in order to discuss the theory provided by this model on the relationship between consumption and income. In addition, we want to have a comparison with the same type of results for the habit formation model that we present in the next section 5.

There are two types of changes in income with a bearing on the solution of this model: anticipated and non-anticipated changes. Anticipated changes are already incorporated in the solution of the model. If at a certain point in time consumption is changing it is because it is not at a steady state. Non-anticipated changes involve time changes because they alter the steady state of the problem.

The optimality conditions for problem (P6) comprise the instantaneous budget constraint, the adjoint equation, the static optimality condition and the initial level for the net wealth

$$\dot{a} = \rho\, a + w - C(q) \tag{9a}$$

$$\dot{q} = 0 \tag{9b}$$

$$c(t) = C(q) \equiv q(t)^{-\frac{1}{\theta}} \tag{9c}$$

$$a(0) = a_0. \tag{9d}$$

where we assume that $r = \rho$ and we want to study the local dynamics in the neighborhood of a steady state.

From equation (9d) we can label financial wealth a as a pre-determined variable and the adjoint variable q, or consumption c, which is monotonously related to q by equation (9c), as a non-predetermined variable.

Equation (9b) implies that q is constant, which implies that c is constant as well, as we saw in the last section. Therefore, there are potentially an infinite number of steady states, from equation (9a), comprising all combinations of a and q that satisfy

$$c = \rho\, a + w$$

However, from the fact that a is pre-determined we can tie down the steady state which interests us to be

$$\bar{c} = C(\bar{q}) = \rho\, a_0 + w.$$

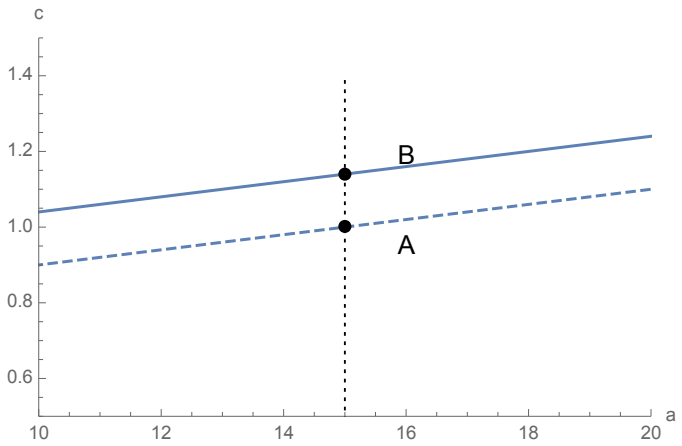

Figure 1: Effect of a non-anticipated increase in income. The line corresponds to the isocline for a, $c = w + \rho\, a$. A shock in income, w, moves the isocline up, leading the economy to jump from point A to point B.

This steady state value for c is dependent on the value of non-financial income w, which lead us to write $\bar{c} = \bar{c}(w)$.

A comparative dynamics exercise asks the following question: given an initial value of w,say w_0 what is the effect on the solution to the problem if w_0 increases to $w_1 = w_0 + dw$, starting from a steady state ?

In the appendix E we prove that the multipliers are

$$dc(t) = dw, \text{for any } t \in (0, \infty)$$
$$da(t) = 0, \text{for any } t \in (0, \infty)$$

that is, consumption immediately and completely adjusts to innovations in income, which means that they are perfectly correlated. There is no transitional dynamics.

A phase diagram is presented in Figure 1.

Therefore, in the benchmark infinite horizon model of the household behavior, in a deterministic setting, non-anticipated changes in income will be immediately spent in consumption, which means that this model displays a counterfactual perfect correlation between consumption and income and a potentially volatile behavior of consumption.

This is a consequence of the assumption $r = \rho$, which is what one would expect to be the case for a representative consumer in the long run.

Therefore we need some mechanism allowing for an incomplete translation of income shocks to

consumption expenditures which could generate a consumption adjustment closer to stylized facts. Unfortunately this comes at a cost of increasing the dimension of the model.

5 Intertemporally dependent preferences

There are several ways to introduce intertemporally dependent preferences. In this section we consider the habit formation model.

There are two types of models dealing with habit formation: the internal habit formation (also called habit persistence) model and the external habit formation model. In the first type of models consumer have an internal pattern of consumption that only changes marginally and in the second they follow an external pattern of consumption. While in the first the pattern of consumption, or habit, is built throughout time internally, in the second type of models it is an externality. This is why the second type of habits involve "going along with the Joneses' (see Abel, 1990). In this sense, classifying the two types of models within the same category can be misleading

Next we present the preferences under (internal) habit formation, in subsection 5.1 and in subsection 5.2 we extend the previous household consumption model with habit formation.

5.1 Preferences under habit formation

We saw, in section 2, that if the intertemporal utility functional is additive, in the utility of consumption for different moments in time, it displays intertemporally independent preferences, in the sense that the history of consumption until time t, i.e., $c^t = \big(c(s)\big)_{s=0}^{t}$, does not influence the valuation of consumption at time t, $u(c(t))$. We also saw that this leads to large shifts in consumption after innovations, which is counter-factual. In the **habit formation** utility functional past consumption affects the evaluation of consumption for every point in time.

Assume that the household has a consumption pattern, that we call habit, and denote by $h(t)$. Habits are formed from past consumption. Therefore, in this model, current consumption $c(t)$ has two effects: first, it is an immediate source of utility, and, second, it marginally changes the pattern of consumption. In this sense there is a "technology" in changing habits through current consumption which we assume to be linear and have a parameter η. However, habits record the past consumption history with some rate of decay which we assume to be also equal to η.

The following version is a simplified version of the model: [10]

$$\mathsf{U}[c] = \int_0^T u\big(c(t), h(t)\big)\, e^{-\rho\, t}\, dt$$

[10] The first papers are Wan, 1970 and Ryder and Heal, 1973.

where $h(t)$ is the habits at time t, which are

$$h(t) = e^{-\eta\, t}\Big(h_0 + \eta \int_0^t e^{\eta\, s}\, c(s)\, ds\Big)$$

where $\eta > 0$ represents both the effect present consumption in the stock of habits and the rate of decay (or forgetting) of habits. This is a stock (pre-determined) variable which can be seen as the solution to the problem

$$\begin{cases} \dot{h} = \eta\,(c(t) - h) & \text{for } t \in \mathrm{T} \\ h(0) = h_0 & \text{for } t = 0. \end{cases}$$

In this case the utility at time t is a function of the consumption and the habits, $u(t) = u\big(c(t), h(t)\big)$, where we assume that the marginal utility of consumption is positive $u_c(c,h) \equiv \dfrac{\partial u(c,h)}{\partial c} > 0$ but the marginal utility of habits is negative $u_h(c,h) \equiv \dfrac{\partial u(c,h)}{\partial h} < 0$. We also assume that the utility function is continuous and smooth.

Using the concepts introduced in section 2, the marginal utility for consumption at time t is now

$$\mathsf{U}_t \equiv \delta \mathsf{U}[c;t] = e^{-\rho t}\Big(u_c\big(c(t),h(t)\big) + \eta \int_t^{\bar{t}} e^{-(\eta+\rho)\,(s-t)} u_h\big(c(s),h(s)\big)\, ds\Big), \text{ for any } t \in \mathrm{T}.$$

where $\bar{t} = \max\ \mathrm{T}$. The intertemporal marginal rate of substitution between t_0 and $t_1 = t_0 + \tau$ is now

$$IMRS_{t_0,t_1} = \frac{e^{\rho\,\tau}\Big(u_c\big(c(t_0),h(t_0)\big) + \eta \int_{t_0}^{\bar{t}} e^{-(\eta+\rho)\,(s-t_0)} u_h\big(c(s),h(s)\big)\, ds\Big)}{u_c\big(c(t_1),h(t_1)\big) + \eta \int_{t_1}^{\bar{t}} e^{-(\eta+\rho)\,(s-t_1)} u_h\big(c(s),h(s)\big)\, ds}.$$

For a stationary consumption path, such that $c(t) = \bar{c}$ and $h(0) = \bar{c}$, implying $h(t) = \bar{c}$, for any $t \in \mathrm{T}$, we find [11]

$$\bar{\mathsf{U}}_t = \delta \mathsf{U}[\bar{c};t] = e^{-\rho t}\, \bar{\mathsf{U}}'$$

where we define

$$\bar{\mathsf{U}}' \equiv u_c(\bar{c}) + \frac{\eta}{\eta+\rho} u_h(\bar{c})$$

and assume that $\bar{\mathsf{U}}' > 0$. For any other moment in time $t' \neq t$ we have the marginal utility $\bar{\mathsf{U}}_{t'} = e^{-\rho t'}\, \bar{\mathsf{U}}'$.

This implies that the $IMRS_{t_0,t_1} = e^{\rho\,\tau}$ is the same as in the additive model, which means that preferences with habit formation display **impatience** as in the additive model.

[11] Setting $h_0 = \bar{c}$, we find $h(t) = \bar{c}$ for any $t \in \mathrm{T}$ and therefore, we write $u_c(\bar{c}) = u_c(\bar{c},\bar{c})$ and $u_h(\bar{c}) = u_h(\bar{c},\bar{c})$.

The change in marginal utility is now given by the second-order functional derivative

$$\mathsf{U}_{t_i,t_j} \equiv \delta^2 \mathsf{U}[c;t_i,t_j] = \\ = \eta\, e^{-\rho t_i}\Big(\int_{t_i} e^{-(\eta+\rho)(t-t_i)}\, u_{hc}\big(c(t),h(t)\big)\, dt + \eta \int_{t_j} e^{-(\eta+\rho)(t-t_i)-\eta(t-t_j)}\, u_{hh}\big(c(t),h(t)\big)\, dt\Big)$$

and for a stationary consumption path, we find

$$\bar{\mathsf{U}}_{t_i,t_j} = \eta\, e^{-\rho\, t_j - \eta(t_j-t_i)}\, \bar{\mathsf{U}}''$$

where we define

$$\bar{\mathsf{U}}'' \equiv u_{hc}(\bar{c}) + \frac{\eta}{2\eta+\rho} u_{hh}(\bar{c}).$$

The Uzawa-Allen elasticities, associated to a stationary consumption path, depend only on the time difference between t and t'

$$\epsilon_{t,t'} = -\frac{\bar{\mathsf{U}}_{t,t'}\,\bar{c}}{\bar{\mathsf{U}}_t} = e^{-(\rho+\eta)\,(t'-t)}\sigma_h(\bar{c})$$

where we define

$$\sigma_h(\bar{c}) = -\frac{\eta\,\bar{\mathsf{U}}''\,\bar{c}}{\bar{\mathsf{U}}'}.$$

As $e^{-(\rho+\eta)\,(t'-t)} \in (0,1)$ then we have intertemporal substitutability if $\bar{\mathsf{U}}'' < 0$, implying $\sigma_h(\bar{c}) > 0$, independence if $\bar{\mathsf{U}}'' = 0$, implying $\sigma_h(\bar{c}) = 0$, and intertemporal complementarity if $\bar{\mathsf{U}}'' > 0$, implying $\sigma_h(\bar{c}) < 0$.

As, in general $\sigma_h(\bar{c}) \neq 0$ then, using the definition of the intertemporal elasticity of substitution given in equation (3), yields

$$IES_{t_0,t_1} = \frac{1}{\sigma_h(\bar{c})}\Big(\frac{1+e^{-\rho\tau}}{1+e^{-\rho\tau}-2\,e^{-(\rho+\eta)\tau}}\Big).$$

Comparing with the additive model, in this case the IES can have any sign, depending on the elasticity $\sigma_h(\bar{c})$, and its magnitude is a function of the lag between the two moments, τ. If there is intertemporal substitution we see that $\lim_{\tau\to 0} IES_{t,t+\tau} = +\infty$ and if $\lim_{\tau\to\infty} IES_{t,t+\tau} = \dfrac{1}{\sigma_h(\bar{c})}$, which means that for any $0 < \tau < \infty$, the intertemporal elasticity of substitution will be smaller than the inverse of $\sigma_h(\bar{c})$, $IES_{t,t+\tau} > \dfrac{1}{\sigma_h(\bar{c})}$.

There are two benchmark utility functions displaying habit formation in the literature: the additive habits model (see Constantinides, 1990),

$$u(c,h) = v\big(c-\zeta\, h\big),\ \text{for}\ \zeta > 0$$

and the multiplicative habits model (see Carroll, 2000)

$$u(c,h) = v\big(c\,h^{-\zeta}\big),\ \text{for } 0<\zeta<1$$

where ζ measures the force of habits, i.e, the relative weight of habits as regards present consumption. In both models the utility of consumption is measured against the change of a monotonous function of habits.

If we assume multiplicative habits such that

$$u(c,h) = \frac{1}{1-\theta}\Big(\big(c\,h^{-\zeta}\big)^{1-\theta} - 1\Big) \tag{10}$$

and evaluate it at a stationary path such that $h(t) = c(t) = \bar{c}$, for any t, then we obtain

$$\begin{aligned}
\bar{\mathsf{U}}' &= \Big(\frac{\eta\,(1-\zeta)+\rho}{\eta+\rho}\Big)\,\bar{c}^{\,\zeta\,(\theta-1)-\theta} > 0\\
\bar{\mathsf{U}}'' &= \zeta\Big(\frac{(\theta-1)\big(\rho+\eta\,(2-\zeta)\big)+\eta}{2\,\eta+\rho}\Big)\,\bar{c}^{\,\zeta\,(\theta-1)-\theta-1} > 0
\end{aligned}$$

if $\theta \geq 1$ and $0<\zeta<1$. Therefore,

$$\sigma_h(\bar{c}) = -\eta\,\zeta\,\frac{(\eta+\rho)\Big((\theta-1)\big(\rho+\eta\,(2-\zeta)\big)+\eta\Big)}{(2\,\eta+\rho)\,(\eta\,(1-\zeta)+\rho)} < 0$$

which means that this model displays **intertemporal complementarity**.

Exercise Prove this.

