Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2019-2020

Lecturer: Paulo Brito
Exam: **Época de Recurso** (Re-sit exam)
6.7.2020 (18.00h-20.15h)

Warning:

- This is an online open book exam. This implies that in the assessment the following two points will be taken into consideration:

  Your answer should be concise, objective and specific. Any notation, calculation, motivation, discussion or explanation **not strictly related to the specific question** it tries to address will either not be considered or have a negative assessment.

- Classification: 1 - 5 points, 2 - 5 points, 3 - 5 points, 4 - 5 points.
- Your exam will only be considered if it is uploaded in Aquila between 20:10 and 20:15.

1. Consider a Solow growth model in which the production function is

$$Y = K^{\alpha} L^{\beta} S^{1-\alpha-\beta}, \text{ for } 0 < \alpha < 1, \text{ and } 0 < \beta < 1-\alpha. \tag{1}$$

where K is private physical capital, S, are public infrastructures, and L is the labor input which is equal to total population. Assume population is constant, the investment expenditures in private capital is $I = \dot{K} + \delta K$ and government expenditures are only used to increase and maintain public capital $G = \dot{S} + \delta S$, where $\delta > 0$ is the depreciation rate. Assuming government expenditures are financed by income taxes and the government follows a balanced budget rule we have $G = \tau Y$, where $\tau \in (0, 1)$ is the tax rate. The equilibrium equation for the product market is $Y = C + I + G$. Defining $y \equiv Y/L$, $k \equiv K/L$ and $s \equiv S/L$ this economy is represented by the system of coupled differential equations over (k, s), where $y = y(k, s)$ is the per-capita product associated to production function (1)

$$\begin{aligned} \dot{k} &= (1 - c - \tau)\, y(k, s) - \delta\, k, \\ \dot{s} &= \tau\, y(k, s) - \delta\, s \end{aligned} \tag{2}$$

where $0 < c < 1 - \tau$ is the marginal (equal to average) propensity to consume, and $k(0) = k_0$ and $s(0) = s_0$ are given. Assume that $(k, s) \in \mathbb{R}^2_{++}$.

   (a) Draw the phase diagram of system (2).

   (b) Find the steady state and characterize analytically the phase diagram. Are there any local bifurcations ? Why ?

   (c) Study the long-run multipliers for an increase in the tax rate, τ. Which effect can we expect on the long-run output level ?

2. Consider the following optimal control problem

$$\min_{u(\cdot)} \int_0^{\infty} \left(x(t)^2 + \frac{\alpha}{2}\, u(t)^2\right) e^{-\rho t} dt$$

subject to $\dot{x} = x(t) - u(t)$, for every $t \in (0, \infty)$ and $x(0) = x_0$ is given. All the parameters are positive, that is $\rho > 0$ and $\alpha > 0$. The terminal value of $x(\cdot)$ is free.

   (a) Write the problem as a calculus of variations problem. Find the first-order necessary conditions for optimality. Are they sufficient ?

   (b) Find the general solution to the Euler-Lagrange equation. Discuss the existence and uniqueness of a solution to the problem for different values of the parameter ρ.

   (c) Assume that $\rho \leq \tilde{\rho} \equiv 1 + \frac{\alpha}{2}$. Find the solution to the problem.

3. Consider the following stochastic resource-depletion problem, where $\{X(t)\}_{t\in\mathbb{R}}$ is the process for the stock of the resource, and $\{C(t)\}_{t\in\mathbb{R}}$ is the process for its use,

$$\max_{C(\cdot)} \mathbb{E}_0\Big[\int_0^{\infty} \ln\left(C(t)\right) e^{-\rho\, t}\Big]$$

subject to

$$dX(t) = -C(t)\, dt + \sigma\, X(t)\, dW(t), \text{ for } t \in (0, \infty)$$

where $\{W(t)\}_{t\in\mathbb{R}}$ is a Wiener process, and $X(0) = x_0 > 0$ is given. The rate of time preference and the volatility parameters, ρ and σ, are both positive and satisfy $\rho > \sigma^2$.

   (a) Write the first-order conditions for optimality according to the stochastic Pontriyagin's maximum principle.

   (b) Find the stochastic process for $C(t)$.

   (c) Using $C(t) = \phi\, X(t)$, for an undetermined constant ϕ , as a trial function find the solution for the optimal $X(t)$.

4. Consider the diffusion equation

$$dX(t) = -X(t)dt + dW(t),\ t \in [0, \infty)$$

where $\{W(t)\}$ is a standard Wiener process.

   (a) Let $X(0) = x_0$, where x_0 is a real number. Find the solution to the initial value problem.

   (b) Write the forward Fokker-Planck-Kolmogorov (FPK) for the density associated to $X(t) = x$, conditional on $X(0) = x_0$, that is $p(t, x) = \mathbb{P}[X(t) = x \,|X(0) = x_0]$.

   (c) Let $P(t, s) = \mathcal{F}[p(t, x)]$ be the Fourier transform of $p(t, w)$. Find $P(t, s)$, which is the solution to the transformed FPK equation together with the initial condition $P(0, s) = \mathcal{F}[\delta(x-x_0)] = e^{-2\,\pi\, i\, s\, x_0}$(tip: $\mathcal{F}[x\, \partial_x p(t, x)] = -\big(P(t, s)+s\, \partial_s P(t, s)\big)$ and $\mathcal{F}[\partial_{xx}\, p(t, x)] = -(2\,\pi\, s)^2\, P(t, s)$).