Advanced Mathematical Economics

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Chapter 1

Linear ODE: the scalar case

1.1 Introduction

A **scalar ODE** is a differential equation in which a (unknown) function $y(x)$ is derived from a known function $F(y'(x), y(x).x) = 0$, where x is the a real independent variable with domain X, i.e., $x \in \mathrm{X} \subseteq \mathbb{R}$, and y is a real valued function of x with range in the one-dimensional space of real numbers, that is $y : \mathrm{X} \to \mathrm{Y} \subseteq \mathbb{R}$ and $y'(x)$ is the derivative of y.

Recalling that

$$y'(x) \equiv \frac{dy(x)}{dx} = \lim_{h\to 0} \frac{y(x+h) - y(x)}{x}$$

then a scalar ODE models phenomena which we describe by the existence of a local interaction. The particular features of that local interaction is encoded in function $f(\cdot)$.

A **scalar ODE in normal form** that interests us takes has the form

$$y'(x) \equiv \frac{dy(x)}{dx} = f(y(x), x),$$

Most of the times we are interested in equations of the form $y'(x) = f(y(x), x, \varphi)$, where $\varphi \in \mathbb{R}^m$ is a vector of parameters and function $f(\cdot)$ is known.

The known function $f(\cdot)$ constrains the family of functions to which function y belongs, which we call $\mathcal{Y}$. If function$f(\cdot)$ is continuous and differentiable, then the solution of solutions of the differential equation are elements of the space of continuous and differentiable functions.

Solving a scalar ODE means finding a function, say $\phi(x)$, such that $\phi : \mathrm{X} \to \mathrm{Y}$ such that $\phi'(x) = f(\phi(x), x, \varphi)$. The question of the existence and uniqueness of solutions is related to number of elements of $\mathcal{Y}$ which satisfy the differential equation.

A **linear scalar ODE in normal form** is an ODE in which $f(\cdot)$ is a linear function of y. The most general form is

$$y'(x) = a(x)\, y + b(x)$$

where $f(y(x), x) = a(x)\, y + b(x)$.

Linear ordinary differential equations are the simplest ordinary differential equations (ODE). They always have one and only one **explicit (or closed form) solution**. This means that we can find exactly one and only one member of the functional space $\mathcal{Y}$ which solves. This means that a unique solution always exist.

Most applications of differential equations have time, t, as the independent variable. In this case, the convention is to use Newton's notation for the derivative, i.e., $\dot{y} \equiv \dfrac{dy(t)}{dt}$ and write the linear ODE as

$$\dot{y} = a(t)\, y + b(t)$$

where $y : \mathrm{T} \to \mathbb{R}$, where usually $\mathrm{T} \subseteq \mathbb{R}$.

The existence and uniqueness of solutions for linear ODE is known, the analysis of the solutions for non-linear ODE can be done by comparing them to the linear ODE. In particular, the qualitative theory for ODE is based upon the local approximation of non-linear ODE by linear ODE and by verifying conditions under which a non-linear ODE is (topologically) equivalent to a linear ODE (at least locally).

There are two major distinctions among scalar linear ODE, depending on the form of $f(.)$. The first, is related to the functional dependence of f on the independent variable x and the second refers to homogeneity property of function $f(\cdot)$ regarding y. We say that the ODE is **autonomous** if $f(\cdot)$ is independent of the independent variable x and it is **non-autonomous** if $F(\cdot)$ depends directly on the independent of it. We say that the ODE is **homogeneous** if F is an homogeneous function of y and **non-homogeneous** otherwise.

In most applications, differential equations are supplemented by some side conditions which may differ according to the problem under study.

In general a model (or a problem) involving an ODE takes the form

$$\begin{cases} y'(x) = f(y(x), x) \\ \mathsf{F}[y] \;\; = \text{constant} \end{cases}$$

where $\mathsf{F}[y]$ is a functional over y, i.e., $\mathsf{F} : \mathcal{F} \to \mathbb{R}$.

The characterization of the solution of a model featuring an ODE has a close relationship to the type of side conditions which are assumed.

For instance, in models in which the independent variable is not time the constraint takes the form $\int_{\mathrm{X}} \beta(y(x), x) dx = 0$. In general we have moment conditions, and we are interested in some global characteristics of the solution curve.

In models in which the independent variable time the constraint sometimes takes to form $\int_{\mathrm{T}} \delta(t - t_0) y(t) dx = y_{t_0}$ where $\delta(\dot{})$ is Dirac's delta generalized function. In this case we fix the value of the function for a particular value of the independent variable time, and want to characterize the evolution across Y across time. This leads to the stability and bifurcation analysis of the model: stability regarding some fixed points of Y, existence of invariant sets, dependence of the solution on parameters.