Exercise Find the IES of the additive habit formation model where

$$u(c,h) = \frac{1}{1-\theta}\Big(\big(c-\zeta\,h\big)^{1-\theta} - 1\Big)$$

and find its intertemporal dependence properties.

5.2 Partial equilibrium under habit formation

The extension of the household problem (P6) with habit formation becomes:

$$\begin{aligned}
&\max_{c} \int_0^{\infty} u(c(t), h(t))\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\, a + w(t) - c(t), \text{ for } t \in \mathbb{R}_+ \\
&\dot{h}(t) = \eta \left(c - h\right) \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&h(0) = h_0 \text{ given} \\
&\lim_{t \to \infty} a(t)\, e^{-r\, t} \geq 0
\end{aligned} \tag{P7}$$

where, if we assume the multiplicative habit formation model, we have $u_c(c,h) > 0$, $u_h(c,h) < 0$, $u_{cc}(c,h) < 0$, $u_{ch}(c,h) = u_{hc}(c,h) > 0$ and $u_{hh}(c,h)$ has an ambiguous sign.

Exercise Compute the those derivatives for the general case and for the case in which $c = h$.

Observe that we now have two state variables, a and h and just one control variable c. This means that we have two initial conditions , one for each pre-determined (or state) variable, $a(0) = a_0$ and $h(0) = h_0$.

Assume from now on that $r = \rho$. This implies, using the intuition from subsection 3.5 that the consumption path is bounded, which implies that the stock of habits is bounded as well.

The current-value Hamiltonian function is now

$$\mathcal{H} = u(c,h) + q_a(\rho\, a + w - c) + q_h\, \eta\, (c - h)$$

where q_a is the adjoint variable associated to the stock of net wealth a and q_h is the adjoint variable associated with the stock of habits h.

Using the Pontriyagin's maximum principle the optimality conditions for problem (P7) are

$$\dot{a} = \rho\, a + w - c, \tag{11a}$$

$$\dot{h} = \eta\,(c - h), \tag{11b}$$

$$\dot{q}_a = 0, \tag{11c}$$

$$\dot{q}_h = (\rho + \eta)\, q_h - u_h(c, h), \tag{11d}$$

$$u_c(c(t), h(t)) = q_a(t) - \eta\, q_h(t), \tag{11e}$$

$$a(0) = a_0, \tag{11f}$$

$$h(0) = h_0 \tag{11g}$$

and the transversality conditions.

As in problem P6, with an intertemporally additive utility functional, the static arbitrage condition (11e) can be implicitly solved for optimal consumption. Consumption is not only a function of shadow value of net financial asset, as in problem P6, but it is also a function of the stock of habits and of its shadow value

$$c = C(h, q_a, q_h),$$

with partial derivatives

$$C_h = -\frac{u_{ch}(c,h)}{u_{cc}(c,h)} > 0,\ C_{q_a} = \frac{1}{u_{cc}(c,h)} < 0,\ C_{q_h} = -\frac{\eta(c,h)}{u_{cc}(c,h)} > 0.$$

Therefore consumption decreases with the shadow value of the financial asset but increases with both the level and the value of the stock of habit.

As we are interested in comparing the dynamic comparative statics properties of this model with the non-habit formation model we deal with the case in which the initial condition is a steady state, and introduce a perturbation in non-financial income from $w = w_0$ to $w = w_1 = w_0 + dw$.

As in that model equation (11c) implies that the steady state exists but there is potentially an infinite number of steady states. Furthermore, equation (11b), evaluated at the steady state yields $\bar{c} = \bar{h}$.

Anchoring again the steady state by the initial value of financial wealth a_0, a steady state only exists if the initial value of habits satisfies $\bar{h} = h_0 = \rho\, a_0 + w_0$, which we assume is the case from now on.

Therefore, the steady state, for $w = w_0$, is determined from the equations

$$\bar{a} = a_0 \tag{12a}$$

$$\bar{h} = h_0 = \rho\, a_0 + w_0 \tag{12b}$$

$$\bar{c}(w_0) = \rho\, a_0 + w_0 \tag{12c}$$

$$\bar{q}_h(w_0) = \frac{u_h(\bar{c}(w_0), h_0)}{\rho + \eta} \tag{12d}$$

$$\bar{q}_a(w_0) = u_c(\bar{c}(w_0), h_0) + \frac{\eta}{\rho + \eta}\, u_h(\bar{c}(w_0), h_0) \tag{12e}$$

This steady state, projected in the space (a, c) is shown by point A in Figure 2.

Assuming a multiplicative habits model with utility function

$$u(c, h) = \frac{\left(c\, h^{\zeta}\right)^{1-\theta} - 1}{1 - \theta}$$

we find the steady state values for q_a and q_h

$$\bar{q}_a = \frac{\rho + \eta\,(1 - \zeta)}{\eta + \rho}\left(w_0 + \rho a_0\right)^{-\theta(1-\zeta)-\zeta}$$

$$\bar{q}_a = -\frac{\zeta}{\eta + \rho}\left(w_0 + \rho a_0\right)^{-\theta(1-\zeta)-\zeta}$$

Exercise Prove this.

Exercise Find the steady state for the additive habits model.

The Jacobian of the MHDS, evaluated at the steady state is a four dimensional matrix which can be found in Appendix F. This Jacobian has four eigenvalues equal to zero, ρ (as in the additive model) and the pais

$$\lambda_s = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}\right)^2 - S},\ \lambda_u = \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2}\right)^2 - S}$$

If $S < 0$ then $\lambda_s < 0 < \lambda_u$ [12] and the model can display transition dynamics converging to a steady state.

[12] These two eigenvalues also satisfy $\lambda_s + \lambda_u = \rho$ and $\lambda_s\, \lambda_u = S$.

In the appendix F we prove that

$$S \equiv= -\eta\Big(\frac{(\rho+\eta)\bar{u}_{cc}+(2\eta+\rho)\big(\bar{u}_{hc}+\eta\bar{u}_{hh}\big)}{\bar{u}_{cc}}\Big).$$

Observe that, using the notation we used in subsection 5.1 we can write

$$S = -\frac{\eta\,(2\eta+\rho)}{\bar{u}_{cc}}\Big(\frac{\rho+\eta}{2\eta+\rho}\bar{u}_{cc}+\bar{\mathsf{U}}''\Big)$$

which means that S is negative if consumption is intertemporally substitutable or independent, i.e., $\bar{\mathsf{U}}'' \leq 0$, or if it is intertemporally complementary, i.e., $\bar{\mathsf{U}}'' > 0$, the concavity of the utility function $\bar{u}_{cc}$ dominates the intertemporal complementarity effect.

For the multiplicative we found that there is intertemporal complementarity bur we also have

$$S = -\frac{\zeta\left(\rho+\eta(1-\zeta)\right)\left(\theta\,(1-\zeta)+\zeta\right)}{\theta} < 0$$

which means that we have the last case: although there is intertemporal complementarity, if $0<\zeta<1$, it is dominated by the decreasing marginal utility relative to instantaneous consumption.

It can be shown that if there is an increase in the wage rate by $dw > 0$ the (linearly approximate) dynamics that unfolds is the following: at the time of the shock consumption increases discontinuously from point A to point B; this introduces a change in the stock of net wealth but also a change in the stock of habits; however, as the stock of habits only changes slowly, the increase in wage is not completely used in the purchase of goods, which generates positive savings; changes in savings increases the stock of net wealth which increases further consumption, the stock of habits and savings.

A new steady state, depicted as point C will only be reached when we have again $\bar{c}(w_1) = \bar{h}(w_1) = \bar{a}(w_1)$. The stability mechanism is brought about by the fact that there is a decay mechanism in habit formation (in equation $\dot{h} = \eta\,(c-h)$) and the degenerate nature of the model is solved not by anchoring the solution to a_0 as in the non-habit formation model, but by a steady state relationship between the stock of habits and net wealth by $\bar{h} = \rho\,\bar{a} + w$.

The lower diagram in Figure 2 show the (approximate) trajectories for income $y(t) = \rho a(t) + w$ and consumption after the shock in non-financial income. As can be seen their behavior is positively correlated, but consumption has a slower adjustment, for the reasons just explained.

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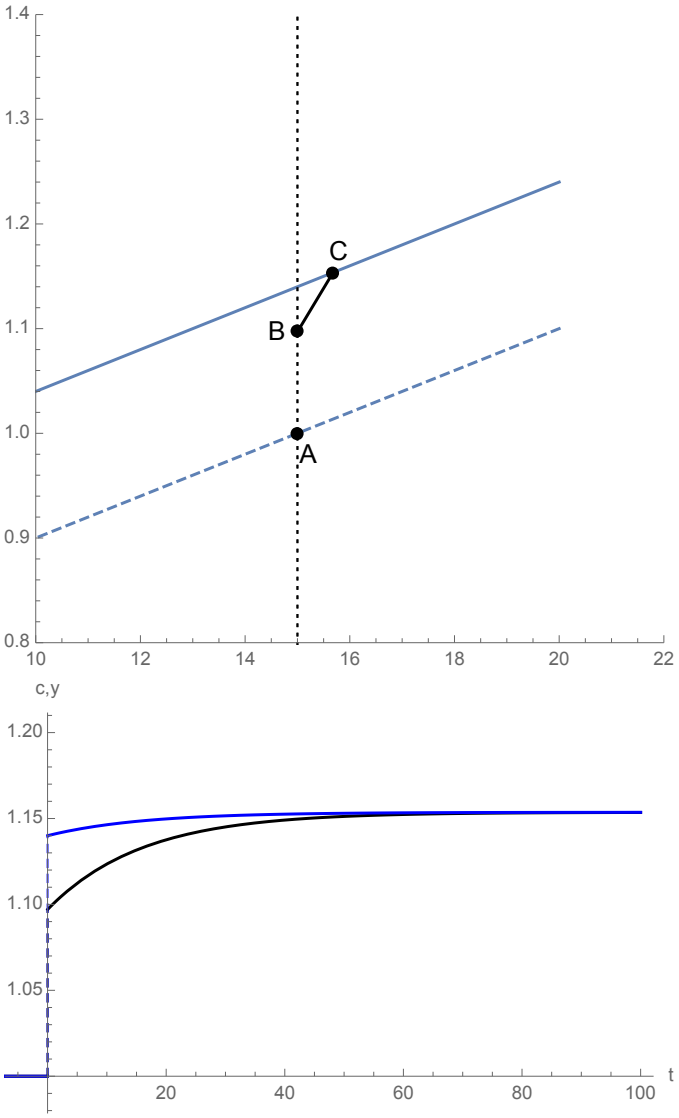

Figure 2: Effect of a non-anticipated increase in income in the habit formation model

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A functional derivatives

There are several different ways to present the idea of the derivative of a functionaldeal, some more mathematically correct and some more informal, as is the case in physics or mechanics. We follow Gel'fand and Fomin, 1963 which presents a good compromise between those two approaches.

Assume we have the space of functions $\mathcal{F}$, i.e., a collection of functions sharing some common property, for instance, continuity, differentiability, boundedness, etc. Every element of $\mathcal{F}$, for instance f, is a mapping between a space X and a subset of the space of real numbers, that is $f : \mathrm{X} \subseteq \mathbb{R} \to \mathbb{R}$. A **functional** is a mapping between the space of functions and the set of real numbers $\mathsf{F} : \mathcal{F} \to \mathbb{R}$.

The following functionals are common examples in economics:

$$\mathsf{F}_1[f] = \int_{\mathrm{X}} f(x)\, dx,$$

$$\mathsf{F}_2[f] = \int_{\mathrm{X}} u(f(x), x)\, dx$$

where $u(\cdot)$ is a function, or

$$\mathsf{F}_3[f] = \int_{\mathrm{X}} u(f'(x), f(x), x)\, dx$$

where $f'(\cdot)$ is the derivative of function $f(\cdot)$.

There are two concepts of derivatives related to functionals.

Consider a perturbation in function f from $f \mapsto f + \delta f$. A Frechet derivative of functions $\mathsf{F}[f]$, denoted by $\delta\mathsf{F}[f]$ is defined from

$$\lim_{\delta f \to 0} \frac{||\mathsf{F}[f + \delta f] - \mathsf{F}[f] - \delta\mathsf{F}[f]||}{\delta f} = 0$$

where $||\cdot||$ is the norm of space $\mathcal{F}$.

A more useful concept is the concept is the Gâteaux derivative. It is an extension of the directional derivative for functions. Assume that we introduce a perturbation on function $f \mapsto f + \varepsilon\varphi$ in which φ is a function, $\varphi : X \to \mathbb{R}$ and $\varepsilon > 0$ is a number.

The first variation of a functional is

$$d\mathsf{F}[f; \varphi] = \mathsf{F}[f + \eta\, \varepsilon] - \mathsf{F}[f]$$

and the functional derivative (in the GâteUx sense) is analogous to the concept of derivative for functions:

$$\delta\mathsf{F}[f; \varphi] = \lim_{\varepsilon \to 0} \frac{d\mathsf{F}[f; \varphi]}{\epsilon}.$$

If the functional involves more than one function we may write $\delta_{f_1}\mathsf{F}[f_1, f_2; \varphi_1, \varphi_2]$.

In regular cases, in order apply this concept, we perform a first order Taylor approximation

$$\mathsf{F}[f + \epsilon\varphi] = \mathsf{F}[f] + \delta\mathsf{F}[f; \varphi]\,\varepsilon + o(\varphi)$$

where $\lim_{\varepsilon\to 0} \dfrac{||o(\varphi)||}{\varepsilon} = 0$.

