New-Keynesian models of capital accumulation

Paulo B. Brito
Advanced Macroeconomics (PhD in Economics): 2022-2023
ISEG
Universidade de Lisboa
pbrito@iseg.ulisboa.pt

19.11.2022

Contents

1 Introduction

In the benchmark dynamic general equilibrium (DGE) model[1] all the markets are competitive (i.e, all agents are price-takers), and there is only one good which was used both for final consumption and for investment. In this model the equilibrium is Pareto optimal, which makes it equivalent to the Ramsey model of optimal capital allocation.

There are two common features of new-Keynesian models (NK):

- first, equilibrium is not Pareto optimal;
- second, there can be multiple asymptotic equilibria and/or multiple equilibrium paths (indeterminacy), which may justify the need to some type of intervention by economic authorities.

New-Keynesian models are DGE models in which some markets are not competitive. Imperfect competition is usually introduced by separating the final good production from intermediate good production and specifying the existence of a continuum or intermediate goods in which every producer is monopolistic in its own market and there is monopolistic competition among producers. In these models the DGE is not necessarily Pareto optimal.

A fundamental reason for developing those models stems from the need to justify the empirical observation on the existence of the Phillips curve, which apparently renders counterfactual two important characteristics of the classic macroeconomics: the separation of the real and nominal variables, and the neutrality of money. An important strand of New-Keynesian models deals with the existence of nominal rigidities and the need to a micro-founded model to address them. In most of the New-Keynesian models of inflation and unemployment the stock of capital is taken as constant.[2] However, in this lecture we deal with new-Keynesian models in which there is capital accumulation. Section 2 presents a simple new-Keynesian model with monopolistic competition.

Multiple steady states exist if the aggregate rate of return of capital displays non-convexities, stemming from local non-existence of decreasing marginal returns. In section 3 we extend the model of section 2 by considering the transition from a MC regime to a Cournotian monopolistic competition regime (CMC) in which there can be entry of firms and monopolies change to oligopolies in the intermediate goods markets. In this case we will see that (deterministic) multiple steady states can exist. This means that, given some shocks to an economy, the general competitive level can increase or decrease.

However, in some cases non-convexities can be introduced by the existence of externalities. Section 4 presents an abbridged version of the Benhabib and Farmer (1994) model. We see that, if the elasticity of labor supply is sufficiently low relative to the share of labor in the aggregate

[1] See `https://pmbbrito.github.io/cursos/phd/am/am2223_ramsey.pdf`.

[2] See Woodford (2003) or Galí (2008).

production the equilibrium can be indeterminate, in the sense that there are an infinite number of equilibrium paths.

Recent concerns on what is perceived as a reduction in competitiveness, and the concomitant increase in markups, in some industrialized economies have increased the appeal of these models.[3]

2 A monopolistic competitive model

This section presents a new-Keynesian (NK) model for an economy with a structure similar to the basic RBC model. It features a decentralized economy with two types of product markets, for a final good and for a continuum of intermediate goods, and two factor markets (labor and capital). The final good market and the factor markets are competitive (in the sense that all participants are price-takers) but there is monopolistic competition in the intermediate goods' markets.

We will see that the equilibrium representation is similar to the Ramsey model, with the difference that there is a markup over the marginal rate of return of capital. That is, there is a wedge between the rate of return of capital, which is determined in the asst market, and the marginal productivity of capital, which is generated in production.

2.1 The model

In this section we present the problems of the three types of representative agents of the model: households, the final good producer and the generic problem for an intermediate good producer. The last subsection defines the DGE for this economy.

We introduce several simplifying assumptions. In particular, we assume that the final product sector only uses intermediate inputs, all physical capital and labor are used in the intermediate goods' sector, the labor supply is inelastic, there are no adjustment costs for both types of inputs, and there is a classic dichotomy between prices and quantities. This means, in the NK terminology, that there are no real or nominal rigidities.

2.1.1 Household problem

The representative household consumes, offers inelastically labor and invests in a risk-free financial asset. It has an initial level of net financial wealth, and receives a flow of labor and financial income. We assume that its preferences are represented by an intertemporal additive utility functional and a CRRA utility function. Its problem is

[3] See Philippon (2019) and Eeckhout (2021).

$$
\begin{aligned}
\max_{C} V[C] = \int_0^\infty & \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t}\, dt \\
& \text{subject to} \\
& \dot{W} = \omega(t)L(t) + r_w(t)W(t) - C(t) \\
& W(0) = W_0, \text{ given} \\
& \lim_{t\to\infty} e^{-R_w(t)} W(t) \geq 0,
\end{aligned} \tag{1}
$$

where $C(\cdot)$, $L(\cdot)$ and $W(\cdot)$ denote consumption, labor supply, and the net financial wealth, respectively. In addition, $\omega(t)$ and $r_w(t)$ denote the wage rate and the real interest rate and $R(t) = \int_0^t r(s)ds$. Observe that, because the wage rate and the rate of return of capital are endogenous at the general equilibrium level, we set them as variable. All the quantity variables are in real terms, deflated by the final good price $P(t) = 1$.

The first order conditions, are, from Pontryiagin's maximum principle

$$
\dot{W} = \omega(t)L(t) + r_w(t)W(t) - C(t) \tag{2}
$$

$$
\frac{\dot{C}}{C} = \frac{1}{\theta}(r_w(t) - \rho) \tag{3}
$$

together with the initial condition $W(0) = W_0$ and the transversality condition $\lim_{t\to\infty} C(t)^{-\theta}\, W(t)\, e^{-\rho\, t} = 0$.

2.1.2 Final producer problem

Final output uses a continuum of varieties of intermediate goods, $j \in [0,1]$, with a constant elasticity of substitution (CES) technology, via a Dixit and Stiglitz (1977) aggregator. Denoting the quantity of intermediate input of variety j, used at time t, by $x(j,t)$ and by $\mathbf{x} = \big(x(j,t)\big)_{j\in[0,1]}$ the ensemble of all inputs, the production function is a functional over $x(\cdot)$,

$$
Y(t) = F\big(\mathbf{x}(t)\big) = \left(\int_0^1 x(j,t)^{\frac{\varepsilon-1}{\varepsilon}}\, dj\right)^{\frac{\varepsilon}{\varepsilon-1}},
$$

where $\varepsilon > 1$ is the elasticity of substitution between.

The final producer is a price-taker in both markets, inputs and output, seeks to maximize the total costs for producing the output quantity $Y(t)$, where given the ensemble of input prices $\mathbf{p} = \big(p(j,t)\big)_{j\in[0,1]}$. The profit can be written as a functional over $\mathbf{x}$,

$$
\pi(\mathbf{x}(t)) = P(t)\, F\big(\mathbf{x}(t)\big) - C(\mathbf{x}(t)),
$$

where $C(\mathbf{x}) = \int_0^1 p(j,t)\, x(j,t)\, dj$ is total cost (again a functional over $x(\cdot)$). The problem for the producer of the final good is

$$\max_{x(\cdot,t)} \pi(\mathbf{x}(t)) \text{ s.t. } F\big(\mathbf{x}(t)\big) = Y(t) \tag{4}$$

The demand for input j is[4]

$$x(j,t) = \left(\frac{p(j,t)}{P(t)}\right)^{-\varepsilon} Y(t), \text{ for every } j \in [0,1], \text{ and } t \in [0,\infty) \tag{5}$$

where because of perfect competition the market price is equal to the marginal cost of producing one unit of output

$$P(t) = P^*(t) = \left(\int_0^1 p(j,t)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \ t \in [0,\infty).$$

Because we have no monetary variables we will set $P(t) = 1$, for all t. We see that the demand of variety j is negatively related to its own price relative to output price, $p(j,t)/P(t)$, and is positively related the level of activity Y. The dependence to the relative price, and the linearity to Y is a consequence of the fact that the production function $F(\cdot)$ displays constant returns to scale (it is homogeneous of degree one).

