Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2019-2020

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Exam: **Época Normal**
17.6.2020 (18.00h-20.15h)

Warning:

- This is an online open book exam. This implies that in the assessment the following two points will be taken into consideration:

  > Your answer should be concise, objective and specific. Any notation, calculation, motivation, discussion or explanation **not strictly related to the specificquestion**it tries to address will either not be considered or have a negative assessment.

- Classification: 1 - 4 points, 2 - 6 points, 3 - 7 points, 4 - 3 points.
- Your exam will only be considered if it is uploaded in Aquila between 20:10 and 20:15.

1. The benchmark model for epidemics is the so-called SIR model. It considers the division of total population, N, between subjected, S, infected, and removed, R, by immunisation or death. Therefore $N(t) = S(t)+I(t)+R(t)$ for every moment in time t. The dynamic model for epidemics is

$$\dot{S} = -\beta\, S\, I \tag{1}$$
$$\dot{I} = \beta\, S\, I - \gamma\, I \tag{2}$$
$$\dot{R} = \gamma\, I \tag{3}$$

   all the parameters are positive, and have the following meaning: β is the infection rate and γ is the recovery rate.

   (a) Prove that if the total population is constant, equation (3) is redundant, i.e, it can be obtained from the other two.

   (b) Find the steady state(s) of the model (consider only equations (1) and (2)). Study analytically and geometrically the dynamics of the model.

   (c) Are there bifurcations in the model ? If yes, of which type ?

   (d) Assume that, for a given value of the parameter β, e.g. $\beta = \beta_0$, the values of (S, I), at some point in time t, are such that $I(t)$ is close zero and $S(t) < \frac{\gamma}{\beta_0}$. Now introduce a perturbation on β, from β_0 to β_1, such that $\beta_1 > \beta_0$, implying $S(t) > \frac{\gamma}{\beta_1}$. Which type of trajectory of the epidemics one should expect instead.

2. Let the value of a firm be given by the present value of cash-flows,

$$\int_0^\infty \left(AK(t) - \frac{I(t)^2}{2} \right) e^{-rt} dt$$

where K is the capital stock, I is gross investment and $r > 0$ is the constant rate of interest. The capital accumulation constraint is $\dot{K} = I - \delta K$ where $\delta > 0$ is the capital depreciation rate. The objective of the firm is to maximize its value by using gross investment as the control variable, and satisfying the capital accumulation and the solvability constraint: $\lim_{t\to\infty} e^{-rt}K(t) \geq 0$

(a) Write the optimality conditions by using the Pontriyagin's maximum principle.

(b) Draw the phase diagram for the maximized Hamiltonian dynamic system (MHDS) in the space (K, I).

(c) Find and explicit solution for the optimal capital stock.

(d) Use instead the principle of dynamic programming. Find the Hamilton-Jacobi-Bellman (HJB) equation.

(e) By solving the HJB equation (hint: use a quadratic trial value function), find an explicit solution for the optimal capital stock such that the marginal value function is bounded asymptotically.

(f) Compare the two approaches for solving the firm's problem.

3. For a given initial level of the capital stock $K(0) = k_0$, the stock of capital, $K(t)$, of a firm evolves according to the diffusion process

$$dK(t) = \big(I(t) - \delta\, K(t)\big)dt + \sigma\, dW(t),\ t > 0$$

where $I(t)$ is gross investment, $\delta > 0$ is the capital depreciation rate, and $\sigma\, dW(t)$ represents additive random shocks to the capital accumulation process, where $\{W(t)\}_{t\in\mathbb{R}_+}$ is a Wiener process.

The firm's objective is to use gross investment to maximize the expected value of the present value of the cash flows,

$$\mathbb{E}_0\left[\int_0^\infty \left(AK(t) - \frac{I(t)^2}{2}\right)e^{-rt}dt\right].$$

(a) Write the Hamilton-Jacobi-Bellman equation.

(b) Solve the HJB equation (hint: use a linear trial value function).

(c) Obtain the stochastic differential equation (SDE) for the optimal capital accumulation equation.

(d) Solve that equation. **Remark**: if you could not obtain the SDE in (c), use instead $dK(t) = \big(\bar{I} - \delta\, K(t)\big)dt + \sigma\, dW(t)$ where $\bar{I} > 0$ is a constant, in this question and in the following questions.

(e) Find the statistics for the optimal capital stock of the firm (expected value and variance).

(f) Provide an intuitive discussion of your results (hint: compare them with the deterministic analog).

(g) Write the Fokker-Planck-Kolmogorov equation associated to the distribution of the capital stock, starting from $K(0) = k_0$, which is fully observed.

(h) Solve this equation for the case in which $\delta = 0$.

4. Consider the first-order partial differential equation (PDE) $y_t(t, x) + ax\, y_x(t, x) = 0$, where $y = y(t, x) \in \mathbb{R}$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

(a) Find the solution to the PDE (hint use the method of characteristics).

(b) Let $y(0, x) = e^{-x^2}$. Find the solution to the initial-value problem. Provide an intuitive characterization of the solution.