Performing a second order Taylor approximation we find

$$\mathsf{F}[f + \epsilon\,\varphi] = \mathsf{F}[f] + \delta\mathsf{F}[f; \varphi]\,\varepsilon + \delta^2\mathsf{F}[f; \varphi^2]\,\varepsilon^2 + o(\varphi^2).$$

However, we can distinguish between the "own" second order functional derivative

$$\delta\mathsf{F}[f + \varepsilon\,\varphi; \varphi] = \delta\mathsf{F}[f; \varphi] + \delta^2\mathsf{F}[f; \varphi^2]\,\varepsilon^2 + o(\varphi^2)$$

where $\lim_{\varepsilon\to 0} \dfrac{||o(\varphi^2)||}{\varepsilon} = 0$, and "crossed" second order derivative

$$\delta\mathsf{F}[f + \varepsilon\,\varphi'; \varphi] = \delta\mathsf{F}[f; \varphi] + \delta^2\mathsf{F}[f; \varphi, \varphi']\,\varepsilon^2 + o(\varphi\,\varphi')$$

where $\lim_{\varepsilon\to 0} \dfrac{||o(\varphi\,\varphi')||}{\varepsilon} = 0$

For the previous examples, and assuming that $\varphi(x) = 0$ for $x \in \partial\mathrm{X}$, where $x \in \partial\mathrm{X}$ is the boundary of X, we have

$$\delta\mathsf{F}_1[f; \varphi] = \int_X f(x)\,\varphi(x)\,dx,$$

$$\delta\mathsf{F}_2[f; \varphi] = \int_X \frac{\partial U\big(f(x), x\big)}{\partial f}\,\varphi(x)\,dx$$

and, if f belongs to a space of differentiable functions, which means that so φ does,

$$\delta\mathsf{F}_3[f; \varphi] = \int_X \frac{\partial U\big(f'(x), f(x), x\big)}{\partial f'}\,\varphi(x) + \frac{\partial U\big(f'(x), f(x), x\big)}{\partial f}\,\varphi'(x)\,dx.$$

For the second case we have the second order functional derivative

$$\delta^2\mathsf{F}_3[f; \varphi] = \int_X \frac{\partial^2 U\big(f(x), x\big)}{\partial f^2}\,\varphi^2(x)\,dx.$$

A particular case of perturbation is the "spike" variation. In this case the functional derivative is sometimes called the Volterra derivative.

Consider an element of the domain of function f, say x' and introduce the variation $df(x) = 0$ if $x \neq x'$ and $df(x) = \varepsilon$ if $x = x'$. We can write this as a perturbation $f \mapsto f + \varepsilon\,\delta(x - x')$ where δ is Dirac's delta generalized function. It has the property $\int_{\mathrm{x}} \delta(x - x')\, f(x)\, dx = f(x')$.

Applying to the previous examples, we find

$$\delta \mathsf{F}_1[f; x'] = \int_X f(x)\, \delta(x - x')\, dx = f(x') \tag{13}$$

$$\delta \mathsf{F}_2[f; x'] = \int_X \frac{\partial U\big(f(x), x\big)}{\partial f}\, \delta(x - x')\, dx = \frac{\partial U\big(f(x), x\big)}{\partial f}\Big|_{x=x'}. \tag{14}$$

B Necessary conditions for problem (P2)

We introduce the **functional**

$$\mathsf{U}([c], a_0) = \int_0^T u(c(t))\, e^{-\rho t} - \lambda(t)\big(\dot{a}(t) + c(t)\big)\, dt$$

where we introduce an adjoint multiplier $\lambda : \mathrm{T} \to \mathbb{R}_+$. Its introduction involves a penalization associated to the reduction in value brought about by the budget constraint. Because are now constrained by the initial value of the stock, we call value **function** to

$$V(a_0) = \max_c \mathsf{U}([c], a_0) = \mathsf{U}([c^*], a_0) = \int_0^T u(c^*(t))\, e^{-\rho t} - \lambda(t)\big(\dot{a}^*(t) + c^*(t)\big)\, dt.$$

Assume we know the optimal path $\big(c^*(t), a^*(t)\big)_{t \in \mathrm{T}}$.

We introduce now perturbations in both functions $c^*(t) \to c^*(t) + \varphi_c(t)$ and $a^*(t) \to a^*(t) + \varphi_a(t)$, such that $\varphi_a(0) = 0$, because $a^*(0) = a_0$ is not free. The value functional is now

$$\mathsf{U}([c^* + \varphi_c], a_0) = \int_0^T u(c^*(t) + \varphi_c(t))\, e^{-\rho t} - \lambda(t)\big(c^*(t) + \varphi_c(t) + \dot{a}^*(t) + \dot{\varphi}_a(t)\big)\, dt.$$

The first variation becomes,

$$\begin{aligned}
\delta \mathsf{U}([c^*]) &= \int_0^T \big(u'(c^*(t)\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t) - \lambda(t)\, \dot{\varphi}_a(t)\, dt \\
&= \int_0^T \big(u'(c^*(t)\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t)\, dt - \int_0^T \lambda(t)\, \dot{\varphi}_a(t) dt \\
&= \int_0^T \big(u'(c^*(t)\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t)\, dt - \lambda(t)\varphi_a(t)\Big|_{t=0}^T + \int_0^T \dot{\lambda}(t)\varphi_a(t) dt \\
&= \int_0^T \big(u'(c^*(t))\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t)\, dt - \lambda(T)\varphi_a(T) + \int_0^T \dot{\lambda}(t)\varphi_a(t) dt
\end{aligned}$$

Using integration by parts and the fact that $\varphi_a(0) = 0$. At the optimum $\big(c^*(t), a^*(t)\big)_{t\in \mathrm{T}}$ is such that $\delta \mathsf{U}([c^*]) = 0$. The first-order conditions are thus: $u'(c^*(t)\, e^{-\rho t} - \lambda(t) = \dot{\lambda}(t) = 0$, for $t \in [0, T]$ and $\lambda(T) = 0$. Integrating $\dot{\lambda}(t) = 0$ we find $\lambda(t) =$ constant for every $t \in [0, T]$. but as $\lambda(T) = 0$ then $\lambda(t) = 0$ for every $t \in [0, T]$. Therefore $u'(c^*(t)\, e^{-\rho t} = 0$ for every $t \in [0, T]$ as in (P1:foc).

C Necessary conditions for problem (P3)

The (penalized) utility functional is

$$\mathsf{U}([c], a_0) = \int_0^T u(c(t))\, e^{-\rho t} - \lambda(t)\big(\dot{a}(t) + c(t)\big) + \eta(t)(a(t) - \underline{a})\, dt$$

where we introduce a multiplier $\eta : \mathrm{T} \to \mathbb{R}_+$ associated to the instantaneous constraint on a such that the complementary slackness conditions hold

$$\eta(t) \geq 0,\ \eta(t)(a(t) - \underline{a}) = 0,\ \text{for every}\, t \in \mathrm{T}.$$

Using the same method as in section B the perturbed value functional is

$$\mathsf{U}([c^*+\varphi_c], a_0) = \int_0^T u(c^*(t)+\varphi_c(t))\, e^{-\rho t} - \lambda(t)\big(c^*(t)+\varphi_c(t)+\dot{a}^*(t)+\dot{\varphi}_a(t)\big) + \eta(t)(a^*(t)+\varphi_a(t)-\underline{a})\, dt.$$

Then, using the same procedure as before

$$\delta \mathsf{U}([c^*]) = \int_0^T \big(u'(c^*(t))\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t)\, dt - \lambda(T)\varphi_a(T) + \int_0^T \big(\dot{\lambda}(t) + \eta(t)\big)\varphi_a(t) dt.$$

Therefore the f.o.c are $u'(c^*(t))\, e^{-\rho t} = \lambda(t)$, $\dot{\lambda}(t) = -\eta(t)$, for $t \in [0, T]$, together with the complementary slackness conditions. However at $t = T$ we have also $\lambda(T) \geq 0$ and $\lambda(T)(a(T)-\underline{a}) = 0$, which is only possible if $\lambda(T^-) = \eta(T)$, or there is a discontinuity on $\lambda(t)$ at $t = T$.

D Necessary conditions for problem (P4)

We can find the necessary (in this case also sufficient) optimality conditions for problem (P4) by using the Pontriyagin maximum Principle. As the Hamiltonian function is

$$H = \frac{c^{1-\theta} - 1}{1 - \theta} + q\,(r\, a - c)$$

we have

$$\begin{cases} \dot{a} = r\,a - c & \text{for } t \in [0,T] \\ \dot{c} = \gamma_c\,c & \text{for } t \in [0,T] \\ a(0) = a_0 \text{ given} & \text{for } t = 0 \\ c(T)^{-\theta}(a(T) - \underline{a}) = 0 & \text{for } t = T \end{cases}$$

where the rate of growth of consumption is $\gamma_c \equiv \dfrac{r-\rho}{\theta}$. Solving the Euler equation we have $c(t) = c(0)\,e^{\gamma_c t}$. Substituting in the budget constraint, together with the initial condition yields

$$a(t) = e^{rt}\Big(a_0 + \frac{c(0)}{\gamma_c - r}\big(1 - e^{\gamma_c - r)\,t}\big)\Big), \text{ for } t \in [0,T]$$

If $c(0) > 0$ and finite, then $c(T) > 0$ and finite, which implies that the transversality constraint only holds if $a(T) = \underline{a}$. Therefore, we can find $c(0)$ by solving the equation

$$\underline{a}\,e^{rT} = a_0 + \frac{c(0)}{\gamma_c - r}\big(1 - e^{\gamma_c - r)\,T}\big)$$

Then we obtain equations (4) and (5).

E Comparative dynamics

Assume we have a non-linear dynamic system

$$\dot{X} = F(X, \varphi)$$

where φ is an exogenous variable and $X = (x_1, x_2)$ in which x_1 is pre-determined and x_2 is non-predetermined.

Let the exogenous variable takes the value φ_0, and let the associated steady state be $\bar{X}(\varphi_0)$.

Now consider a variation in the exogenous variable from φ_0 to $\varphi_1 = \varphi_0 + d\varphi$. If the syatem is at the steady state $\bar{X}(\varphi_0)$ it will be perturbed away from it. Let $dX(t) = X(t) - \bar{X}(\varphi_0)$ be the variation of X when away from the steady state.

The effects resulting from the perturbation $d\varphi$ can be studied from the solutions of the **variational system**. Taking a time derivative, and observing that $d\dot{X}(t) = \dot{X}$ (because $\dot{\bar{X}} = 0$, yields the linear ordinary differential equation

$$\dot{X} = \bar{F}_x(\varphi_0)\,dX(t) + \bar{F}_\varphi(\varphi_0)\,d\varphi.$$

where the Jacobians are

$$\bar{F}_x(\varphi_0) \equiv F_x\big(\bar{X}(\varphi_0), \varphi_0\big),\ \bar{F}_\varphi(\varphi_0) \equiv F_\varphi\big(\bar{X}(\varphi_0), \varphi_0\big).$$

The **comparative dynamics multipliers** are the solutions, $dX(t)$ to this system.

For MHDS systems with only one pre-determined variable (or state variable) the Jacobian $\bar{F}_x(\varphi_0)$ has two eigenvalues, λ_s and λ_u, that satisfy the relationship $\lambda_s \leq 0 < \lambda_u$. This means that the steady state is a saddle point, if $\lambda_s < 0$ or an unstable saddle-node, if $\lambda_s = 0$.

Therefore, two generic cases can occur, which have consequences on the method for determining $dX(t)$, depending on the Jacobian having a non-zero or a zero determinant. In the first case, we have $\lambda_s < 0$ and the dynamics will not depend on $x_1(0)$, the initial value of the pre-determined variable and in the second case $\lambda_s = 0$ and the dynamics will depend on $x_1(0)$.

Next we deal with the two cases separately.

Non-zero eigenvalues case

As $\det\big(\bar{F}_x(\varphi_0)\big) < 0$ then the Jacobian has a classic inverse, $\bar{F}_x(\varphi_0)^{-1}$, which allows us to determine the **long-run multipliers** as

$$d\bar{X} = -\bar{F}_x(\varphi_0)^{-1}\, \bar{F}_\varphi(\varphi_0)\, d\varphi = X_\varphi(\varphi_0)\, d\varphi$$

and the general solution to the variational system is

$$dX(t) = d\bar{X} + k_s\, P^s\, e^{\lambda_s\, t} + k_u\, P^u\, e^{\lambda_u\, t} \tag{15}$$

where P^s and P^u are the eigenvectors associated to the eigenvalues $\lambda_s < 0$ and $\lambda_u > 0$, respectively, and k_s and k_u are two arbitrary constants.

The two arbitrary constants provide us with two degrees of freedom allowing us to introduce two properties in the solution: first, we can force it converge to a new steady state $\bar{X}(\varphi_1)$, and, second, to make the pre-determined variable x_1 be continuous at the time of the shock, such that $dx_1(0) = x_1(0) - \bar{x}_1(\varphi_0) = 0$. The state variable, while being constant at the time of the shock, will start to change as a consequence of the shock, thus $\dot{dx}_1(0) \neq 0$.

The first condition is satisfied if we set set $k_u = 0$, which yields $dX(t) = d\bar{X} + k_s\, P^s\, e^{\lambda_s\, t}$, or, invector notation,

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} d\bar{x}_1 \\ d\bar{x}_2 \end{pmatrix} + k_s \begin{pmatrix} P_1^s \\ P_2^s \end{pmatrix} e^{\lambda_s\, t}.$$

The second condition is satisfied if, at time $t = 0$, we set $dx_1(0) = 0$, or equivalently, $d\bar{x}_1 + k_s\, P_1^s = 0$. Solving for the other arbitrary constant yields $k_s = -\dfrac{d\bar{x}_1}{P_1^s}$.

Substituting both constants in equation (15) allows us to obtain the **comparative dynamics variations**

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} d\bar{x}_1\big(1 - e^{\lambda_s\, t}\big) \\ d\bar{x}_2 - d\bar{x}_1\Big(\dfrac{P_2^s}{P_1^s}\Big)\, e^{\lambda_s\, t} \end{pmatrix}.$$

Recalling that $dX(t) = X(t) - \bar{X}(\varphi_0)$ we have equivalently

$$\begin{aligned} x_1(t) &= \bar{x}_1(\varphi_0) + d\bar{x}_1\big(1 - e^{\lambda_s\, t}\big) \\ x_2(t) &= \bar{x}_2(\varphi_0) + d\bar{x}_2 - d\bar{x}_1\Big(\frac{P_2^s}{P_1^s}\Big)\, e^{\lambda_s\, t}. \end{aligned}$$

The meaning of this formula is the following: assuming that at time $t = 0$ the economy is at a steady state associated to the level of the exogenous variable φ_0, $(\bar{x}_1(\varphi_0), \bar{x}_2(\varphi_0))$, a change in the exogenous variable to level $\varphi_1 = \varphi_0 + d\varphi$ changes the steady state by $(d\bar{x}_1, d\bar{x}_2)$; as the variable x_1 is pre-determined the adjustment is not immediate; the variables $(x_1(t), x_2(t))$ trace out the path of the economy following that shock.