2.1.3 Intermediate producer of variety j

We assume that the number of industries, that produce all intermediate inputs, is fixed and is normalized to one. Furthermore, its industry $j \in [0,1]$ has only one firm. This means that every producer is a monopolist in its own industry. However, because varieties are substitutable in the production of the final good, there is some degree of competition at the aggregate level. This is a case of monopolistic competition (MC) [5] because, although the firm in any industry is a price-setter on the market for its output, j, it has to compete with all other industries in the supply of intermediate products to the producer of the final good, whose demand function is given in equation (5). An increase in the price of product j will generate a reduction in its demand and a redirection to other varieties.

We assume a Cobb-Douglas technology for producing variety j. The production uses labour and capital such that

$$x(j,t) = A\, k(j,t)^{\alpha} \ell(j,t)^{1-\alpha} \tag{6}$$

where $k(j,t)$ and $\ell(j,t)$ are the capital and labour inputs for producing $x(j,t)$.

[4] See Appendix A.

[5] Chamberlin (1933) is credited as the first to introduce this framework.

In order to simplify the model we assume a homogeneous technology across industries. In particular, it is assumed that the TFP is not product specific, that is $A(j) = A$ for every $j \in [0, 1]$.[6]

The instantaneous profit for producer j is, in real terms, deflated by the price of the final good,

$$\pi(j,t) = s(j,t) - w(t)\ell(j,t), \text{for } (j,t) \in \left[0,1\right] \times (0,\infty)$$

where real sales of variety j, using the price of the final product as a numéraire, is

$$s(j,t) \equiv \frac{x(j,t)p(j,t)}{P(t)}.$$

and $w(t)$ is the real wage. Substituting the demand function (5) then

$$s(j,t) = \frac{x(j,t)p(j,t)}{P(t)} = \left(\frac{x(j,t)}{Y(t)}\right)^{-\frac{1}{\varepsilon}} x(j,t).$$

If we write $\mu \equiv 1/\varepsilon \in (0,1)$ as the Lerner index, therefore,

$$s(j,t) = x(j,t)^{1-\mu} Y(t)^{\mu}. \tag{7}$$

If we substitute the production function (6), the profit of the intermediate producer j becomes

$$\pi(j,t) = \left(Ak(j,t)^{\alpha}\ell(j,t)^{1-\alpha}\right)^{1-\mu} Y(t)^{\mu} - \omega(t)\ell(j,t). \tag{8}$$

As we assume that capital is used in the production of intermediate goods, the problem of each intermediate producer is dynamic. We assume the classic Jorgenson (1967) model for the producer in which there are no adjustment costs in investment.

The firm's objective is to maximize the present-value of the cash-flow discounted by the market interest rate, subject to the accumulation equation for capital:

$$\begin{aligned} \max_{i(j,t),\ell(j,t)} \int_0^{\infty} & \left(\pi(j,t) - i(j,t)\right) e^{-R(t)}\, dt \\ & \text{subject to} \\ & \frac{dk(j,t)}{dt} = i(j,t) - \delta k(j,t), \text{ for each } t \in [0,\infty) \\ & k(j,0) = k_{j,0}, \text{ given} \end{aligned} \tag{9}$$

[6] An extension of the static Dixit and Stiglitz (1977) model, assuming heterogeneity in productivity across industries in Melitz (2003), became a benchmark in the international trade and industrial organization literatures.

for $R(t) = \int_0^t r(s)\, ds$, where $r(t)$ is the market rate of return of capital, and we assume that $k(j,t)$ is asymptotically non-negative. The optimum demand for labor and capital are **symmetric**, in the sense that they are the same for the producer of any intermediate good,[7]

$$\begin{aligned} \ell^*(j,t) &= \ell^*(t) = (1-\mu)\left(\frac{1-\alpha}{w(t)}\right) s^*(t), \text{ for each } j \in [0,1] \\ k^*(j,t) &= k^*(t) = (1-\mu)\left(\frac{\alpha}{r(t)+\delta}\right) s^*(t), \text{ for each } j \in [0,1]. \end{aligned} \tag{10}$$

Therefore, the optimal return from sales is (see Appendix B) symmetric across varieties $s^*(j,t) = s^*(t)$ for any $j \in [0,1]$, where

$$s^*(t) = \left[(1-\mu)\, A \left(\frac{\alpha}{r(t)+\delta}\right)^{\alpha} \left(\frac{1-\alpha}{w(t)}\right)^{1-\alpha}\right]^{\frac{1-\mu}{\mu}} Y(t), \text{ for each } j \in [0,1]$$

is the optimal sales level, which is a function of the cost of labor and capital inputs and of the output of the final good. Then, the profit for intermediate producers is also symmetric

$$\pi^*(j,t) = \pi^*(t) = \Big(1 - (1-\mu)\,(1-\alpha)\Big)\, s^*(t) = \big(\alpha + \mu - \alpha\,\mu\big)\, s^*(t)$$

is positive, because $\alpha + \mu - \alpha\,\mu \in (0,1)$. Differently from the final producer case, the profit is different from zero because there is monopolistic competition (MC): the intermediate producer of variety j is a monopolist in its own market but competes with the producer of all the other varieties that enter in the production of the final output $Y(t)$. It has a Lerner-markup given by $\mu = \frac{\varepsilon-1}{\varepsilon}$ which decreases with the elasticity of substitution ε.

Therefore, the optimal supply of intermediate inputs is also symmetric across industries, that is $x^*(j,t) = x^*(t)$ for all $j \in [0,1]$ where, from equation (7),

$$x^*(t) = Y(t)^{-\frac{\mu}{1-\mu}}\, s^*(t)^{\frac{1}{1-\mu}}, \text{ for each } j \in [0,1],$$

is also a function of the final output and of the profit of the intermediate producer.

2.1.4 Aggregation

Because different inputs are measured in real terms, they can be measured in different physical units. Therefore, in order to build the aggregate capital stock and labor input we need to choose an appropriate aggregator.

If we use the Dixit and Stiglitz (1977) aggregator we have the aggregate demand for capital and labor

$$K(t) = \left(\int_0^1 k^*(j,t)^{\frac{\varepsilon-1}{\varepsilon}}\, dj\right)^{\frac{\varepsilon}{\varepsilon-1}} = k^*(t)$$

[7] See Appendix B.

and

$$L(t) = \Big(\int_0^1 \ell^*(j,t)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \Big)^{\frac{\varepsilon}{\varepsilon-1}} = \ell^*(t).$$

where $k^*(t)$ and $\ell^*(t)$ are given in equation (10). Total investment is also obtained from

$$I(t) = \Big(\int_0^1 i^*(j,t)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \Big)^{\frac{\varepsilon}{\varepsilon-1}}$$

although it is indeterminate, as is well known in a Jorgenson (1967) model, we can determine it in equilibrium.

