Advanced macroeconomics 2022-2023
Problem set 1: consumption, savings, and asset accumulation

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Questions marked with asterisks have a higher degree of difficulty.

1 Intertemporal utility

1 Consider the uncertain horizon finite lifetime utility functional

$$\mathsf{U}[c] = \int_0^T \log c(t)\, e^{-\rho\, t}\, dt$$

where $\rho > 0$, $theta > 0$, and T is finite.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

2 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T -\frac{1}{\zeta}\, e^{-\zeta\, c(t)}\, e^{-\rho t}\, dt,\ \rho > 0,\ \zeta > 0.$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

3 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \frac{c(t)^{1-\theta} - 1}{1-\theta}\, e^{-\rho t}\, dt,\ \rho > 0,\ \theta > 0.$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

4 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \left(c(t) - \frac{\beta}{2}c(t)^2\right) e^{-\rho t}\, dt,\ \rho > 0,\ \beta > 0.$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

5 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T c(t)\, e^{-\rho t}\, dt,\ \rho > 0,\ \theta > 0.$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

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$$\mathsf{U}[c] = \int_0^T \frac{c(t)^{1-\theta} - 1}{1-\theta}\, dt,\ \rho > 0,\ \theta > 0.$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

7 Consider the uncertain horizon finite lifetime utility functional in which the terminal time has a Poisson distribution with instantaneous mortality rate $\mu > 0$,

$$\mathsf{U}[c] = \int_0^\infty \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{-(\rho+\mu)\,t}\, dt$$

where $\rho > 0$ and $theta > 0$.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

8 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \log\big(c(t)\big)\, e^{\int_0^t \log\left(c(s)\right) ds}\, dt,$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

9 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \log\big(c(t) - \zeta h(t)\big)\, e^{-\rho t}\, dt,\ \rho > 0,\ \zeta > 0,$$

where

$$\dot{h} = \eta\,(c - h)$$

and $h(0) = h_0$, where h_0 is given.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

10 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \log\left(\frac{c(t)}{h(t)^\zeta}\right) e^{-\rho t}\, dt,\ \rho > 0,\ \zeta > 0,$$

where

$$\dot{h} = \eta\,(c - h)$$

and $h(0) = h_0$, where h_0 is given.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

11* Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \log\left(\frac{c(t)}{h(t)^\zeta}\right) e^{-\rho t}\, dt,\ \rho > 0,\ \zeta > 0,$$

where

$$\dot{h} = \eta\,(c - h)$$

and $h(0) = h_0$, where h_0 is given.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

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$$\mathsf{U}[c] = \int_0^T \log\left(\frac{c(t)}{h(t)^\zeta}\right) e^{-\rho t}\, dt,\ \rho > 0,\ \zeta > 0,$$

where

$$\dot{h} = \eta\, c - \delta\, h,\ \text{for } 0 < \delta. < 1$$

and $h(0) = h_0$, where h_0 is given.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

2 Household problems

1 Assume that a consumer has utility functional

$$\mathsf{U}[c] = \int_0^T \left(c(t) - \frac{\beta}{2}c(t)^2\right) e^{-\rho t}\, dt,\ \rho > 0,\ \beta > 0.$$

and no constraints on consumer.

(a) Find the optimal consumption function

(b) Would the solution change if consumer had an initial wealth $w(0) = w_0$ and no further constraints on wealth ?

(c) Would the solution change if consumer had an initial wealth $w(0) = w_0 > 0$ and had a constraint on wealth such that $w(t) \geq 0$?

(d) Discuss the previous results.

2 Consider the problem

$$\max_c \int_0^T \log\left(c(t)\right) e^{-\rho t} dt$$

subject to

$$\dot{a}(t) = r\, a - c(t),\ \text{for } t \in \mathrm{T}$$

$$a(t) \in [\underline{\mathrm{a}}, \infty),\ \text{for every } t \in [0, T]$$

$$a(0) = a_0 > \max\{0, \underline{\mathrm{a}}\} \text{ given}$$

(a) Find the optimality conditions.

(b) Find the solution to the problem. Under which conditions it is optimum to saturate the borrowing constraint at the terminal time T ?

(c) Provide an intuition for your results.

3** Consider the problem

$$\max_c \int_0^\infty \log\left(c(t)\right) e^{-\rho t} dt$$

subject to

$$\dot{a}(t) = \begin{cases} r_1\, a + y - c(t), & \text{for } 0 \leq t < t_s \\ r_2\, a + y - c(t), & \text{for } t_s \leq t < \infty \end{cases}$$

$$a(0) = a_0 > \text{ given}$$

$$\lim_{t\to\infty} e^{-r_2\, t} a(t) \geq 0$$

where $y > 0$ and $r_1 > \rho > r_2$.

(a) Find the optimality conditions.

(b) Find the solution to the problem.

(c) Draw the phase diagram.

(d) Provide an intuition for your results.

3 Comparative dynamics

1 Consider the problem

$$\begin{aligned}
&\max_c \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = (1-\tau)\big(r\,a + w\big) - c(t), \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&\lim_{t\to\infty} a(t)\, e^{-r\,t} \geq 0
\end{aligned}$$

where $0 < \tau < 1$ is the income tax rate.

(a) Find the optimality conditions.

(b) Study the comparative dynamics effects for an anticipated, permanent and constant increase in the income tax rate τ.

(c) Provide an intuition for your results.

2 Consider the problem

$$\begin{aligned}
&\max_c \int_0^\infty -\frac{e^{-\xi c(t)}}{\xi}\, e^{-\rho t} dt,\ \xi > 0 \\
&\text{subject to} \\
&\dot{a}(t) = (1-\tau)\big(r\,a + w\big) - c, \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&\lim_{t\to\infty} a(t)\, e^{-r\,t} \geq 0,
\end{aligned}$$

where $0 < \tau < 1$ is the income tax rate.

(a) Find the optimality conditions.

(b) Study the comparative dynamics effects for an anticipated, permanent and constant increase in the income tax rate τ.

(c) Provide an intuition for your results.

4 Habit formation

1 Consider the problem

$$
\begin{aligned}
&\max_{c} \int_0^{\infty} \log\big(c(t) - \zeta\, h(t)\big)\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\, a + w - c(t), \ \text{for } t \in \mathbb{R}_+ \\
&\dot{h}(t) = \eta\,\big(c - h\big) \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&h(0) = h_0 \text{ given} \\
&\lim_{t \to \infty} a(t)\, e^{-r\, t} \geq 0
\end{aligned}
$$

for $\rho > 0$, $0 < \zeta < 1$ and $\eta > 0$.

(a) Find the first order conditions

(b) Under which conditions there will be transitional dynamics

(c) . Discuss the dynamics of consumption and income for a non-anticipated, permanent and constant increase in non-financial income w.

(d) Is consumption response perfectly correlated with income ? Why ?

2 Consider the problem

$$
\begin{aligned}
&\max_{c} \int_0^{\infty} \log\big(c(t) h(t)^{-\zeta}\big)\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\, a + w - c(t), \ \text{for } t \in \mathbb{R}_+ \\
&\dot{h}(t) = \eta\,\big(c - h\big) \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&h(0) = h_0 \text{ given} \\
&\lim_{t \to \infty} a(t)\, e^{-r\, t} \geq 0
\end{aligned}
$$

for $\rho > 0$, $0 < \zeta < 1$ and $\eta > 0$.

(a) Find the first order conditions

(b) Under which conditions there will be transitional dynamics

(c) . Discuss the dynamics of consumption and income for a non-anticipated, permanent and constant increase in non-financial income w.

(d) Is consumption response perfectly correlated with income ? Why ?