Evaluating for $t \to \infty$ yields the **long-run multipliers**

$$\frac{X(\infty) - \bar{X}(\varphi_0)}{d\varphi} = \frac{d\bar{X}}{d\varphi} = \begin{pmatrix} \dfrac{d\bar{x}_1}{d\varphi} \\ \dfrac{d\bar{x}_2}{d\varphi} \end{pmatrix}, \tag{16}$$

and evaluating at $t = 0$ we obtain the **impact multiplier for the non-predetermined variable**

$$\frac{x_2(0) - \bar{x}_2(\varphi_0)}{d\varphi} = \frac{P_1^s\, d\bar{x}_2 - d\bar{x}_1\, P_2^s}{P_1^s\, d\varphi}.$$

The difference between the initial and the (approximated) final steady state after the shock in φ, which we denote by $d\bar{x}_i = \bar{x}_i(\varphi_1) - \bar{x}_i(\varphi_0)$, for $i = 1, 2$, for the two variables have a close relationship,

$$P_1^s\,\big(x_2(t) - \bar{x}_2(\varphi_1)\big) = P_2^s\,\big(x_1(t) - \bar{x}_1(\varphi_1)\big),$$

which is given by the slope of the eigenspace associated to the negative eigenvalue (the stable eigenspace).

In the presence of a zero eigenvalue

When $\det\big(\bar{F}_x(\varphi_0)\big) = 0$ the Jacobian has eigenvalues $\lambda_s = 0 < \lambda_u$ and there is a not a classic inverse for the Jacobian. We can use the Moore-Penrose inverse to determine the long run multipliers

$$d\bar{X} = -\bar{F}_x(\varphi_0)^+ \bar{F}_\varphi(\varphi_0)\, d\varphi + \Big(I - \bar{F}_x(\varphi_0)^+\, \bar{F}_x(\varphi_0)\Big)\, Z$$

where I is the identity matrix, $Z = (z_1, z_2)^\top$ is a vector of constants and we use

$$\bar{F}_x(\varphi_0) = P\,\Lambda\,P^{-1}\;\bar{F}_x(\varphi_0)^+ = P\,\Lambda^+\,P^{-1}$$

where the Jordan form, the Moore-Penrose inverse and the eigenvector matrices are

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_u \end{pmatrix},\; \Lambda^+ = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{\lambda_u} \end{pmatrix},\; P = \begin{pmatrix} P_1^s & P_1^u \\ P_2^s & P_2^u \end{pmatrix}.$$

Because, differently from the classic inverse $\bar{F}_x(\varphi_0)^+\,\bar{F}_x(\varphi_0) \neq I$ then the expression for the multipliers would allows us to obtain a linear equation in one of the elements of vector Z, say z_2. We can determine it by using the predetermine nature of x_1 by setting $d\bar{x}_1 = 0$.

The solution to the variational system is now

$$dX(t) = d\bar{X} + k_s\,P^s + k_u\,P^u\,e^{\lambda_u t}$$

where

$$d\bar{X} = \begin{pmatrix} 0 \\ d\bar{x}_2 \end{pmatrix}$$

which again contains two arbitrary constants, k_s and k_u. To eliminate unbounded trajectories, we set again $k_u = 0$ and determine k_s such that $dx_1(0) = 0$. This yields the variations

$$dx_1(t) = 0,\ \text{for all } t \in [0, \infty)$$
$$dx_2(t) = d\bar{x}_2,\ \text{for all } t \in [0, \infty)$$

where the shock in φ is completely absorbed by x_2. This means that the values of the perturbed variables are

$$x_1(t) = x_{1,0},\ \text{for all } t \in [0, \infty)$$
$$x_2(t) = \bar{x}_2(\varphi_1),\ \text{for all } t \in (0, \infty)$$

where we set $d\bar{x}_2 = \bar{x}_2(\varphi_1, x_{1,0}) - \bar{x}_2(\varphi_0, x_{1,0})$ because, as we saw in the main text that the value of the steady state for the non-predetermined variable depends on the initial value of the predetermined variable. This means that the non-predetermined immediately "jumps" to the new steady state.

Application to problem P6

For the problem having first-order conditions in equations (9a)-(9d) we have the initial steady state

$$\bar{X}(w_0) = \begin{pmatrix} \bar{a}_0 \\ \bar{q}_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ (\rho a_0 + w_0)^{-\theta}) \end{pmatrix}$$

and the Jacobian for an increase in the wage rate $dw = w_1 - w_0$ is

$$\begin{pmatrix} \dot{a} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \rho & -C'(\bar{q}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} da(t) \\ q(t) \end{pmatrix} + \begin{pmatrix} dw \\ 0 \end{pmatrix}.$$

The first Jacobian has eigenvalues $\lambda_s = 0$ and $\lambda_u = \rho$, which means that we have to use the formulas derived for the case in which there is one zero eigenvalue.

In order to find the long run variation introduced by the shock in w, from equation (16), we have to do some preliminary work: we find the

$$P = \begin{pmatrix} C'(\bar{q}_0) & 1 \\ \rho & 0 \end{pmatrix}$$

the Moore-Penrose inverse of the Jacobian

$$\bar{F}_x(w_0)^+ = P\,\Lambda^+\,P^{-1} = \begin{pmatrix} C'(\bar{q}_0) & 1 \\ \rho & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\lambda_u} \end{pmatrix} \frac{1}{\rho} \begin{pmatrix} 0 & 1 \\ \rho & -C'(\bar{q}_0) \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} 1 & -\frac{C'(\bar{q}_0)}{\rho} \\ 0 & 0 \end{pmatrix}$$

and

$$I - \bar{F}_x(w_0)^+\,\bar{F}_x(w_0) = \begin{pmatrix} 0 & -\frac{C'(\bar{q}_0)}{\rho} \\ 0 & 0 \end{pmatrix}$$

Therefore, the general expression for the long run multipliers is

$$\begin{pmatrix} d\bar{a} \\ d\bar{q} \end{pmatrix} = \begin{pmatrix} -\dfrac{dw}{\rho} + \frac{C'(\bar{q}_0)}{\rho}\, z_2 \\ z_2 \end{pmatrix}$$

where z_2 is an arbitrary constant.

As we require $d\bar{a} = 0$ then $z_2 = \dfrac{dw}{C'(\bar{q}_0)}$ then

$$\begin{pmatrix} d\bar{a} \\ d\bar{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{dw}{C'(\bar{q}_0)} \end{pmatrix}.$$

We obtain the short run variations from

$$\begin{pmatrix} da(t) \\ dq(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{dw}{C'(\bar{q}_0)} \end{pmatrix} + k_s \begin{pmatrix} C'(\bar{q}_0) \\ \rho \end{pmatrix}$$

Setting again $da(t) = 0$ yields $k_s = 0$, and, therefore, the short-run variations are

$$\begin{pmatrix} da(t) \\ dq(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{dw}{C'(\bar{q}_0)} \end{pmatrix}.$$

As

$$dc(t) = d\bar{c} = C'(\bar{q}_0)\, dq(t) = \frac{C'(\bar{q}_0)}{C'(\bar{q}_0)}\, dw = dw$$

where $dw = w_1 - w_0$.

Writing $\bar{c}(w_1) = d\bar{c} + \bar{c}(w_0)$ yields the linear approximation for the behavior of consumption after the shock $c(t) = \bar{c}(w_0) + d\bar{c} = \bar{c}(w_1) = \rho\, a_0 + w_1$. We conclude that consumption changes discontinuously from $\bar{c}(w_0) = \rho\, a_0 + w_0$ to $\bar{c}(w_1) = \rho\, a_0 + w_1$, as shown in Figure 1.

F Comparative dynamics for the habit formation model

We start from the steady state in equations (12a)-(12e), for the initial level of the exogenous variable $w = w_0$ and consider a positive change to $w_1 = w_0 + dw$.

Next we introduce the notation for the partial derivatives $\bar{u}_i = u_i\big(\bar{c}(w_0), \bar{h}(w_0)\big)$ for $i = c, h$ and $\bar{u}_{ij} = u_{ij}\big(\bar{c}(w_0), \bar{h}(w_0)\big)$ for $i, j = c, h$. We define accordingly $\bar{C}_h$, $\bar{C}_{q_a}$ and $\bar{C}_{q_h}$.

Using this notation for the partial derivatives, evaluated at the initial steady state, and the notation in section D for the Jacobians we obtain the Jacobian

$$\bar{F}_x(w_0) = \begin{pmatrix} \rho & -\bar{C}_h & -\bar{C}_{q_a} & -\bar{C}_{q_h} \\ 0 & \eta\big(\bar{C}_h - 1\big) & \eta\, \bar{C}_{q_a} & \eta\, \bar{C}_{q_h} \\ 0 & 0 & 0 & 0 \\ 0 & -\big(\bar{u}_{hc}\, \bar{C}_h + \bar{u}_{hh}\big) & -\bar{u}_{hc}\, \bar{C}_{q_a} & \rho + \eta - \bar{u}_{hc}\, \bar{C}_{q_h} \end{pmatrix}$$

This Jacobian has the characteristic equation$\det\big(\bar{F}_x(w_0) - \lambda\, I\big) = 0$. Expanding, yields the polynomial equation

$$\lambda\,(\lambda - \rho)\big(\lambda^2 - \rho\,\lambda + S\big) = 0.$$

This is because $\mu\, \bar{C}_h = \bar{u}_{hc}\, \bar{C}_{q_h}$ and we have

$$\begin{aligned} S &= \eta\,\Big(\big(\bar{C}_h - 1\big)\,\big(\rho + \eta - \bar{u}_{hc}\, \bar{C}_{q_h}\big) + \bar{C}_{q_h}\big(\bar{u}_{hc}\, \bar{C}_h + \bar{u}_{hh}\big) \\ &= \eta\,(\rho + \eta)\,\Big(\bar{C}_h - 1 + \frac{\bar{C}_{q_h}}{\rho + \eta}\big(\bar{u}_{hc} + \bar{u}_{hh}\big)\Big) \\ &= -\eta\Big(\frac{(\rho + \eta)\bar{u}_{cc} + (2\eta + \rho)\big(\bar{u}_{hc} + \eta \bar{u}_{hh}\big)}{\bar{u}_{cc}}\Big) \end{aligned}$$

There are four real eigenvalues $\{\lambda_s, 0, \rho, \lambda_u\}$ where

$$\lambda_s = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}\right)^2 - S}$$
$$\lambda_u = \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2}\right)^2 - S}$$

If $S < 0$ then $\lambda_s < 0 < \rho < \lambda_u$ and the steady state is a degenerate saddle-point. Additionally we have $\lambda_s + \lambda_u = \rho$ and $\lambda_s \, \lambda_u = S$.

Looking at the expression for S, we can write it as

$$S = -\frac{\eta\pi(\underline{u})}{\bar{u}_{cc}}$$

where $\pi(\underline{u}) \equiv (\rho+\eta)\bar{u}_{cc} + (2\eta+\rho)\big(\bar{u}_{hc} + \eta\bar{u}_{hh}\big) = (\rho+\eta)\bar{u}_{cc} + (2\eta+\rho)\bar{\mathsf{U}}''$. Then, $S < 0$ if and only if $\pi(\underline{u}) < 0$ which requires $\bar{\mathsf{U}}'' < -\dfrac{\rho+\eta}{2\eta+\rho}\bar{u}_{cc}$ which only holds if consumption is intertemporally substitutable or independent and, if there is intertemporal complementarity, it is not to large compared with the concavity as regards consumption c.

From now on we assume this condition holds.

As we have a zero eigenvalue we can adapt the method explained in the last section. The generalized long-run multipliers are

$$d\bar{X} = -\bar{F}_x(w_0)^+ \bar{F}_w(w_0)\, dw + \Big(I - \bar{F}_x(w_0)^+ \, \bar{F}_x(w_0)\Big)\, Z$$

where the Jacobian for the exogenous variable is

$$\bar{F}_w(w_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

and the Moore-Penrose inverse is $\bar{F}_x(w_0)^+ = P\,\Lambda^+\,P^{-1}$ where

$$\Lambda^+ = \begin{pmatrix} \dfrac{1}{\lambda_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\rho} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\lambda_u} \end{pmatrix},$$

and the eigenvector matrix concatenating the eigenvectors in the same order as in the Jordan matrix Λ, i,e, $P = [P^s P^0 P^\rho P^u]$ is

$$P = \begin{pmatrix} -\dfrac{\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u H(\bar{u})} & 1 & 1 & \dfrac{\bar{u}_{hc}\left(\eta\bar{u}_{hc} + (\lambda_u+\eta)baru_{cc}\right) - \eta H(\bar{u})}{\bar{u}_{cc}\, H(\bar{u})\lambda_s} \\ -\dfrac{\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u H(\bar{u})} & \rho & 0 & \dfrac{\eta\bar{u}_{hc} + (\lambda_u+\eta)baru_{cc}}{H(\bar{u})} \\ 0 & \dfrac{\rho\,\chi(\bar{u})}{\rho+\eta} & 0 & 0 \\ 1 & \dfrac{\rho\,(\bar{u}_{hc}+\bar{u}_{hh})}{\rho+\eta} & 0 & 1 \end{pmatrix},$$

where $H(\bar{u}) = \bar{u}_{cc}\,\bar{u}_{hh} - \bar{u}_{hc}^2$ and $\chi(\underline{u}) \equiv (\rho+\eta)\bar{u}_{cc} + \bar{u}_{hc}(2\eta+\rho) + \eta\bar{u}_{hh}$

Performing the calculations yields the generalized variation

$$d\bar{X} = \begin{pmatrix} d\bar{a} \\ d\bar{h} \\ d\bar{q}_a \\ d\bar{q}_h \end{pmatrix} = \begin{pmatrix} -\dfrac{dw}{\rho} + \dfrac{(\rho+\eta)}{\rho\,\chi(\bar{u})}\, z_3 \\ +\dfrac{(\rho+\eta)}{\chi(\bar{u})}\, z_3 \\ z_3 \\ \dfrac{\bar{u}_{hc}+\bar{u}_{hh}}{\chi(\bar{u})}\, z_3 \end{pmatrix}$$

We set $d\bar{a} = 0$ to find the value for z_3 and substituting back we obtain the particular long-run variation

$$d\bar{X} = \begin{pmatrix} d\bar{a} \\ d\bar{h} \\ d\bar{q}_a \\ d\bar{q}_h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \dfrac{\chi(\bar{u})}{\rho+\eta} \\ \dfrac{\bar{u}_{hc}+\bar{u}_{hh}}{\rho+\eta} \end{pmatrix} dw$$

The short run variation $dX(t) = X(t) - \bar{X}(w_0)$, introduced by the perturbation in w can be obtained from the general solution of the variational system,

$$dX(t) = d\bar{X} + k_s\, P^s\, e^{\lambda_s t} + k_0 P^0 + k_\rho P^\rho e^{\rho t} + k_u P^u e^{\lambda_u t}$$

where k_s, k_0, k_ρ and k_u are arbitrary constants.