The supply of the intermediate goods, given the demand at equilibrium of the intermediate inputs is

$$\begin{aligned} X(t) &= \Big(\int_0^1 x(t)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \Big)^{\frac{\varepsilon}{\varepsilon-1}} \\ &= \left(\int_0^1 \left(\left(A\, k(t)^{\alpha}\, \ell(t)^{1-\alpha} \right)^{1-\mu} Y(t)^{\mu} \right)^{1-\mu} dj \right)^{\frac{1}{1-\mu}} \\ &= \left(A\, k(t)^{\alpha}\, \ell(t)^{1-\alpha} \right)^{1-\mu} Y(t)^{\mu}. \end{aligned}$$

2.1.5 General equilibrium

The **dynamic general equilibrium** (DGE) is defined by the aggregate flows of consumption, $\big(C(t)\big)_{t\in[0,\infty)}$, output $\big(Y(t)\big)_{t\in[0,\infty)}$, and by the allocation flows of production, capital input, labor input and prices of intermediate-goods, $\Big(\big(x(j,t), k(j,t), \ell(j,t), p(j,t)\big)_{j\in[0,1]}\Big)_{t\in[0,\infty)}$, the rate of return of capital $\big(r(t)\big)_{t\in[0,\infty)}$, and the wage rate $\big(\omega(t)\big)_{t\in[0,\infty)}$ such that

1. households solve their problem, in equation (1), given the rate of return and the wage rate;
2. the final producer solve its problem, in equation (4), given the prices of the intermediate goods;
3. every intermediate producer, $j \in [0,1]$, solve its problem, in equation (9), given the rate of return and the wage rate;
4. the balance sheet and consistency conditions hold: $W(t) = K(t)$ (households own firms);
5. the market clearing conditions are satified, for the final good market

   $$Y(t) = C(t) + I(t), \text{ for each } t \in [0,\infty)$$

   and for the factor markets

   $$L(t) = 1, \text{ and } r_w(t) = r(t) - \delta, \text{ for each } t \in [0,\infty).$$

2.2 General equilibrium representation and dynamics

The equilibrium in the market for the intermediate goods yields

$$Y(t) = X(t) = \left(A\,K(t)^{\alpha}\,L(t)^{1-\alpha}\right)^{1-\mu}\,Y(t)^{\mu}$$

Using the the equilibrium condition for the labor market yields

$$Y(t) = A\,K(t)^{\alpha}.$$

The total income distributed to households is $\omega(t)\,L(t) + r_w(t)\,W(t) = \omega(t) + r(t)\,K(t)$. But, at the aggregate, $\omega(t)L(t) + (r(t)+\delta)\,k(t) = (1-\mu)\,S(t)$.

As $Y(t) = X(t) = S(t)^{1-\mu}\,Y(t)^{\mu}$ then $S(t) = Y(t)$ then

$$\omega(t)\,L(t) + r_w(t)\,W(t) = (1-\mu)Y(t) - \delta\,K(t),$$

and the budget constraint of the household $\dot{W} = \omega(t)\,L(t) + r_w(t)\,W(t) - C(t)$ is equivalent, at the equilibrium, to

$$\dot{K} = (1-\mu)Y(t) - \delta\,K(t) - C(t).$$

We could obtain the same condition from the final good's market equilibrium $Y(t) = C(t)+I(t) = C(t) + \dot{K} - \delta\,K(t)$.

At last, the equilibrium condition in the capital market becomes

$$r_w(t) = r(t) - \delta = \alpha\,(1-\mu)\frac{s^*(t)}{K(t)} - \delta = \alpha\,(1-\mu)A\,K(t)^{\alpha-1} - \delta,$$

which allows us to write the Euler equation as $\dot{C} = \frac{C}{\theta}\big(r(t) - \rho - \delta\big)$.

2.3 Characterizing the equilibrium

Therefore all the variables defining the DGE the equilibrium is the solution to the system

$$\begin{aligned}\dot{K} &= Y(t) - C(t) - \delta K(t)\\ \dot{C} &= \frac{C(t)}{\theta}\big(r(t) - \rho - \delta\big)\end{aligned}$$

where

$$r(t) = (1-\mu)\,\alpha\,A\,K(t)^{\alpha-1}, \text{ and } Y(t) = (1-\mu)\,A\,K(t)^{\alpha}$$

is similar to the one we found for the Ramsey model, with a distortion (a wedge) $m \equiv 1-\mu$ between the market rate of return of capital r and the marginal productivity of capital $\alpha\,A\,K(t)^{\alpha-1}$. In this MC model the wedge is exogenous because the markup is ecogenous.

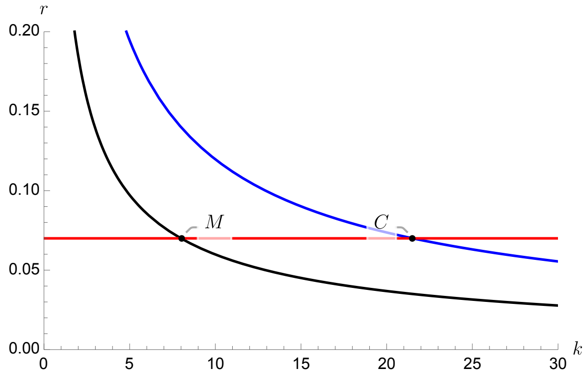

Figure 1: Competitive monopolistic M and competitive C equilibria

The dynamics of the model is similar to the Ramsey model with the difference that it converges to the steady state (K_M^*, C_M^*)

$$\begin{aligned} K_M^* &= \left(\frac{(1-\mu)\,\alpha\,A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \\ C_M^* &= \frac{\rho+\delta\,(1-\alpha)}{\alpha}\,K_M^*. \end{aligned} \tag{11}$$

In Figure 1 we compare the steady state between the competitive (Ramsey) and monopolistic case. Clearly the steady state stock of capital in this NK model is smaller than in the competitive case: $K_M^* < K_C^* = \left(\frac{\alpha\,A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}$. This implies that the steady state consumption level is smaller as well $C_M^* < C_C^* = \frac{\rho+\delta\,(1-\alpha)}{\alpha}\,K_C^*$. The first case refers to the monopolistic case and the second to the competitive case (se Figure 1)

3 A Cournotian monopolistic competitive model

Until now, we have dealt with two limit market environments: either there is full competition (the DGE-Ramsey case) or monopoly in the intermediate product sector (the previous case). Furthermore, in the first case the markup is equal to zero and in the second markups are positive but constant.

A more realistic situation is the one in which there is an intertemediate level of competition and markups are endogenous, and can vary across sectors. In this case we should allow for entry in every sector.

The consideration of a market environment halfway between perfect competition and monopolistic competition, that is oligopoly, requires the introduction of potencial strategic interactions between incumbent firms and entrants, that will make the model very hard to address. Thus our presentation will follow a typical macroeconomic approach by assuming that the coordination and information dimensions of the firms' problems is conveyed only by prices. This is usually the approach, for example, in growth theory (see Acemoglu (2009) in expansion of varieties and Schumpeterian models).

We assume that here are $z(t) \in (0,1]$ industries j, that is $j \in [0, z(t)]$. We also assume that the number of firms in any industry, denoted by $n(j,t)$, can be larger than one. If $n(j,t) = 1$ this means that there is only one producer, a monopolist, in industry j at time t.

Entry can take two forms: if $z(t) < 1$ then an entrant can start a new industry, thus becoming a monopolist; but if $z(t) = 1$ then entry can only be done by entering an existing industry, thus increasing the level of competition in industry j. Of course exit can have the inverse type of effect.

In this section we present an abridged version of Brito et al. (2013).

3.1 The model

Next we present the problems for the representative household, for the producer of the final good, for a firm in an intermediate good industry, equilibrium at the industry level and the general equilibrium.