To distinguish the solution of an ODE from the solution to a model (or a problem) involving an ODE we call **general solution** to the solution of an ODE and **particular solution** to the solution of the latter. Although linear scalar ODE have one unique solution, models (or problems) involving them may not have solutions (if the constraint cannot be satisfied by the solution of the ODE). That is, the fact that a general solution exists and is unique does not imply that the particular solution exists.

We present in 1.2 the solution and some examples in which the independent variable is not time, and in section 1.3 we briefly present some results for ODEs as functions of other independent variables.

1.2 General solutions of scalar ODE's

Let us start with the homogeneous equation

$$y'(x) = a(x)y,\ y : \mathrm{X} \to \ \mathrm{Y} \tag{1.1}$$

where both X are Y are subsets of $\mathbb{R}$.

Proposition 1. *The unique solution of ODE* (1.1) *is a function* y

$$y(x) = k\, e^{\int_{\mathrm{X}} a(x)\, dx} \tag{1.2}$$

where k *is an arbitrary element of* Y

Proof. Recalling that we denoted $y'(x) = \frac{dy}{dx} = a(x)y$, we use the method of separation of variables[1] writing $\frac{dy}{y} = a(x)$. Integrating, we have

$$\int_{\mathrm{Y}} \frac{dy}{y} = \int_{\mathrm{X}} a(x)\, dx \Leftrightarrow \log y + k_y = \int_{\mathrm{X}} a(x)\, dx + k_x$$

where k_y and k_x are constant of integration. Taking exponentials on both sides and writing $k = \exp{(k_x - k_y)}$ yields the general solution (1.2). □

There are two particular cases of this equation,

[1] There are several methods we can employ to find the proof (separation of variables, Laplace transforms, Fourier transforms, transforming into an integral equation, using the concept of generating function, just to name a few)

1. If $a(x) = 0$, the solution (1.2) becomes $y(x) = k$;

2. If $a(x) = a$, a constant, the solution (1.2) becomes $y(x) = k\, e^{x}$;

where k is an arbitrary element of Y, in both cases.

Now, we consider the non-homogeneous equation

$$y'(x) = a(x)y + b(x),\ y : \mathrm{X} \to\ \mathrm{Y} \tag{1.3}$$

where, again, both X are Y are subsets of $\mathbb{R}$.

Proposition 2. *The unique solution of ODE* (1.3) *is a function* y

$$y(x) = y(x_0)\, e^{\int_{x_0}^{x}\ a(s)\, ds} + \int_{x_0}^{x} e^{\int_{x_0}^{s} a(z)\, dz}\, b(s)ds \tag{1.4}$$

where y_0 *is an arbitrary element of* Y *which we associate to* x_0*, i.e.,* $y(x_0) = y_0$

Proof. We apply the variation of constant method[2]. First, we consider the solution for the homogeneous equation, such that $b(x) = 0$ for all $x \in \mathrm{X}$. From equation (1.2) its solution for the fixed interval (x_0, x), for $x_0 < x$ is

$$y_h(x, y_0) = y_0\, e^{\int_{x_0}^{x}\ a(s)ds}$$

But we expect the solution to equation (1.3) to be, for an arbitrary element of $x > x_0$.

$$y(x) = y_h(x, y_0(x)) = y_0(x)e^{\int_{x_0}^{x}\ a(s)ds}. \tag{1.5}$$

Taking derivatives of the last equation we get

$$y'(x) = y_0'(x)e^{\int_{x_0}^{x}\ a(s)ds} + y_0(x)a(x)e^{\int_{x_0}^{x}\ a(s)ds}$$

which should be equal to equation (1.3). Equating the right-hand sides of both equations we get the ODE

$$y_0'(x) = b(x)e^{-\int_{x_0}^{x}\ a(s)ds}.$$

Applying the separation of variables to solve this equation we find

$$y_0(x) = y_0(x_0) + \int_{x_0}^{x} b(s)e^{-\int_{x_0}^{s}\ a(z)dz}ds.$$

Substituting in equation (1.5) and because $y_0(x_0) = y(x_0) = y_0$ we finally get solution (1.4) $\square$

Next we see some applications of models involving non-autonomous ODE's, which include particular side constraimnts

[2] Due to Lagrange (1811).

1.2.1 Some applications

The Gaussian distribution

We can derive the standard Gaussian density probability from the ODE problem,

$$y'(x) = -x\, y(x),\ x \in \mathrm{X} = (-\infty, \infty) \tag{1.6a}$$

$$\int_{-\infty}^{\infty} y(x)\ dx = 1 \tag{1.6b}$$

While equation (1.6a) means that the rate of decay between to adjacent points in X is equal to the value of x in which we measure it, equation (1.6a) constraints $Y(x) = \int_{-\infty}^{x} y(s)ds$ to be a distribution.