Eliminating the explosive components by setting $k_\rho = k_u = 0$ and solving for k_s and k_0 such that $da(0) = 0$ and $dh(0) = 0$, yields

$$\bar{k}_s = -\frac{(\eta+\lambda_u)\,\bar{u}_{hc} + \eta\bar{u}_{hh}}{(\eta+\lambda_u)\big(\lambda_u\bar{u}_{cc} + \rho\bar{u}_{hc}\big) + \eta\rho\bar{u}_{hh}}\, dw$$

$$\bar{k}_0 = -\frac{\lambda_u H(\bar{u})}{(\eta+\lambda_u)\big(\lambda_u\bar{u}_{cc} + \rho\bar{u}_{hc}\big) + \eta\rho\bar{u}_{hh}}\, dw$$

Therefore the short run variation is

$$
\begin{aligned}
da(t) &= -\frac{\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})}(1-e^{\lambda_s t})\,dw, \text{for } t\in[0,\infty)\\
dh(t) &= \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)}{\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})}(1-e^{\lambda_s t})\,dw, \text{for } t\in[0,\infty)\\
dq_a(t) &= \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)\pi(\bar{u})}{(\rho+\eta)\left(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})\right)}dw, \text{for } t\in[0,\infty)\\
dq_h(t) &= \frac{\lambda_u\,\bar{u}_{hc}\big((\eta+\lambda_u)\bar{u}_{hc} + \eta\bar{u}_{hh}\big)}{(\rho+\eta)\left(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})\right)}\times\\
&\quad\times\left((\bar{u}_{cc}\bar{u}_{hh}(\eta+\lambda_u) + \eta\bar{u}_{hc}^2 - (\rho+\eta)H(\bar{u})e^{\lambda_s t}\right)dw, \text{for } t\in[0,\infty).
\end{aligned}
$$

The variation in consumption can be obtained as

$$dc(t) = \bar{C}_h\,dh(t) + \bar{C}_{q_a}\,dq_a(t) + \bar{C}_{q_h}\,dq_h(t).$$

In the expressions for the variations of the state variables, we see the effect of the existence of a zero eigenvalue: we find that

$$\big(\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}\big)\,dh(t) + \lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)\,da(t) = 0.$$

We also find that

$$
\begin{aligned}
da(0) &= 0\\
dh(0) &= 0\\
dq_a(0) = dq_a(t) &= \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)\pi(\bar{u})}{(\rho+\eta)\left(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})\right)}dw\\
dq_h(0) &= \frac{\lambda_u\,\bar{u}_{hc}\big((\eta+\lambda_u)\bar{u}_{hc} + \eta\bar{u}_{hh}\big)\Big((\bar{u}_{cc}\bar{u}_{hh}(\eta+\lambda_u) + \eta\bar{u}_{hc}^2 - (\rho+\eta)H(\bar{u})\Big)}{(\rho+\eta)\left(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})\right)}\,dw.
\end{aligned}
$$

and

$$da(\infty) = -\frac{\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})}\, dw, \text{for } t \in [0,\infty)$$

$$dh(\infty) = \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)}{\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})}(1-e^{\lambda_s t})\, dw, \text{for } t \in [0,\infty)$$

$$dq_a(\infty) = \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)\pi(\bar{u})}{(\rho+\eta)\left(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})\right)} dw, \text{for } t \in [0,\infty)$$

$$dq_h(\infty) = \frac{\lambda_u\,\bar{u}_{hc}\big((\eta+\lambda_u)\bar{u}_{hc} + \eta\bar{u}_{hh}\big)\Big(\bar{u}_{cc}\bar{u}_{hh}(\eta+\lambda_u) + \eta\bar{u}_{hc}^2\Big)}{(\rho+\eta)\left(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})\right)} dw.$$

which are the long run multipliers. To determine the levels of the new steady state after the shock we can write $\bar{X}(w_1) = dX(\infty) + \bar{X}(w_0)$. This is point C shown in figure 2.

In[1026]:=
```
(* Habit formation model
  See https://pmbbrito.github.io/cursos/phd/am/am2021_consumption.pdf
*)
```

In[1031]:=
```
ClearAll[a, a0, h, hx, t, tx]
(* derivatives of the consumption function *)
Chs := -Uhc / Ucc
Cqas := 1 / Ucc
Cqhs := -η / Ucc
(* elements of the Jacobian *)
J11 := ρ
J12 := -Ch
J13 := -Cqa
J14 := -Cqh
J22 := η (Ch - 1)
J23 := η Cqa
J24 := η Cqh
J42 := -(Uhc Ch + Uhh)
J43 := -Uhc Cqa
J44 := ρ + η - Uhc Cqh
```

In[1045]:=
```
(* Jacobians *)
J = {{J11, J12, J13, J14}, {0, J22, J23, J24}, {0, 0, 0, 0}, {0, J42, J43, J44}}
Dimensions[J]
Jw = {{dw}, {0}, {0}, {0}}
```

Out[1045]=
$$\{\{\rho, -\text{Ch}, -\text{Cqa}, -\text{Cqh}\}, \{0, (-1+\text{Ch})\,\eta, \text{Cqa}\,\eta, \text{Cqh}\,\eta\}, \{0, 0, 0, 0\}, \{0, -\text{Ch}\,\text{Uhc} - \text{Uhh}, -\text{Cqa}\,\text{Uhc}, -\text{Cqh}\,\text{Uhc} + \eta + \rho\}\}$$

Out[1046]=
$$\{4, 4\}$$

Out[1047]=
$$\{\{\text{dw}\}, \{0\}, \{0\}, \{0\}\}$$

In[1048]:=
```
{{ρ, -Ch, -Cqa, -Cqh}, {0, (-1 + Ch) η, Cqa η, Cqh η},
 {0, 0, 0, 0}, {0, -Ch Uhc - Uhh, -Cqa Uhc, -Cqh Uhc + η + ρ}}
```

Out[1048]=
$$\{\{\rho, -\text{Ch}, -\text{Cqa}, -\text{Cqh}\}, \{0, (-1+\text{Ch})\,\eta, \text{Cqa}\,\eta, \text{Cqh}\,\eta\}, \{0, 0, 0, 0\}, \{0, -\text{Ch}\,\text{Uhc} - \text{Uhh}, -\text{Cqa}\,\text{Uhc}, -\text{Cqh}\,\text{Uhc} + \eta + \rho\}\}$$

In[1204]:=
```
(* * * * * *  * Derivation "by hand"  * * * * * *)
TrJ = Simplify[J22 + J44 /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}]
DetJ = Simplify[J22 J44 - J24 J42 /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}]
```

Out[1204]=
$$\rho$$

Out[1205]=
$$-\frac{\eta\,(\text{Uhh}\,\eta + \text{Ucc}\,(\eta+\rho) + \text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}$$

In[1206]:=
```
(* JOrdan form of the Jacobian *)
Λ = {{λs, 0, 0 , 0} ,{0, 0, 0, 0}, {0, 0, ρ, 0}, {0, 0, 0, λu}}
Dimensions[Λ]
```

Out[1206]= $\{\{\lambda s, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, \rho, 0\}, \{0, 0, 0, \lambda u\}\}$

Out[1207]= $\{4, 4\}$

In[1208]:=
```
(* Eigenvector matrix: transposed *)
detP0 := J12 ( J23 J44 - J24 J43) - J13 (J22 J44 - J24 J42) + J14 (J22 J43 - J23 J42)
PT := {{- (J14 J42 - J12 (J44 - λs)) / (λu J42), (λs - J44) / J42, 0, 1},
  {1, -ρ (J23 J44 - J24 J43) / detP0,
   ρ (J22 J44 - J24 J42) / detP0, -ρ (J22 J43 - J23 J42) / detP0},
  {1, 0, 0, 0},
  {(- J14 J42 - J12 (J44 - λu)) / (λs J42), (λu - J44) / J42, 0, 1}}
```

In[1064]:=
```
{1, ρ, (ρ (Uhh η + Ucc (η + ρ) + Uhc (2 η + ρ))) / (η + ρ), ((Uhc + Uhh) ρ) / (η + ρ)}
```

Out[1064]= $\left\{1, \rho, \frac{\rho \left(\text{Uhh}\, \eta + \text{Ucc}\, (\eta + \rho) + \text{Uhc}\, (2\, \eta + \rho)\right)}{\eta + \rho}, \frac{(\text{Uhc} + \text{Uhh})\, \rho}{\eta + \rho}\right\}$

In[1217]:=
```
P = Simplify[Transpose[PT] /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}]
Ps1 := P[[All, 1]]
P0 = Simplify[P[[All, 2]] /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}]
Pρ = Simplify[P[[All, 3]] /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}]
Pu1 := P[[All, 4]]
Ps = Simplify[Ps1 /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}, λs + λu == ρ]
Pu = Simplify[Pu1 /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}, λu + λs == ρ]
```

Out[1217]= $\left\{\left\{\frac{\text{Uhh}\, \eta + \text{Uhc}\, (\eta - \lambda s + \rho)}{(\text{Uhc}^2 - \text{Ucc}\, \text{Uhh})\, \lambda u}, 1, 1, \frac{2\, \text{Uhc}^2\, \eta - \text{Ucc}\, \text{Uhh}\, \eta + \text{Ucc}\, \text{Uhc}\, (\eta - \lambda u + \rho)}{\text{Ucc}\, (-\text{Uhc}^2 + \text{Ucc}\, \text{Uhh})\, \lambda s}\right\},\right.$
$\left\{\frac{\text{Uhc}\, \eta + \text{Ucc}\, (\eta - \lambda s + \rho)}{-\text{Uhc}^2 + \text{Ucc}\, \text{Uhh}}, \rho, 0, \frac{\text{Uhc}\, \eta + \text{Ucc}\, (\eta - \lambda u + \rho)}{-\text{Uhc}^2 + \text{Ucc}\, \text{Uhh}}\right\},$
$\left.\left\{0, \frac{\rho \left(\text{Uhh}\, \eta + \text{Ucc}\, (\eta + \rho) + \text{Uhc}\, (2\, \eta + \rho)\right)}{\eta + \rho}, 0, 0\right\}, \left\{1, \frac{(\text{Uhc} + \text{Uhh})\, \rho}{\eta + \rho}, 0, 1\right\}\right\}$

Out[1219]= $\left\{1, \rho, \frac{\rho \left(\text{Uhh}\, \eta + \text{Ucc}\, (\eta + \rho) + \text{Uhc}\, (2\, \eta + \rho)\right)}{\eta + \rho}, \frac{(\text{Uhc} + \text{Uhh})\, \rho}{\eta + \rho}\right\}$

Out[1220]= $\{1, 0, 0, 0\}$

Out[1222]= $\left\{\frac{\text{Uhc}\, \eta + \text{Uhh}\, \eta + \text{Uhc}\, \lambda u}{\text{Uhc}^2\, \lambda u - \text{Ucc}\, \text{Uhh}\, \lambda u}, -\frac{\text{Uhc}\, \eta + \text{Ucc}\, (\eta + \lambda u)}{\text{Uhc}^2 - \text{Ucc}\, \text{Uhh}}, 0, 1\right\}$

Out[1223]= $\left\{\frac{2\, \text{Uhc}^2\, \eta - \text{Ucc}\, \text{Uhh}\, \eta + \text{Ucc}\, \text{Uhc}\, (\eta - \lambda u + \rho)}{\text{Ucc}\, (-\text{Uhc}^2 + \text{Ucc}\, \text{Uhh})\, \lambda s}, \frac{\text{Uhc}\, \eta + \text{Ucc}\, (\eta - \lambda u + \rho)}{-\text{Uhc}^2 + \text{Ucc}\, \text{Uhh}}, 0, 1\right\}$

In[1069]:=
```
(* Confirmation of the last derivation by using Mathematica functions *)
Simplify[Tr[J]]
Simplify[Det[J]]
EJ = Simplify[Eigenvalues[J] /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}]
EP = Simplify[Eigenvectors[J] /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}]
```

Out[1069]= $-\text{Cqh}\,\text{Uhc} + \text{Ch}\,\eta + 2\,\rho$

Out[1070]= 0

Out[1071]= $\left\{0,\ \rho,\ \frac{1}{2}\left(\rho - \sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta + \text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right),\right.$
$\left.\frac{1}{2}\left(\rho + \sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta + \text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right)\right\}$