3.1.1 Household's problem

As households are homogeneous and are the owners of capital, we can simplify the setup of the model by directly considering the budget constraint of households to be

$$\dot{K} = w(t) + r(t)\,K(t) + \pi(t) - C - \delta\;K \tag{12}$$

where total income comes from labor, capital income, which includes return from capital and rents associated to imperfect competition (plus-values). Therefore, savings are $w(t)+r(t)\,K(t)+\pi(t)-C$ and gross investment is $\dot{K} + \delta\;K$.

Thus, the household's problem is

$$\begin{aligned} &\max_{c(\cdot),\ell(\cdot)} \int_0^\infty \ln\left(C(t)\right) e^{-\rho\,t}\,dt \\ &\qquad\text{subject to} \\ &\qquad\dot{K} = w(t) + r(t)\,K(t) + \pi(t) - C - \delta\;K \\ &\qquad k(0) = k_0 \end{aligned}$$

Optimal consumption satisfies the Keynes-Ramsey rule

$$\dot{C} = C\,\big(r(t) - \rho - \delta\big), \tag{13}$$

and the transversality condition, $\lim_{t\to\infty} \dfrac{K(t)}{C(t)} e^{-\rho t} = 0$, should be verified.

3.1.2 Final production sector

The mass of intermediate goods at time t is $0 < z(t) \le 1$. The final good uses a continuum of intermediate goods $j \in [0, z(t)]$ with technology specified by the CES production function

$$Y(t) = z(t)^{\frac{1}{1-\varepsilon}} \Big(\int_0^{z(t)} X(j,t)^{\frac{\varepsilon-1}{\varepsilon}}\, dj \Big)^{\frac{\varepsilon}{1-\varepsilon}}. \tag{14}$$

For every $t \in [0, \infty)$, we find the demand for variety j to be

$$X^d(j,t) = p(j,t)^{-\varepsilon} \frac{Y(t)}{z(t)} \tag{15}$$

where we assumed that the price of the final good is

$$P(t) = \Big(\frac{1}{z(t)} \int_0^{z(t)} p(j,t)^{1-\varepsilon}\, dj \Big)^{\frac{1}{1-\varepsilon}} = 1.$$

3.1.3 Intermediate production sector

Industry level aggregates

Industry j is assumed to have $n(j,t)$ firms, each producing a homogeneous good of variety $j \in [0, z(t)]$, for $0 < z(t) \le 1$. To simplify we assume a continuum of firms, which means that we have a monopoly if $0 \le n(j,t) \le 1$ and an oligopoly if $n(j,t) > 1$. The price of variety j is $p(j,t)$ which is determined in an imperfectly competitive environment. The total output of variety j is the sum of the production by all firms in the industry j

$$X(j,t) = \int_0^{n(j,t)} x(i,j,t)\, di. \tag{16}$$

where $x(i,j,t)$ is the output of firm i producing variety j at time t.

We already know that the demand for variety j is given by equation (15). The supply is deduced from the production function of firm i in industry j, using capital and labor, $k(i,j,t)$ and $\ell(i,j,t)$, with a Cobb-Douglas technology

$$x(i,j,t) = A\, k(i,j,t)^{\alpha}\, \ell(i,j,t)^{1-\alpha} - \phi \tag{17}$$

where ϕ is a fixed cost. Both the productivity A and the shares of capital and labor are assumed to be the same for all firms, independently from it industry that they belong[8].

The total inputs of capital and labor in the industry are

$$K(j,t) = \int_0^{n(j,t)} k(i,j,t)\,di, \tag{18}$$

and

$$L(j,t) = \int_0^{n(j,t)} \ell(i,j,t)\,di. \tag{19}$$

Problem for an intermediate production firm

A firm i in industry j is assumed to solve the static problem, at each point in time t,

$$\begin{aligned} \max_{\ell(\cdot),k(\cdot)} \pi(i,j,t) &= p(j,t)\,x(i,j,t) - w(t)\ell(i,j,t) - r(t)\,k(i,j,t) \\ &\text{subject to} \\ &x(i,j,t) = A\,k(i,j,t)^{\alpha}\,\ell(i,j,t)^{1-\alpha} - \phi \end{aligned} \tag{20}$$

where $w(t)$ is the real wage and $r(t)$ is the rental rate of return of capital. It is assumed that there are no costs of adjustment in both inputs, which implies that firms can choose their quantities at every point in time.

In Appendix C we prove that the first order conditions are

$$\begin{aligned} (1-\mu(i,j,t))\,(1-\alpha)\,A\left(\frac{k(i,j,t)}{\ell(i,j,t)}\right)^{\alpha} &= \frac{w(t)}{p(j,t)} \\ (1-\mu(i,j,t))\,\alpha\,A\left(\frac{k(i,j,t)}{\ell(i,j,t)}\right)^{\alpha-1} &= \frac{r(t)}{p(j,t)} \end{aligned} \tag{21}$$

where the Lerner index for firm i in industry j.

$$\mu(i,j,t) = \frac{x(i,j,t)}{\varepsilon\,x(j,t)} \in (0,1).$$

Solving equations for the ratio $\dfrac{k(i,j,t)}{\ell(i,j,t)}$ we find

$$1-\mu(i,j,t) = \frac{1}{A\,p(j,t)}\left(\frac{w(t)}{1-\alpha}\right)^{1-\alpha}\left(\frac{r(t)}{\alpha}\right)^{\alpha} \text{ for every } i \in [0, n(j,t)].$$

[8] Making the productivity and technology industry- of firm-specific is one of the components of heterogeneous agent new-Keynesian (HANK) models, allowing for studying problems of capital misallocation, but is beyond the level of complexity required for this note.

Therefore, the Lerner markup is equal for all firms in industry j, that is $\mu(i,j,t) = \mu(j,t)$. This implies there should exist a symmetry among the firms producing variety j, that is the production of all firms in the market for variety j produce the same output. This implies, using equation (16), that $x(i,j,t) = \dfrac{X(j,t)}{n(j,t)}$ that the markup is dependent on the number of firms

$$\mu(j,t) = \frac{1}{\varepsilon\, n(j,t)}.$$

Furthermore, the inputs of capital and labor should also be equal across firm. From (18), and (19) we have $k(i,j,t) = \frac{K(j,t)}{n(j,t)}$ and $\ell(i,j,t) = \frac{L(j,t)}{n(j,t)}$. Therefore, the optimal production of firm i in industry j is

$$X(j,t) = A\,K(j,t)^{\alpha}\,L(j,t)^{1-\alpha} - n(j,t)\,\phi$$

and the profit for every firm is

$$\pi(j,t) = \pi(i,j,t)\,n(j,t) = p(j,t)\Big(\mu(j,t)\,A\,K(j,t)^{\alpha}\,L(j,t)^{1-\alpha} - n(j,t)\,\phi\Big).$$

Symmetric inter-sectoral equilibrium There is still some industry specific output and use of inputs because it depends on the number of firms, $n(j,t)$, that can be industry-dependent.

From now on we **assume a symmetric equilibrium**. This has two implications:

1. the number of firms is equal in all industries

$$n(j,t) = n(t), \text{ implying } \mu(j,t) = \mu(t) = \frac{1}{\varepsilon n(t)} \text{ for all } j \in [0, z(t)].$$

2. the capital and labor inputs, $K(j,t)$ and $L(j,t)$ are symmetric across industries.