Out[1072]= $\left\{\left\{\frac{\eta+\rho}{\text{Uhc}\,\rho + \text{Uhh}\,\rho},\ \frac{\eta+\rho}{\text{Uhc}+\text{Uhh}},\ \frac{\text{Uhh}\,\eta + \text{Ucc}\,(\eta+\rho) + \text{Uhc}\,(2\,\eta+\rho)}{\text{Uhc}+\text{Uhh}},\ 1\right\},\right.$
$\{1,\ 0,\ 0,\ 0\},$
$\left\{\frac{2\,\text{Uhh}\,\eta + \text{Uhc}\left(2\,\eta+\rho+\sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta+\text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right)}{(\text{Uhc}^2 - \text{Ucc}\,\text{Uhh})\left(\rho+\sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta+\text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right)},\right.$
$\left.-\frac{2\,\text{Uhc}\,\eta + \text{Ucc}\left(2\,\eta+\rho+\sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta+\text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right)}{2\,(\text{Uhc}^2 - \text{Ucc}\,\text{Uhh})},\ 0,\ 1\right\},$
$\left\{\frac{-2\,\text{Uhh}\,\eta + \text{Uhc}\left(-2\,\eta-\rho+\sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta+\text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right)}{(\text{Uhc}^2 - \text{Ucc}\,\text{Uhh})\left(-\rho+\sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta+\text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right)},\right.$
$\left.\left.\frac{-2\,\text{Uhc}\,\eta + \text{Ucc}\left(-2\,\eta-\rho+\sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta+\text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right)}{2\,(\text{Uhc}^2 - \text{Ucc}\,\text{Uhh})},\ 0,\ 1\right\}\right\}$

In[1073]:=
```
-Cqh Uhc + Ch η + 2 ρ
```

Out[1073]= $-\text{Cqh}\,\text{Uhc} + \text{Ch}\,\eta + 2\,\rho$

In[1074]:=
```
{0, ρ, 1/2 (ρ - Sqrt[(2 η + ρ)^2 + 4 η (Uhh η + Uhc (2 η + ρ))/Ucc]),
 1/2 (ρ + Sqrt[(2 η + ρ)^2 + 4 η (Uhh η + Uhc (2 η + ρ))/Ucc])}
```

Out[1074]= $\left\{0,\ \rho,\ \frac{1}{2}\left(\rho - \sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta + \text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right),\right.$
$\left.\frac{1}{2}\left(\rho + \sqrt{(2\,\eta+\rho)^2 + \frac{4\,\eta\,(\text{Uhh}\,\eta + \text{Uhc}\,(2\,\eta+\rho))}{\text{Ucc}}}\right)\right\}$

In[1232]:=
```
(* Generalized inverse Jplus *)
PI := Inverse[P]
ΛI = {{1 / λs, 0, 0 , 0} , {0, 0, 0, 0}, {0, 0, 1 / ρ, 0}, {0, 0, 0, 1 / λu}}
Jplus := Simplify[P.ΛI .Inverse[P] /. {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}]
Dimensions[Jplus]
```

Out[1233]= $\left\{\left\{\frac{1}{\lambda s}, 0, 0, 0\right\}, \{0, 0, 0, 0\}, \left\{0, 0, \frac{1}{\rho}, 0\right\}, \left\{0, 0, 0, \frac{1}{\lambda u}\right\}\right\}$

Out[1235]= {4, 4}

In[1080]:=
```
(* Linear approximation of the solution after a shock in w
 starting from SS0, for a 10% increase in w
   X(t) = X(0) + F(Xss,t) where Xss is the new steady state *)
(* Long run multipliers *)
```

In[1081]:=
```
(* dXLPg:=
 Simplify[-Jplus.(Jw )+(IdentityMatrix[4]-Jplus.J).{{k1},{k2},{k3},{k4}}/.
   {Ch→Chs,Cqa→Cqas,Cqh→Cqhs},{λs + λu==ρ,λs λu==DetJ}] *)
dXLPg := Simplify[-Jplus.(Jw ) +
    (IdentityMatrix[4] - P.ΛI.Λ.PI).{{k1}, {k2}, {k3}, {k4}} /.
   {Ch → Chs, Cqa → Cqas, Cqh → Cqhs}, {λs + λu == ρ, λs λu == DetJ}]
Dimensions[
 dXLPg]
```

Out[1082]= {4, 1}

In[1236]:=
```
dXLP1 = Simplify[Factor[dXLPg[[1, 1]]]]
dXLP2 = dXLPg[[2, 1]]
dXLP3 = dXLPg[[3, 1]]
dXLP4 = dXLPg[[4, 1]]
```

Out[1236]= $\frac{\text{k3}\ (\eta+\rho) - \text{dw}\left(2\ \text{Uhc}\ \eta + \text{Uhh}\ \eta + \text{Uhc}\ \rho + \text{Ucc}\ (\eta+\rho)\right)}{\rho\left(\text{Uhh}\ \eta + \text{Ucc}\ (\eta+\rho) + \text{Uhc}\left(2\ \eta+\rho\right)\right)}$

Out[1237]= $\frac{\text{k3}\ (\eta+\rho)}{\text{Uhh}\ \eta + \text{Ucc}\ (\eta+\rho) + \text{Uhc}\left(2\ \eta+\rho\right)}$

Out[1238]= k3

Out[1239]= $\frac{\text{k3}\left(\text{Uhc} + \text{Uhh}\right)}{\text{Uhh}\ \eta + \text{Ucc}\ (\eta+\rho) + \text{Uhc}\left(2\ \eta+\rho\right)}$

In[1246]:=
```
(* auxiliar *)
dXLPaux1[k3_] = dXLP1
Auxeq1 := Simplify[Solve[{dXLPaux1[x] == 0}, {x}]]
k3ep = x /. Auxeq1[[1, 1]]
```

Out[1246]=
$$\frac{\text{k3}\ (\eta+\rho) - \text{dw}\ \left(2\ \text{Uhc}\ \eta + \text{Uhh}\ \eta + \text{Uhc}\ \rho + \text{Ucc}\ (\eta+\rho)\right)}{\rho\ \left(\text{Uhh}\ \eta + \text{Ucc}\ (\eta+\rho) + \text{Uhc}\ \left(2\ \eta+\rho\right)\right)}$$

Out[1248]=
$$\frac{\text{dw}\ \left(\text{Uhh}\ \eta + \text{Ucc}\ (\eta+\rho) + \text{Uhc}\ \left(2\ \eta+\rho\right)\right)}{\eta+\rho}$$

In[1249]:=
```
(* Long-run multipliers *)
dXLP = Simplify[dXLPg /. {k3 → k3ep}]
```

Out[1249]=
$$\left\{\{0\}, \{\text{dw}\}, \left\{\frac{\text{dw}\ \left(\text{Uhh}\ \eta + \text{Ucc}\ (\eta+\rho) + \text{Uhc}\ \left(2\ \eta+\rho\right)\right)}{\eta+\rho}\right\}, \left\{\frac{\text{dw}\ \left(\text{Uhc} + \text{Uhh}\right)}{\eta+\rho}\right\}\right\}$$

In[1250]:=
```
(* Short run  multipliers *)
dXtg = Simplify[dXLP + kk1 P0 + kk2 Ps Exp[λs t]]
```

Out[1250]=
$$\left\{\left\{\text{kk1} + \frac{e^{t\,\lambda s}\ \text{kk2}\ \left(\text{Uhh}\ \eta + \text{Uhc}\ (\eta+\lambda u)\right)}{\left(\text{Uhc}^2 - \text{Ucc}\ \text{Uhh}\right)\ \lambda u}\right\}, \left\{\text{dw} - \frac{e^{t\,\lambda s}\ \text{kk2}\ \left(\text{Uhc}\ \eta + \text{Ucc}\ (\eta+\lambda u)\right)}{\text{Uhc}^2 - \text{Ucc}\ \text{Uhh}} + \text{kk1}\ \rho\right\},\right.$$
$$\left\{\frac{\left(\text{dw} + \text{kk1}\ \rho\right)\ \left(\text{Uhh}\ \eta + \text{Ucc}\ (\eta+\rho) + \text{Uhc}\ \left(2\ \eta+\rho\right)\right)}{\eta+\rho}\right\},$$
$$\left.\left\{\frac{\text{dw}\ \left(\text{Uhc} + \text{Uhh}\right) + \text{kk1}\ \left(\text{Uhc} + \text{Uhh}\right)\ \rho + e^{t\,\lambda s}\ \text{kk2}\ (\eta+\rho)}{\eta+\rho}\right\}\right\}$$

In[1251]:=
```
dX1tg[kk1_, kk2_] = Simplify[dXtg [[1, 1]] /. t → 0]
dX2tg[kk1_, kk2_] = Simplify[dXtg [[2, 1]] /. t → 0]
Auxeq2 := Solve[{dX1tg[x, y] == 0, dX2tg[x, y] == 0}, {x, y}]
kk1ep = x /. Auxeq2[[1, 1]]
kk2ep = y /. Auxeq2[[1, 2]]
```

Out[1251]=
$$\text{kk1} + \frac{\text{kk2}\ \left(\text{Uhh}\ \eta + \text{Uhc}\ (\eta+\lambda u)\right)}{\left(\text{Uhc}^2 - \text{Ucc}\ \text{Uhh}\right)\ \lambda u}$$

Out[1252]=
$$\text{dw} + \frac{\text{kk2}\ \left(\text{Uhc}\ \eta + \text{Ucc}\ (\eta+\lambda u)\right)}{-\text{Uhc}^2 + \text{Ucc}\ \text{Uhh}} + \text{kk1}\ \rho$$

Out[1254]=
$$-\frac{\text{dw}\ \left(\text{Uhc}\ \eta + \text{Uhh}\ \eta + \text{Uhc}\ \lambda u\right)}{\text{Ucc}\ \eta\ \lambda u + \text{Uhc}\ \eta\ \lambda u + \text{Ucc}\ \lambda u^2 + \text{Uhc}\ \eta\ \rho + \text{Uhh}\ \eta\ \rho + \text{Uhc}\ \lambda u\ \rho}$$

Out[1255]=
$$\frac{\text{dw}\ \left(\text{Uhc}^2 - \text{Ucc}\ \text{Uhh}\right)\ \lambda u}{\text{Ucc}\ \eta\ \lambda u + \text{Uhc}\ \eta\ \lambda u + \text{Ucc}\ \lambda u^2 + \text{Uhc}\ \eta\ \rho + \text{Uhh}\ \eta\ \rho + \text{Uhc}\ \lambda u\ \rho}$$

In[1256]:=
```
(*
Solution of the linearized model
 First: short run multipliers
    Second: level values
        ac, hc, qac, qhc initial steady state
*)
dXt := Simplify[dXtg /. {kk1 → kk1ep, kk2 → kk2ep}]
Xt = Simplify[{{ac}, {hc}, {qac}, {qhc}} + dXt]
```

Out[1257]=
$$\left\{\left\{ac + \frac{dw\left(-1+e^{t\,\lambda s}\right)\left(Uhh\,\eta + Uhc\,(\eta+\lambda u)\right)}{Ucc\,\lambda u\,(\eta+\lambda u) + Uhh\,\eta\,\rho + Uhc\,\lambda u\,\rho + Uhc\,\eta\,(\lambda u+\rho)}\right\},\right.$$
$$\left\{hc - \frac{dw\left(-1+e^{t\,\lambda s}\right)\lambda u\left(Uhc\,\eta + Ucc\,(\eta+\lambda u)\right)}{Ucc\,\lambda u\,(\eta+\lambda u) + Uhh\,\eta\,\rho + Uhc\,\lambda u\,\rho + Uhc\,\eta\,(\lambda u+\rho)}\right\},$$
$$\left\{qac + \frac{dw\,\lambda u\left(Uhc\,\eta + Ucc\,(\eta+\lambda u)\right)\left(Uhh\,\eta + Ucc\,(\eta+\rho) + Uhc\left(2\,\eta+\rho\right)\right)}{(\eta+\rho)\left(Ucc\,\lambda u\,(\eta+\lambda u) + Uhh\,\eta\,\rho + Uhc\,\lambda u\,\rho + Uhc\,\eta\,(\lambda u+\rho)\right)}\right\},$$
$$\left\{qhc + \left(dw\,\lambda u\left(Uhc\,Uhh\,\eta + Ucc\,Uhc\,(\eta+\lambda u) + \right.\right.\right.$$
$$\left.\left.\left.Ucc\,Uhh\left(\eta - e^{t\,\lambda s}\,\eta + \lambda u - e^{t\,\lambda s}\,\rho\right) + Uhc^2\left(\eta + e^{t\,\lambda s}\,\eta + e^{t\,\lambda s}\,\rho\right)\right)\right) /$$
$$\left.\left.\left((\eta+\rho)\left(Ucc\,\lambda u\,(\eta+\lambda u) + Uhh\,\eta\,\rho + Uhc\,\lambda u\,\rho + Uhc\,\eta\,(\lambda u+\rho)\right)\right)\right\}\right\}$$

```
(* * * * * * * * * * * MULTIPLICATIVE HABITS * * * * * * * * * * *)

Um[c_, h_] = ((c / (h^ζ))^(1 - θ) - 1) / (1 - θ)
```

In[1264]:=
```
(* derivatives *)
Umc = D[Um[c, h], c]
Umh = D[Um[c, h], h]
Umcc = D[D[Um[c, h], c], c]
Umhc = Simplify[D[Um[c, h], h, c]]
Umhh = Simplify[D[Um[c, h], h, h]]
```

Out[1264]=
$$h^{-\zeta}\left(c\,h^{-\zeta}\right)^{-\theta}$$

Out[1265]=
$$-c\,h^{-1-\zeta}\left(c\,h^{-\zeta}\right)^{-\theta}\zeta$$

Out[1266]=
$$-h^{-2\,\zeta}\left(c\,h^{-\zeta}\right)^{-1-\theta}\theta$$

Out[1267]=
$$h^{-1-\zeta}\left(c\,h^{-\zeta}\right)^{-\theta}\zeta\,(-1+\theta)$$

Out[1268]=
$$-\frac{\left(c\,h^{-\zeta}\right)^{1-\theta}\zeta\,(-1+\zeta\,(-1+\theta))}{h^2}$$

In[1269]:=
```
(* consumption function: partial derivatives *)
Cmh = Simplify[Chs /. {Ucc → Umcc, Uhc → Umhc}]
Cmqa = Simplify[Cqas /. {Ucc → Umcc}]
Cmqh = Simplify[Cqhs /. {Ucc → Umcc}]
```