Defining the aggregate capital stock and labor by

$$K(t) = \int_0^{z(t)} K(j,t)\,dj$$

and

$$L(t) = \int_0^{z(t)} L(j,t)\,dj$$

and assuming that the total labor supply is constant and equal to $L(t) = 1$, for any t, then

$$K(j,t) = \frac{K(t)}{z(t)}, \text{ and } L(j,t) = \frac{1}{z(t)}.$$

This implies

$$X(j,t) = \frac{X(t)}{z(t)} = \frac{1}{z(t)}\Big(A\,K(t)^{\alpha} - z(t)\,n(t)\,\phi\Big).$$

This has several implications: the equilibrium level of final output is, from the production function (14)

$$Y(t) = X(t) = A\,K(t)^{\alpha} - z(t)\,n(t)\,\phi$$

the prices for all intermediate varieties j is constant and equal to $p(j,t) = 1$, for every $j \in [0, z(t)]$ and for $t \in [0, \infty)$, and the profits for all industries is symmetric

$$\Pi(j,t) = \frac{\Pi(t)}{z(t)},\ \text{for}\ \Pi(t) = \mu(t)\,AK(t)^{\alpha} - z(t)\,n(t)\,\phi.$$

Entry and exit of firms

The dynamics of entry and exit of firms is still undefined, which implies that the markup is still undefined, as well.

Next we assume that there are zero costs of entry.

In order to determine the markup, we consider two regimes:

1. there is monopolistic competition (MC) if there is a monopolist per industry, that is $n(j,t) = n(t) = 1$, but there is still room for change in the number of industries, that is $z(t) < 1$. In this case the markup is as in the previous section

   $$\mu(t) = \frac{1}{\varepsilon},\ \text{for each}\ t \in [0, \infty).$$

   An entrant has to start a new industry, implying that the zero-profit condition determines $z \in (0,1)$: $z(t) = \frac{AK(t)^{\alpha}}{\phi\,\varepsilon}$;

2. there is Cournotian monopolistic competition (CMC) when the number of varieties is saturated and entry can only be done by entering an existing industry, thus increasing competition: $z(t) = 1$ and $n(t) > 1$. Entry increases competition within industries, and therefore determines n. As $n\,\varepsilon\,\mu = 1$ the **free entry condition**, profits equal entry costs, $(\Pi = 0)$ determines the markup. In this case $\pi = \mu\,A\,K^{\alpha} - z\,n\,\phi$ with $\mu = \frac{1}{\varepsilon n}$ which yields an endogenous markup

   $$\mu = m(K) = \sqrt{\frac{\phi}{\varepsilon\,A\,K^{\alpha}}}.$$

3. at the boundary we have $z = n = 1$.

Therefore, the markup is

$$\mu = m(K) = \sqrt{\frac{\phi}{\varepsilon A K^{\alpha}}} = \begin{cases} \frac{1}{\varepsilon} & \text{if } 0 < K \le \left(\frac{\varepsilon\,\phi}{A}\right)^{\frac{1}{\alpha}} \\ \sqrt{\frac{\phi}{\varepsilon\, A\, K^{\alpha}}} & \text{if } K > \left(\frac{\varepsilon\,\phi}{A}\right)^{\frac{1}{\alpha}} \end{cases} \tag{22}$$

General equilibrium

Aggregating over industries, as in the previous section, we find the final good output is

$$Y = (1-\mu)AK^{\alpha}$$

if we assume the free entry condition.

Furthermore at equilibrium the rate of return of capital is

$$r(K) = (1-\mu)\alpha A K^{\alpha-1}$$

an the wage rate is

$$w(K) = (1-\mu)(1-\alpha)AK^{\alpha}$$

Substituting in the budget constraint of the household yields

$$\dot{K} = (1-\mu)AK^{\alpha} - C - \delta\, K$$

We could also obtain the same equation from the market clearing condition for the final good's market

$$Y(t) = C(t) + I(t) = C(t) + \dot{K} + \delta\, K(t)$$

3.2 Characterizing general equilibrium

We can represent the general equilibrium by a dynamic system with two regimes:

$$\dot{K} = Y(K) - \delta\, K - C$$
$$\dot{C} = C\left(r(K) - (\rho+\delta)\right).$$

where the aggregate output is

$$Y(K) = (1-\mu(K))\, A\, K^{\alpha} = \begin{cases} \left(1-\frac{1}{\varepsilon}\right) A\, K^{\alpha} & \text{if } 0 < K \le \tilde{K} \\ \left(1 - m(K)\right) A\, K^{\alpha} & \text{if } K > \tilde{K} \end{cases}$$

and the rate of return of capital is

$$r(K) = (1-\mu(K))\,\alpha\,A\,K^{\alpha-1} = \begin{cases} \Big(1-\frac{1}{\varepsilon}\Big)\,\alpha\,A\,K^{\alpha-1} & \text{if } 0 < K \le \tilde{K} \\ \Big(1-m(K)\Big)\,\alpha\,A\,K^{\alpha-1} & \text{if } K > \tilde{K}. \end{cases}$$

where the critical value for the capital stock is

$$\tilde{K} \equiv \Big(\frac{\varepsilon\,\phi}{A}\Big)^{\frac{1}{\alpha}}.$$

This critical value is reduced by an increase in TFP (A) and is increased by reductions in the elasticity of substitutions between varieties, $1/\varepsilon$, and by increases in the sunk cost.

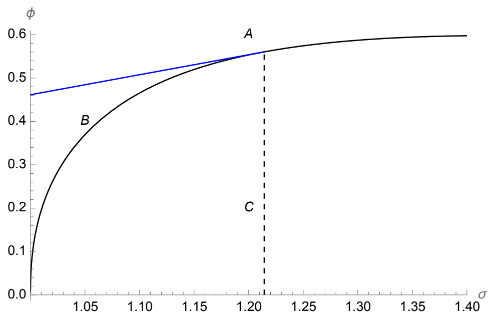

Figure 2: Bifurcation diagram

In Brito et al. (2013) it is proved that there exists a critical value for the inverse of the Frisch elasticity:

$$\bar{\varepsilon} \equiv \frac{2-\alpha}{2\,(1-\alpha)} > 1$$

and a two critical value for the fixed cost

$$\tilde{\phi} \equiv \frac{1}{\varepsilon}\Big(A\Big(\frac{\alpha\,(\varepsilon-1)}{(\rho+\delta)\,\varepsilon}\Big)^{\alpha}\Big)^{\frac{1}{1-\alpha}}$$

and

$$\bar{\phi} \equiv \frac{\varepsilon}{\bar{\varepsilon}^2}\Big(A\Big(\frac{\alpha\,(\bar{\varepsilon}-1)}{(\rho+\delta)\,\bar{\varepsilon}}\Big)^{\alpha}\Big)^{\frac{1}{1-\alpha}}$$

such that three regimes can exist (see Figure 2):

A if $\varepsilon > 1$ and $\phi > \max\{\bar{\phi},\tilde{\phi}\}$ then there is a single steady state in which there is MC (this is the case in the previous section). This steady state is also a saddle point (see Figure 3)

B if $1 < \varepsilon < \bar{\varepsilon}$ and $\tilde{\phi} < \phi < \bar{\phi}$ there are three steady states: a MC steady state and two CMC steady states, one for a low level of entry and the second for a high level of entry. The first steady state is close to a monopoly steady state. One of the CMC steady states corresponds to a low level of entry, a high markup, and to a low level of capital stock, and is unstable. The second CMC steady state has a high level of entry, a low level of markups. and the steady state level of capital capital stock is also high, and it is a saddle point. Therefore, it is (locally) close to a Ramsey case (see Figure 3);

C if $\varepsilon > 1$ and $0 < \phi < \max\{\bar{\phi}, \tilde{\phi}\}$ then there is a single steady state in which there is CMC. This case is closer to the Ramsey model. This steady state is also a saddle point. (see Figure 3)

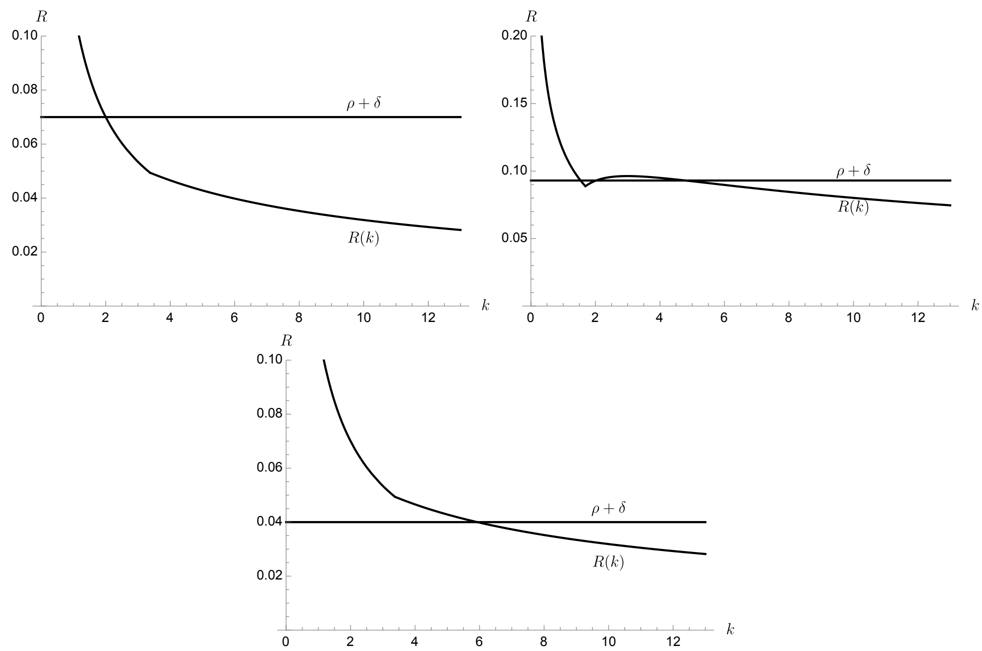

Figure 3: Steady state for the three cases: case A upper left diagram, case B upper right diagram, and case C, lower diagram

In the case B there are multiple steady states.

If the initial point $K(0) = K_0$ is lower than the middle equilibrium point (K_M) the economy will converge to the lower steady state in which there is a smaller number of sectors $z(\infty) < 1$ and there is a monopolist in every sector. Observe that, even if the K_0 corresponds to a case in which $z(0) = 1$ and $n(0) > 1$ there will be a reduction in the number of firms until it reaches a point in which $z(t) = n(t) = 1$ and, from that point on, some sectors disapear.

If the initial point $K(0) = K_0$ is higher then the middle equilibrium point (K_M) then the number of industries will be kept constant $z(t) = 1$ and there will be entry in all industries until the aggregate stock of capital satisfies

$$r(K_L) = (1 - \mu(K_L))\, \alpha\, A\, K_L^{\alpha-1} = \rho + \delta$$

which implies a steady state number of firms per industry equal to $n(K_L) = \sqrt{\frac{A\,K_L^{\alpha}}{\phi\,\varepsilon}}$.

A reduction in productivity can have the effect of changing the transition dynamics from the second case to the first case.

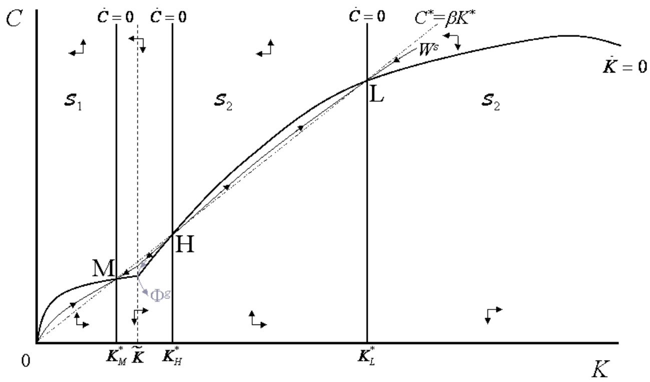

Figure 4: Phase diagram for the case B

In those Figures, an increase (a decrease) in the critical value for the capital stock $\tilde{K}$ can change the convergence to a competitive (non-competitive) steady state, as L (M), to a convergence to a less (more) competitive steady state, as M (as L).

4 A new-Keynesian model with externalities

In all those cases the steady state is locally determinate: given an initial level for the stock of capital there is a unique trajectory converging to the steady state, even though, with a different initial level for the stock of capital the economy can converge to a **different** steady state.

Next we present a case in which there is one unique steady state but there are multiple trajectories converging to it: that is, the equilibrium is **indeterminate**.

Benhabib and Farmer (1994) is an important paper featuring a new-Keynesian model in which there are increasing returns to scale and externalities similar to Romer (1990) model in growth theory. In particular, it presents conditions under which the general equilibrium can be indeterminate even though agents have perfect foresight.

4.1 The model

The **household production function** is

$$y = Ak^a\,\ell^b, \text{for } a > 0,\ b > 0,$$

where productivity is a function of the aggregate capial and labor inputs

$$A = K^{\alpha-a}\,L^{\beta-b},$$

That is, the aggregate capital and labor inputs generate a positive externality over the individual household. This can account for several different factors: infrastructures, network externalities, etc.

The **aggregate production function** is

$$Y = K^{\alpha}\,L^{\beta},\ \text{for } \alpha > 0,\ \beta > 0.$$

W assume that $\alpha+\beta > 1$, meaning that there are increasing returns to scale at the aggregate level, and $\alpha > a$ and $\beta > b$.

We can simplify the problem by assuming that the representative household consumes and does home-production and decides over investment

The **general equilibrium** for this economy is represented by the paths of capital $\big(k(t)\big)_{t\geq 0}$, labor effort $\big(\ell(t)\big)_{t\geq 0}$, consumption $\big(c(t)\big)_{t\geq 0}$ such that:

1. the representative household solves the problem

$$\begin{aligned}
&\max_{c(\cdot),\ell(\cdot)} \int_0^{\infty} \ln\left(c(t)\right) - \frac{\ell(t)^{1+\xi}}{1+\xi}\, e^{-\rho\, t}\, dt \\
&\text{subject to} \\
&\dot{k} = Ak^{\alpha}\,\ell^{\beta} - c - \delta\ k \\
&k(0) = k_0
\end{aligned}$$

   given the aggregate paths of capital $\big(K(t)\big)_{t\geq 0}$ and labor input $\big(L(t)\big)_{t\geq 0}$;

2. the micro-macro consistency conditions

$$C(t) = c(t),\ K(t) = k(t),\ \text{and } L(t) = \ell(t),\ \text{for every } t \in [0,\infty)$$

   hold.

First-order conditions for the household

$$\dot{c} = c\left(a\,\frac{\hat{y}}{k} - \rho - \delta\right)$$
$$\hat{\ell}^{1+\xi} = \frac{b\,\hat{y}}{c}$$

where $\hat{y} = A(K,L)\,k^a\,\ell^b$. Substituting the micro-macro consistency conditions, defining

$$\epsilon \equiv \beta - (1+\xi)$$

and representing the dynamic system in (K, L) we have (see Brito et al. (2017))

$$\dot{K} = K^{\alpha}L^{\beta}(1 - bL^{\epsilon-\beta})\ ,$$
$$\dot{L} = \frac{L}{\epsilon}\,\left[K^{\alpha-1}L^{\beta}\left(a - \alpha(1 - bL^{\epsilon-\beta})\right) - \rho\right],$$

where ϵ can take any sign.