Out[1269]=
$$\frac{c\,\zeta\,(-1+\theta)}{h\,\theta}$$

Out[1270]=
$$-\frac{c\,h^{\zeta}\left(c\,h^{-\zeta}\right)^{\theta}}{\theta}$$

Out[1271]=
$$\frac{c\,h^{\zeta}\left(c\,h^{-\zeta}\right)^{\theta}\eta}{\theta}$$

```
(* Determination of the steady state:
   initial: Xbar(0)
    the final SS is determined by the multipliers: Xbar(1) = Xbar(0) + dbarX
  *)
```

In[1124]:=
```
(* eq1:=h^-ζ (c h^-ζ)^-θ-qh+η qa
    eq2:=qh (ρ+η)-c h^(-1-ζ) (c h^-ζ)^-θ ζ
*)
eq1 := Umc - qa + η qh
eq2 := qh (ρ + η) - Umh
eq3 := c - h
eq4 := ρ a + w - c
SSaux = Solve[{eq1 == 0, eq2 == 0, eq3 == 0, eq4 == 0}, {qa, qh, c, h}]
```

Out[1128]=
$$\left\{\left\{qa \to (w+a\,\rho)^{-\gamma}\left((w+a\,\rho)^{1-\gamma}\right)^{-\theta} - \frac{\gamma\,\eta\,(w+a\,\rho)^{-\gamma}\left((w+a\,\rho)^{1-\gamma}\right)^{-\theta}}{\eta+\rho},\right.\right.$$
$$\left.\left. qh \to -\frac{\gamma\,(w+a\,\rho)^{-\gamma}\left((w+a\,\rho)^{1-\gamma}\right)^{-\theta}}{\eta+\rho},\ c \to w+a\,\rho,\ h \to w+a\,\rho\right\}\right\}$$

```
(* simplification of qa and qh *)
Simplify[SSaux[[1, 1]][[2]]]
Simplify[SSaux[[1, 2]][[2]]]
```

Out[1130]=
$$\frac{(\eta-\gamma\,\eta+\rho)\,(w+a\,\rho)^{-\gamma}\left((w+a\,\rho)^{1-\gamma}\right)^{-\theta}}{\eta+\rho}$$

Out[1131]=
$$-\frac{\gamma\,(w+a\,\rho)^{-\gamma}\left((w+a\,\rho)^{1-\gamma}\right)^{-\theta}}{\eta+\rho}$$

```
(* this is needed to control a0 *)
qassx[ax_, w_, ρ_, θ_, ζ_, η_] = SSaux[[1, 1]][[2]] /. a → ax
qhssx[ax_, w_, ρ_, θ_, ζ_, η_] = SSaux[[1, 2]][[2]] /. a → ax
cssx[ax_, w_, ρ_, θ_, ζ_, η_] = SSaux[[1, 3]][[2]] /. a → ax
hssx[ax_, w_, ρ_, θ_, ζ_, η_] = SSaux[[1, 4]][[2]] /. a → ax
assx[ax_] = ax
```

Out[1134]= $(w + ax\,\rho)^{-\gamma}\left((w + ax\,\rho)^{1-\gamma}\right)^{-\theta} - \frac{\gamma\,\eta\,(w + ax\,\rho)^{-\gamma}\left((w + ax\,\rho)^{1-\gamma}\right)^{-\theta}}{\eta + \rho}$

Out[1135]= $-\frac{\gamma\,(w + ax\,\rho)^{-\gamma}\left((w + ax\,\rho)^{1-\gamma}\right)^{-\theta}}{\eta + \rho}$

Out[1136]= $w + ax\,\rho$

Out[1137]= $w + ax\,\rho$

Out[1138]= ax

In[1141]:=
```
(* Stability condition with multiplicative habits *)
Factor[Simplify[Simplify[DetJ /. {Uhh → Umhh, Ucc → Umcc, Uhc → Umhc}] /.
    {c → w + ax ρ, h -> w + ax ρ}]]
```

Out[1141]= $-\frac{\eta\,(-\gamma - \theta + \gamma\,\theta)\,(-\eta + \gamma\,\eta - \rho)}{\theta}$

In[1142]:=
```
-(-ζ - θ + ζ θ) η (-η + ζ η - ρ)
               θ
```

Out[1142]= $-\frac{\eta\,(-\zeta - \theta + \zeta\,\theta)\,(-\eta + \zeta\,\eta - \rho)}{\theta}$

In[1282]:=
```
(* Short run levels X(t) for the multiplicative habits model  *)
Xtm = Simplify[
  Simplify[Xt /. {Uhh → Umhh, Ucc → Umcc, Uhc → Umhc}] /. {c → w + ax ρ, h -> w + ax ρ}]
```

Out[1282]=
$$\left\{\left\{ac + \frac{dw\left(-1 + e^{t\,\lambda s}\right)\,\zeta\,(\zeta\,\eta\,(-1+\theta) - \eta\,\theta + \lambda u - \theta\,\lambda u)}{\theta\,\lambda u\,(\eta + \lambda u) + \zeta^2\,\eta\,(-1+\theta)\,\rho - \zeta\,(\eta\,(-1+\theta)\,\lambda u + \eta\,\theta\,\rho + (-1+\theta)\,\lambda u\,\rho)}\right\},\right.$$
$$\left\{hc - \frac{dw\left(-1 + e^{t\,\lambda s}\right)\,\lambda u\,(\zeta\,(\eta - \eta\,\theta) + \theta\,(\eta + \lambda u))}{\theta\,\lambda u\,(\eta + \lambda u) + \zeta^2\,\eta\,(-1+\theta)\,\rho - \zeta\,(\eta\,(-1+\theta)\,\lambda u + \eta\,\theta\,\rho + (-1+\theta)\,\lambda u\,\rho)}\right\},$$
$$\left\{qac + \left(dw\,(\zeta\,(-1+\theta) - \theta)\,\lambda u\,(\zeta\,\eta\,(-1+\theta) - \theta\,(\eta + \lambda u))\right.\right.$$
$$((-1+\zeta)\,\eta - \rho)\,(w + ax\,\rho)^{-1-\zeta}\left((w + ax\,\rho)^{1-\zeta}\right)^{-\theta}\Big) \Big/$$
$$\left.\left.\left((\eta + \rho)\,\left(\theta\,\lambda u\,(\eta + \lambda u) + \zeta^2\,\eta\,(-1+\theta)\,\rho - \zeta\,(\eta\,(-1+\theta)\,\lambda u + \eta\,\theta\,\rho + (-1+\theta)\,\lambda u\,\rho)\right)\right)\right\},$$
$$\left\{qhc + \left(dw\,\zeta\,(\zeta\,(-1+\theta) - \theta)\,\lambda u\,(w + ax\,\rho)^{-1-\zeta}\left((w + ax\,\rho)^{1-\zeta}\right)^{-\theta}\right.\right.$$
$$\left(\zeta\,\eta\,(-1+\theta) - \theta\,(\eta + \lambda u) + e^{t\,\lambda s}\,(\eta + \rho)\right)\Big) \Big/$$
$$\left.\left.\left.\left((\eta + \rho)\,\left(\theta\,\lambda u\,(\eta + \lambda u) + \zeta^2\,\eta\,(-1+\theta)\,\rho - \zeta\,(\eta\,(-1+\theta)\,\lambda u + \eta\,\theta\,\rho + (-1+\theta)\,\lambda u\,\rho)\right)\right)\right\}\right\}$$

In[1283]:=
```
(* impact levels X(0) = (ac, hc, qac, qah)  *)
Simplify[Xtm /. t → 0]
```

Out[1283]=
$$\left\{\{\text{ac}\},\{\text{hc}\},\left\{\text{qac}+\left(\text{dw}\,(\zeta\,(-1+\theta)-\theta)\,\lambda\text{u}\,(\zeta\,\eta\,(-1+\theta)-\theta\,(\eta+\lambda\text{u}))\,((-1+\zeta)\,\eta-\rho)\,(\text{w}+\text{ax}\,\rho)^{-1-\zeta}\left((\text{w}+\text{ax}\,\rho)^{1-\zeta}\right)^{-\theta}\right)\Big/\left((\eta+\rho)\left(\theta\,\lambda\text{u}\,(\eta+\lambda\text{u})+\zeta^2\,\eta\,(-1+\theta)\,\rho-\zeta\,(\eta\,(-1+\theta)\,\lambda\text{u}+\eta\,\theta\,\rho+(-1+\theta)\,\lambda\text{u}\,\rho)\right)\right)\right\},\left\{\text{qhc}+\frac{\text{dw}\,\zeta\,(\zeta\,(-1+\theta)-\theta)\,\lambda\text{u}\,((-1+\zeta)\,\eta\,(-1+\theta)-\theta\,\lambda\text{u}+\rho)\,(\text{w}+\text{ax}\,\rho)^{-1-\zeta}\left((\text{w}+\text{ax}\,\rho)^{1-\zeta}\right)^{-\theta}}{(\eta+\rho)\left(\theta\,\lambda\text{u}\,(\eta+\lambda\text{u})+\zeta^2\,\eta\,(-1+\theta)\,\rho-\zeta\,(\eta\,(-1+\theta)\,\lambda\text{u}+\eta\,\theta\,\rho+(-1+\theta)\,\lambda\text{u}\,\rho)\right)}\right\}\right\}$$

In[1284]:=
```
(* long run multipliers *)
Simplify[Evaluate[Xtm /. e^(t λs) -> 0]]
```

Out[1284]=
$$\left\{\left\{\text{ac}-\frac{\text{dw}\,\zeta\,(\zeta\,\eta\,(-1+\theta)-\eta\,\theta+\lambda\text{u}-\theta\,\lambda\text{u})}{\theta\,\lambda\text{u}\,(\eta+\lambda\text{u})+\zeta^2\,\eta\,(-1+\theta)\,\rho-\zeta\,(\eta\,(-1+\theta)\,\lambda\text{u}+\eta\,\theta\,\rho+(-1+\theta)\,\lambda\text{u}\,\rho)}\right\},\left\{\text{hc}+\frac{\text{dw}\,\lambda\text{u}\,(\zeta\,(\eta-\eta\,\theta)+\theta\,(\eta+\lambda\text{u}))}{\theta\,\lambda\text{u}\,(\eta+\lambda\text{u})+\zeta^2\,\eta\,(-1+\theta)\,\rho-\zeta\,(\eta\,(-1+\theta)\,\lambda\text{u}+\eta\,\theta\,\rho+(-1+\theta)\,\lambda\text{u}\,\rho)}\right\},\left\{\text{qac}+\left(\text{dw}\,(\zeta\,(-1+\theta)-\theta)\,\lambda\text{u}\,(\zeta\,\eta\,(-1+\theta)-\theta\,(\eta+\lambda\text{u}))\,((-1+\zeta)\,\eta-\rho)\,(\text{w}+\text{ax}\,\rho)^{-1-\zeta}\left((\text{w}+\text{ax}\,\rho)^{1-\zeta}\right)^{-\theta}\right)\Big/\left((\eta+\rho)\left(\theta\,\lambda\text{u}\,(\eta+\lambda\text{u})+\zeta^2\,\eta\,(-1+\theta)\,\rho-\zeta\,(\eta\,(-1+\theta)\,\lambda\text{u}+\eta\,\theta\,\rho+(-1+\theta)\,\lambda\text{u}\,\rho)\right)\right)\right\},\left\{\text{qhc}+\frac{\text{dw}\,\zeta\,(\zeta\,(-1+\theta)-\theta)\,\lambda\text{u}\,(\zeta\,\eta\,(-1+\theta)-\theta\,(\eta+\lambda\text{u}))\,(\text{w}+\text{ax}\,\rho)^{-1-\zeta}\left((\text{w}+\text{ax}\,\rho)^{1-\zeta}\right)^{-\theta}}{(\eta+\rho)\left(\theta\,\lambda\text{u}\,(\eta+\lambda\text{u})+\zeta^2\,\eta\,(-1+\theta)\,\rho-\zeta\,(\eta\,(-1+\theta)\,\lambda\text{u}+\eta\,\theta\,\rho+(-1+\theta)\,\lambda\text{u}\,\rho)\right)}\right\}\right\}$$

In[1149]:=
```
(* Calibration *)
w0 = 0.7
w1 = 0.7 * 1.2
ρ0 = 0.02
η0 = 0.1
ζ0 = 0.5
θ0 = 2
a0 = 0.3 / 0.02
```

Out[1149]= 0.7

Out[1150]= 0.84

Out[1151]= 0.02

Out[1152]= 0.1

Out[1153]= 0.5

Out[1154]= 2

Out[1155]= 15.

In[1156]:=
```
(* Initial SS *)
qass0 = qassx[a0, w0, ρ0, θ0, ξ0, η0]
qhss0 = qhssx[a0, w0, ρ0, θ0, ξ0, η0]
css0 = cssx[a0, w0, ρ0, θ0, ξ0, η0]
hss0 = hssx[a0, w0, ρ0, θ0, ξ0, η0]
ass0 = a0
```

Out[1156]= $1.^{-1+2\,\gamma} - 0.833333 \times 1.^{-2+2\,\gamma}\,\gamma$

Out[1157]= $-8.33333 \times 1.^{-2+2\,\gamma}\,\gamma$

Out[1158]= 1.

Out[1159]= 1.

Out[1160]= 15.

In[1161]:=
```
(* Auxiliary results *)
Ucc0 = -h^(-2 ξ) (c h^(-ξ))^(-1-θ) θ /. {ξ → ξ0, θ → θ0, h → hss0, c → css0}
Uhc0 = h^(-1-ξ) (c h^(-ξ))^(-θ) ξ (-1 + θ) /. {ξ → ξ0, θ → θ0, h → hss0, c → css0}
Uhh0 = -((c h^(-ξ))^(1-θ) ξ (-1 + ξ (-1 + θ)))/h^2 /. {ξ → ξ0, θ → θ0, h → hss0, c → css0}
```

Out[1161]= −2.