There is a unique positive steady-state

$$\bar{K} = (a/\rho)^{\frac{1}{1-\alpha}}\,\bar{L}^{\frac{\beta}{1-\alpha}},\ \text{and}\ \bar{L} = b^{\frac{1}{\beta-\epsilon}}.$$

If we determine the Jacobian, evaluated at the steady state we obtain the trace and determinant

$$\mathrm{tr}DF(\bar{K},\bar{L}) = \frac{\rho^2(1-\alpha)(\beta-\epsilon)}{a\epsilon},\ \det DF(\bar{K},\bar{L}) = \frac{\rho\,(\beta a - \alpha\,(\beta-\epsilon))}{a\epsilon}.$$

We can easily see that both these quantities can take infinite values for $\epsilon = 0$, which is a case we exclude from now on. With the previous assumptions on the parameters ($\alpha > a$, $0 < \alpha < 1$ and $\rho > 0$) we find that the dimension of the stable manifold depends on ϵ: for $\epsilon < 0$ the steady-state is either a stable focus or node, and for $\epsilon > \epsilon$ it is a saddle point, as can be seen in Figure 5 which represents the phase diagram in the space (K, L):

On the left-hand-side (LHS) panel, we represent the phase diagram for $\epsilon > \beta$. We can observe that there is a unique steady-state equilibrium represented by point E. Since there are two negative eigenvalues associated with this stationary point, all DGE paths converge asymptotically to E. However, the steady-state is locally and globally indeterminate, as there is an infinite number of initial values for L, for a given K_0, leading to the long-run equilibrium. We can also see that L adjusts very fast so that the trajectory quickly approaches the isocline $\dot{L} = 0$ and then K starts adjusting more slowly until the steady-state is reached.

On the right-hand-side (RHS) panel, we represent the phase diagram for $\epsilon < 0$, also small. Now, the unique steady-state is locally and globally determinate, as there is one positive and one negative eigenvalue associated with it. For each initial level for the capital stock, K_0, there is only one value of L, such that there is convergence to the steady-state is asymptotically verified . Notice

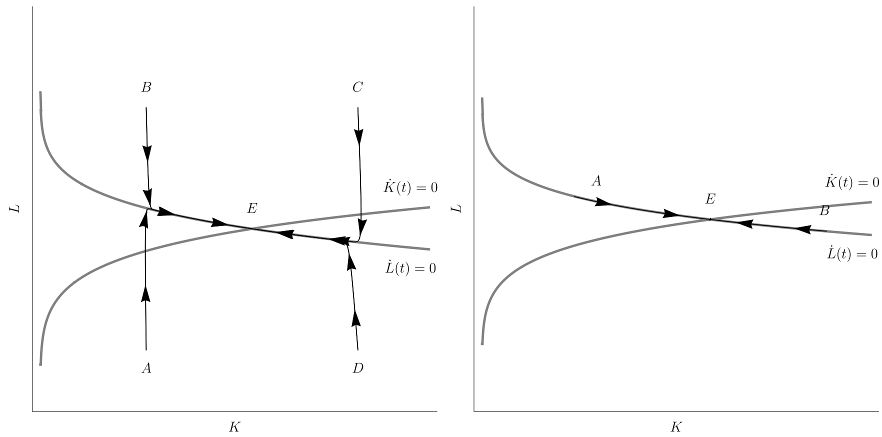

Figure 5: Phase diagrams for the Benhabib and Farmer (1994) model: LHS case $\epsilon > 0$, RHS case $\epsilon < 0$

that the stable manifold associated to the steady-state, $\mathcal{W}^s$, stays very close to the $\dot{L} = 0$ isocline, meaning that labor adjusts faster than capital, as in the case $\epsilon > 0$.[9]

The first (second) case occurs for a high (low) share of labor in the aggregate product or for a high (low) Frisch elasticity (which is again equal to $1/\xi$). Therefore, indeterminacy is more probable if the adjustment of labor supply is very high. For benchmark values of the parameters $\beta = 0.7$ and $\varepsilon = 2$ the second case can be taken as a benchmark.

5 Conclusion

We have presented in this note three main aspects of new-Keynesian models:

1. existence of distortions generating an inefficient general equilibrium
2. possible existence of multiple steady states: dependence of the equilibrium path on the initial level of pre-determined variables and of the parameters of the model
3. possible existence of indeterminacy: multiple paths converging to a steady state.

We can also distinguish between local indeterminacy and global indeterminacy. There is global indeterminacy when we combine the existence of multiple steady states with the existence of at

[9] For $\epsilon = 0$, there is a degenerate case where the adjustment of L is automatic, so that the stable manifold coincides with the $\dot{L} = 0$.

least one steady state which is locally determinate (i.e, in a model there are steady states which are locally determinate and others which are locally indeterminate).[10]

References

Acemoglu, D. (2009). *Introduction to Modern Economic Growth.* Princeton University Press.

Benhabib, J. and Farmer, R. (1994). Indeterminacy and increasing returns. *Journal of Economic Theory*, 63:19–41.

Brito, P., Costa, L., and Dixon, H. (2013). Non-smooth dynamics and multiple equilibria in a Cournot-Ramsey model with endogenous markups. *Journal of Economic Dynamics and Control*, 37(11):2287–2306.

Brito, P., Costa, L., and Dixon, H. (2017). From Sunspots to Black Holes: singular dynamics in macroeconomic models. In Nishimura, K. and all, editors, *Sunspots and Non-linear Dynamics*, chapter 3. Springer.

Brito, P. and Venditti, A. (2010). Local and global indeterminacy in two sector models of endogenous growth. *Journal of Mathematical Economics*, 46(5):893–911.

Chamberlin, E. (1933). *The Theory of Monopolist Competition.* Harvard University Press, Cambridge. republished 1960.

Dixit, A. and Stiglitz, J. (1977). Monopolistic competition and optimum product diversity. *American Economic Review*, 67(3):297–308.

Eeckhout, J. (2021). *The Profit Paradox. How Thriving Firms Threaten the Future of Work.* Princeton University Press.

Galí, J. (2008). *Monetary Policy, Inflation, and the Business Cycle.* Princeton.

Jorgenson, D. (1967). The theory of investment behavior. In Ferber, R., editor, *Determinants of Investment Behavior*, pages 129–175. NBER.

Melitz, M. J. (2003). The impact of trade on intra-industry reallocations and aggregate industry productivity. *Econometrica*, 71(6):1695–1725.

Philippon, T. (2019). *The Great Reversal.* The Belknap Press of Harvard University Press, Cambridge, Mass.

[10]For an example in growth theory see Brito and Venditti (2010).

Romer, P. (1990). Endogenous technological changes. *Journal of Political Economy*, 98(5):S71–S102.