Out[1162]= 0.5

Out[1163]= 0.25

In[1164]:=
```
λs0 = Simplify[1/2 (ρ - Sqrt[(2 η + ρ)^2 + 4 η (Uhh η + Uhc (2 η + ρ))/Ucc]) /.
    {η → η0, ρ → ρ0, Ucc -> Ucc0, Uhc -> Uhc0, Uhh -> Uhh0}]
λu0 = Simplify[1/2 (ρ + Sqrt[(2 η + ρ)^2 + 4 η (Uhh η + Uhc (2 η + ρ))/Ucc]) /.
    {η → η0, ρ → ρ0, Ucc -> Ucc0, Uhc -> Uhc0, Uhh -> Uhh0}]
```

Out[1164]= −0.0631437

Out[1165]= 0.0831437

In[1166]:=
```
(* multiplicadores LP *)
Simplify[dXLPg + kx0 P0 + kx1 Ps /.
  {ac -> ass0, hc → hss0, qac → qass0, qhc → qhss0, dw -> w1 - w0, η → η0,
   ρ → ρ0, Ucc -> Ucc0, Uhc -> Uhc0, Uhh -> Uhh0, λs -> λs0, λu -> λu0}]
```

Out[1166]= $\{\{-7. - 57.1429\,k3 + kx0 + 1.8694\,kx1\},\ \{-1.14286\,k3 + 0.02\,kx0 + 0.421717\,kx1\},\ \{k3 - 0.0175\,kx0\},\ \{-7.14286\,k3 + 0.125\,kx0 + kx1\}\}$

In[1167]:=
```
(* new approximate solution *)
```

In[1168]:=
```
Xtsol =
 Simplify[Xt /. {ac -> ass0, hc → hss0, qac → qass0, qhc → qhss0, dw -> w1 - w0, η → η0,
    ρ → ρ0, Ucc -> Ucc0, Uhc -> Uhc0, Uhh -> Uhh0, λs -> λs0, λu -> λu0}]
```

Out[1168]=
$$\left\{\left\{15.681 - 0.680971\, e^{-0.0631437\, t}\right\}, \left\{1.15362 - 0.153619\, e^{-0.0631437\, t}\right\}, \{0.865583 - 0.833333\, \gamma\}, \left\{0.960121 - 0.364272\, e^{-0.0631437\, t} - 8.33333\, \gamma\right\}\right\}$$

In[1169]:=
```
(*atsolt[t_]=15.680970982036422-0.68097098203642 Exp[ λs0 t]*)
atsolt[t_] = Xtsol[[1, 1]]
htsolt[t_] = Xtsol[[2, 1]]
qatsolt[t_] = Xtsol[[3, 1]]
qhtsolt[t_] = Xtsol[[4, 1]]
```

Out[1169]=
$$15.681 - 0.680971\, e^{-0.0631437\, t}$$

Out[1170]=
$$1.15362 - 0.153619\, e^{-0.0631437\, t}$$

Out[1171]=
$$0.865583 - 0.833333\, \gamma$$

Out[1172]=
$$0.960121 - 0.364272\, e^{-0.0631437\, t} - 8.33333\, \gamma$$

In[1173]:=

In[1174]:=
```
(* Solution for consumption *)
Cmhss = Cmh /. {c → css0, h → hss0, θ → θ0, ξ → ξ0}
Cmqass = Cmqa /. {c → css0, h → hss0, θ → θ0, ξ → ξ0}
Cmqhss = Cmqh /. {c → css0, h → hss0, θ → θ0, ξ → ξ0, η → η0}
ctsol = Simplify[css0 + Cmhss (Xtsol[[2, 1]] - hss0) +
   Cmqass (Xtsol[[3, 1]] - qass0) + Cmqhss (Xtsol[[4, 1]] - qhss0)]
```

Out[1174]=
$$0.5\, \gamma$$

Out[1175]=
$$-0.5$$

Out[1176]=
$$0.05$$

Out[1177]=
$$1.11521 + e^{-0.0631437\, t}\left(-0.0182136 - 0.0768097\, \gamma\right) + 0.0768097\, \gamma$$

In[1178]:=
```
ctsolt[t_] = 1.15362 - 0.056618 Exp[ λs0 t]
```

Out[1178]=
$$1.15362 - 0.056618\, e^{-0.0631437\, t}$$

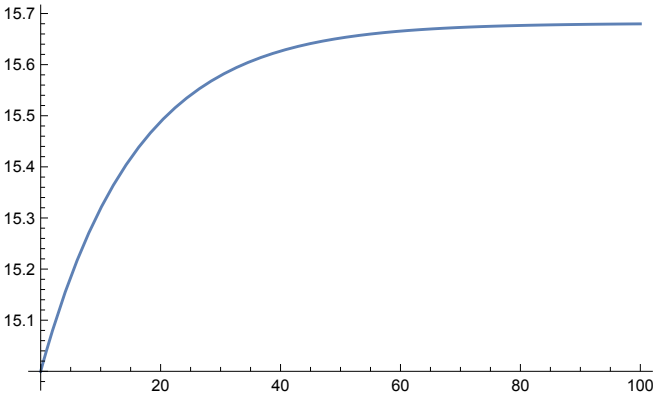
In[1179]:=
```
(* New steady state *)
css1 = ctsolt[Infinity]
ass1 = atsolt[Infinity]
```

Out[1179]=
$$1.15362$$

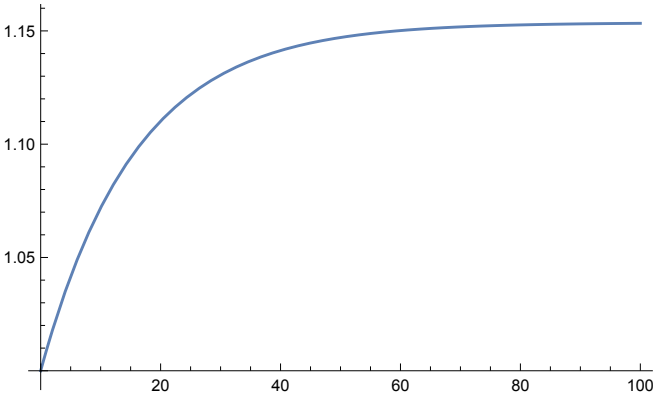
Out[1180]=
$$15.681$$

In[1181]:=
```
(* Trajectories *)
Plot[atsolt[t], {t, 0, 100}]
Plot[htsolt[t], {t, 0, 100}]
Plot[ctsolt[t], {t, 0, 100}]
Plot[qhtsolt[t], {t, 0, 100}]
```

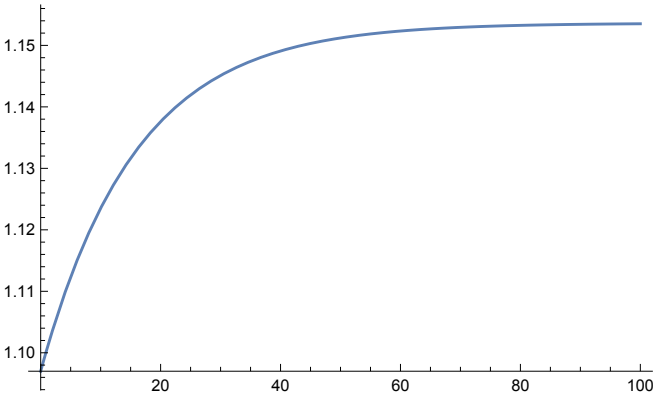
Out[1181]=



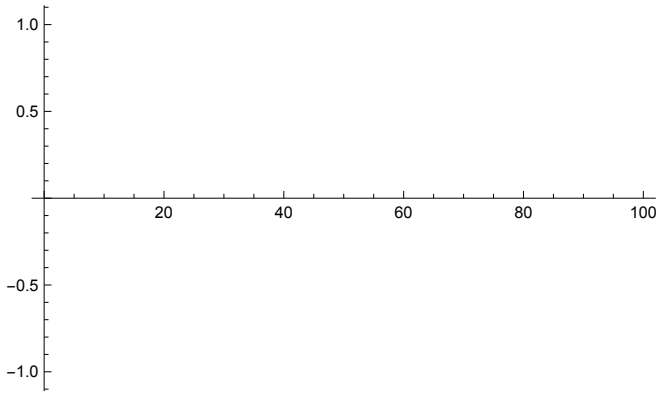
Out[1182]=



Out[1183]=



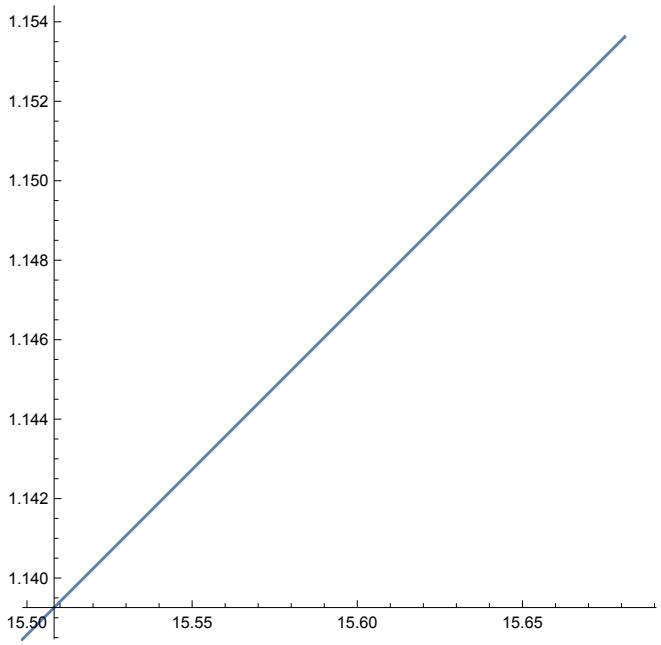
Out[1184]=



In[1185]:=

```
ParametricPlot[{atsolt[t], ctsolt[t]}, {t, 0, 200}, AspectRatio → 1]
```

Out[1185]=

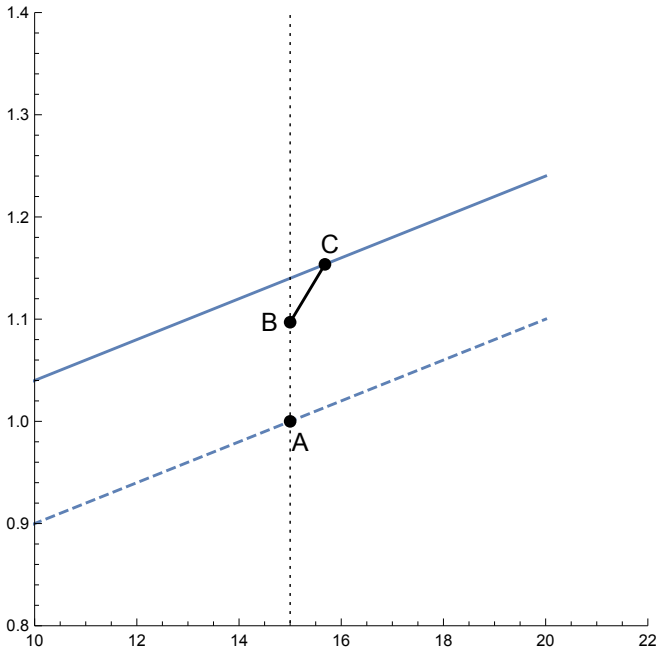


In[1186]:=

```
(* Linearized phase diagram *)
iso1[a_, ρ_, w_] := ρ a + w
Fiso0 := Plot[iso1[a, ρ0, w0], {a, 0, 20},
  PlotRange → {{10, 20}, {0.5, 1.5}}, PlotStyle → Dashed, AxesLabel → {"a", "c"}]
Fiso1 := Plot[iso1[a, ρ0, w1], {a, 0, 20}, PlotRange → {{0, 20}, {0.5, 1.5}}]
(* ic:=Graphics[InfiniteLine[{ass0,0},{ass0,1.4}]]*)
ic = Graphics[{Black, Dotted, Line[{{ass0, 0}, {ass0, 1.4}}]}]
Lss0 := Graphics[{PointSize[Large], Black, Point[{ass0, css0}]}]
Lss1 := Graphics[{PointSize[Large], Black, Point[{ass1, css1}]}]
Lin0 := Graphics[{PointSize[Large], Black, Point[{ass0, ctsolt[0]}]}]
Leg0 := Graphics[Text[Style["A", FontSize → 14, Black], {ass0 + 0.2, css0 - 0.02}]]
Leg1 := Graphics[Text[Style["C", FontSize → 14, Black], {ass1 + 0.1, css1 + 0.02}]]
Leg2 := Graphics[Text[Style["B", FontSize → 14, Black], {ass0 - 0.4, ctsolt[0]}]]
Traj := ParametricPlot[{atsolt[t], ctsolt[t]}, {t, 0, 200},
  PlotRange → {{10, 22}, {0.8, 1.4}}, AspectRatio → 1, PlotStyle → Black]
Show[Traj, Fiso0, Fiso1, Lss0, Lss1, Lin0, ic, Leg0, Leg1, Leg2, AspectRatio → 1]
```

Out[1189]=

Out[1197]=



In[1198]:=
```
(* Trajecgtories of income and consumption *)
TC := Plot[Piecewise[{{ctsolt[t], t > 0}, {1, t < 0}}],
  {t, -10, 100}, PlotStyle → Black, PlotRange → Full]
jumpc := Graphics[{Black, Dashed, Line[{{0, 1}, {0, ctsolt[0]}}]}]
TY := Plot[Piecewise[{{ρ0 atsolt[t] + w1, t > 0}, {1, t < 0}}],
  {t, -10, 100}, PlotStyle → Blue, PlotRange → Full]
jumpy := Graphics[{Blue, Dashed, Line[{{0, 1}, {0, ρ0 atsolt[0] + w1}}]}]
T0 := Plot[1, {t, -1, 0}, PlotRange → Full]
Show[TC, TY, jumpc, jumpy,
 PlotRange → {{-5, 100}, {0.99, 1.2}}, AxesLabel → {"t", "c,y"}]
```

Out[1203]=