Woodford, M. (2003). *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

A Solution of final producer problem (4)

The Lagrangean is, ignoring the time argument,

$$L(\mathbf{x},\lambda) = P\,Y - C(\mathbf{x}) + \lambda\left(F(\mathbf{x}) - Y\right)$$

where λ is the Lagrange multiplier (an unknown constant). The first order conditions are

$$\begin{cases} \dfrac{\delta L(\mathbf{x})}{\delta x(j)} = 0 \iff \dfrac{\delta C(\mathbf{x})}{\delta x(j)} = \lambda \dfrac{\delta F(\mathbf{x})}{\delta x(j)} \iff p(j) = \lambda\left(\dfrac{F(\mathbf{x})}{x(j)}\right)^{\frac{1}{\varepsilon}}, \text{ for every } j \in [0,1] \\ \dfrac{\partial L(\mathbf{x})}{\partial \lambda} = 0 \iff F(\mathbf{x}) = Y. \end{cases}$$

Then $\lambda^{\varepsilon}\,Y = p(j)^{\varepsilon}\,x(j)$ for any $j \in [0,1]$. Then, the demand for input j is

$$x(j) = \lambda^{\varepsilon} p(j)^{-\varepsilon}\,Y,$$

depends on the Lagrange multiplier. But substituting in the constraint,

$$F(\mathbf{x}) = \left(\int_0^1 \left(\lambda^{\varepsilon} p(j)^{-\varepsilon}\,Y\right)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}} = Y$$

and solving for λ, we find that

$$\lambda^* = \left(\int_0^1 p(j)^{1-\varepsilon}\,dj\right)^{\frac{1}{1-\varepsilon}} = P^*$$

is the shadow cost of one unit of output.

Substituting in the profit functional yields

$$\pi^*(t) = Y(t)\left(P(t) - P^*(t)\right).$$

As the firm is price taker then $P^*(t) = P(t)$ and there is zero profit for every t. Therefore, the demand function is as in equation (5).

B Solution of the intermediate producer j problem (9)

The current-value Hamiltonian is

$$H(j,t) = s(j,t) - i(j,t) - \omega(t)\ell(j,t) + q(j,t)\left(i(j,t) - \delta k(j,t)\right).$$

where $s(j,t) = \left(Ak(j,t)^{\alpha}\ell(j,t)^{1-\alpha}\right)^{1-\mu} Y(t)^{\mu}$ is real sales.

The static optimality conditions are

$$\frac{\partial H(j,t)}{\partial \ell(j,t)} = (1-\mu)(1-\alpha)\frac{s^*(j,t)}{\ell(j,t)} - \omega(t) = 0 \tag{23}$$

$$\frac{\partial H(j,t)}{\partial i(j,t)} = -1 + q(j,t) = 0 \tag{24}$$

and the Euler equation and the transversality conditions are

$$\begin{aligned}\frac{dq(j,t)}{dt} &= r(t)q(j,t) - \frac{\partial H(t)}{\partial k(j,t)} \\ &= (r(t)+\delta)\, q(j,t) - (1-\mu)\,\alpha\, \frac{s^*(j,t)}{k(j,t)}\end{aligned} \tag{25}$$

and $\lim_{t\to\infty} e^{-R(t)}\, q(j,t)\, k(j,t) = 0$ for all j.

Equation (24) yields

$$q(j,t) = q(t) = 1, \text{ for every } j \in [0,1] \text{ and } t \in [0,\infty),$$

that is the shadow value of capital is the same for all sectors $q(t) = q(j,t)$ and it is constant $q(t) = 1$. Therefore, the gross rate of return of capital is obtained from (25)

$$r(t) + \delta = (1-\mu)\,\alpha\, \frac{s^*(j,t)}{k^*(j,t)}$$

Equation (23) also involves an arbitrage condition but now in the labor market

$$\omega(t) = (1-\mu)(1-\alpha)\, \frac{s^*(j,t)}{\ell^*(j,t)},$$

Substituting in the definition of optimal sales,

$$\begin{aligned}s^*(j,t) &= \left(A \left((1-\mu) \left(\frac{\alpha}{r(t)+\delta} \right) s^*(j,t), \right)^{\alpha} \left((1-\mu) \left(\frac{1-\alpha}{w(t)} \right) s^*(j,t) \right)^{1-\alpha} \right)^{1-\mu} Y(t)^{\mu} \\ &= \left[(1-\mu)\, A \left(\frac{\alpha}{r(t)+\delta} \right)^{\alpha} \left(\frac{1-\alpha}{w(t)} \right)^{1-\alpha} \right]^{1-\mu} s^*(j,t)^{1-\mu}\, Y(t)^{\mu},\end{aligned}$$

yields

$$s^*(j,t) = s^*(t) = \left[(1-\mu)\, A \left(\frac{\alpha}{r(t)+\delta} \right)^{\alpha} \left(\frac{1-\alpha}{w(t)} \right)^{1-\alpha} \right]^{\frac{1-\mu}{\mu}} Y(t),$$

which means that profits are symmetric across variety producers. Therefore, factor demand functions are symmetric as well

$$\ell^*(j,t) = \ell^*(t) = (1-\mu) \left(\frac{1-\alpha}{w(t)} \right) s^*(t), \text{ and } k^*(j,t) = k^*(t) = (1-\mu) \left(\frac{\alpha}{r(t)+\delta} \right) s^*(t).$$

Therefore there is also symmetry in the supply of varieties,

$$\begin{aligned}x^*(j,t) = x^*(t) &= Ak(t)^{\alpha} \ell(t)^{1-\alpha} \\ &= A \left(\frac{\alpha}{r(t)+\delta} \right)^{\alpha} \left(\frac{1-\alpha}{w(t)} \right)^{1-\alpha} s^*(t) \\ &= \left(\frac{\pi*(t)}{Y(t)} \right)^{\frac{\mu}{1-\mu}} s^*(t) \\ &= Y(t)^{-\frac{\mu}{1-\mu}}\, s^*(t)^{\frac{1}{1-\mu}}, \text{ for any } j \in [0,1], \text{ for each } t \in [0,\infty).\end{aligned}$$

The total cost to the firm,

$$(r(t)+\delta)\,k^*(j,t)+\omega(t)\,\ell^*(j,t)=(1-\mu)\,s^*(j,t),$$

is distributed to the household.

C Solution of the intermediate producer j problem (20)

Observe, first that the inverse demand function is

$$p(j,t)=\Big(X(j,t)\frac{z(t)}{Y(t)}\Big)^{-\frac{1}{\varepsilon}}=\Big(\frac{Y(t)}{z(t)}\Big)^{\frac{1}{\varepsilon}}\Big(\int_0^{n(j,t)}x(i,j,t)\,di\Big)^{-\frac{1}{\varepsilon}}$$

because $X(j,t)$ is the aggregate output of industry j as in equation (16). Therefore, the change in the price of the good produced in industry j from a unit increase in production of firm i is

$$\frac{\delta p(j,t)}{\delta x(i,j,t)}=-\frac{1}{\varepsilon}\frac{p(j,t)}{X(j,t)}.$$

The first-order conditions are, therefore

$$\Big(p(j,t)+\frac{\delta p(j,t)}{\delta x(i,j,t)}\,x(i,j,t)\Big)\,\frac{\partial x(i,j,t)}{\partial k(i,j,t)}=r(t)$$
$$\Big(p(j,t)+\frac{\delta p(j,t)}{\delta x(i,j,t)}\,x(i,j,t)\Big)\,\frac{\partial x(i,j,t)}{\partial \ell(i,j,t)}=w(t),$$

But

$$p(j,t)+\frac{\delta p(j,t)}{\delta x(i,j,t)}\,x(i,j,t)=\Big(1-\frac{x(i,j,t)}{\varepsilon\,X(j,t)}\Big)\,p(j,t)$$

and

$$\frac{\partial x(i,j,t)}{\partial k(i,j,t)}=\alpha\,A\,\Big(\frac{k(i,j,t)}{\ell(i,j,t)}\Big)^{\alpha-1}$$

and

$$\frac{\partial x(i,j,t)}{\partial \ell(i,j,t)}=(1-\alpha)\,A\,\Big(\frac{k(i,j,t)}{\ell(i,j,t)}\Big)^{\alpha}$$

yields equations (21).