The solution to the problem is

$$y(x) = \frac{\sqrt{2\,\pi}}{e^{-\frac{x^2}{2}}} \tag{1.7}$$

To prove this, we already know that the solution to equation (1.6a) is

$$y(x) = k\, e^{\int_{-\infty}^{x} s\, ds} = k\, e^{-\frac{x^2}{2}}.$$

If we substitute this general solution in the constraint (1.6b) we find k as the solution of the equation

$$\int_{-\infty}^{\infty} k\, e^{-\frac{x^2}{2}}\ dx = 1 \Leftrightarrow k\sqrt{2\pi} = 1,$$

which yields (1.7).

The CRRA utility function

The function

$$u(x) = \frac{x^{1-\sigma} - 1}{1-\sigma}$$

has many uses, not only in economics. In economics, in deterministic models it is called iso-elastic utility function and in stochastic models it os called constant relative risk aversion (CRRA) Bernoulli utility function CRRA. In applied-mathematical handbooks it is called a generalized logarithmic function. [3]

[3] If $\sigma = 1$ it can be showed that it is $u(x) = \ln(x)$.

Using the analysis in Pratt (1964) it can be showed that it is a solution of the problem

$$-\frac{u''(x)\,x}{u'(x)} = \sigma,\, x \in \mathrm{X} = [1, \infty) \tag{1.8a}$$

$$\sigma \int_1^\infty \frac{u'(x)}{x}\, dx = 1 \tag{1.8b}$$

$$u(1) = 0 \tag{1.8c}$$

The first equation is a definition of the relative risk aversion, of that the symmetric of the elasticity of $u(\cdot)$ is constant and equal to σ, the first constraint conditions the relative slope of $u(\cdot)$ on all its domain and the last constraint fixes the value of utility of consumption at one (this condition makes transparent that a logarithm is hidden behind the utility function).

Equation (1.8a) is a second order ODE. We can transform it into a first order ODE by defining $y(x) = u'(x)$. Then $y'(x) = u''(x)$ and the ODE becomes equivalent to

$$\frac{y'(x)}{y(x)} = -\frac{\sigma}{x}.$$

We can simplify further if we define $z(x) = \log(y(x)$ (thus $z(x) = \log(u'(x))$, which implies

$$z'(x) = \frac{dz}{dx} = -\frac{\sigma}{x} \Rightarrow z(x) = \sigma\, \log(x) + k_z$$

where k_z is an arbitrary constant in the domain of z. Then

$$u'(x) = y(x) = e^{\sigma\, \log(x) + k_z} = k_y x^{-\sigma}$$

where k_y is an arbitrary constant in the domain of y. Integrating the last equation we find the general solution for $u(\cdot)$

$$u(x) = k_y \frac{x^{1-\sigma}}{1-\sigma} + k_u$$

We can find the two arbitrary constants k_y and k_u by substituting in equations (1.8b) and (1.8c) and solving. Note that

$$\sigma \int_1^\infty \frac{u'(x)}{x}\, dx = \sigma\, k_z \int_1^\infty x^{-\sigma-1}\, dx = \sigma\, k_z \left(\frac{x^{-\sigma}}{-\sigma}\Big|_{x=1}^{\infty} \right) = k_z$$

Therefore $k_z = 1$, and the last constraint writes $u(x)\Big|_{x=0} = \frac{1}{1-\sigma} + k_u = 0$ implies $k_u = -\frac{1}{1-\sigma}$.

1.3 Equations having time as the independent variable

A scalar ODE is **autonomous** if the coefficients are constant, i.e, they are independent of the exogenous variable t,

$$\dot{y} = \lambda y + \beta \tag{1.9}$$

where $(\lambda, \beta) \in \mathbb{R}^2$ are known constants. A scalar ODE is **non-autonomous** if the coefficients depend on the the independent variable [4]

$$\dot{y} = \lambda(t)y + \beta(t), \tag{1.10}$$

where $(\lambda, \beta) : \mathrm{T} \to \mathbb{R}^2$ are known functions of t.

A scalar ODE is **homogeneous** if $F(y, .)$ is a homogeneous function of y and it is **non-homogeneous** if $F(y, .)$ is non-homogeneous. Homogeneity of degree n holds if, for a constant ξ we have $F(\xi y) = \xi^n F(y)$. Therefore

$$\dot{y} = \lambda y, \text{ and } \dot{y} = \lambda(t)y$$

are homogeneous and

$$\dot{y} = \lambda y + \beta, \text{ and } \dot{y} = \lambda(t)y + \beta(t)$$

are **non-homogeneous**.

Solving an autonomous ODE means finding a function $y(t) = \phi(t, y; .)$, mapping $\phi : \mathrm{T} \to \mathrm{Y}$, depending on the parameters λ and β and on an arbitrary element $y \in \mathrm{Y}$. **Characterizing** the solution roughly means tracking the behavior of the flow of the elements of Y, denoted as $(y(t))_{t\in\mathrm{T}}$, when the independent variable changes.

This behavior is determined by function $F(y)$. In order to take account of it we introduce the following definitions:

- **equilibrium point** (or steady state): it is a point in range of y such that $F(y) = 0$, that is

$$\bar{y} = \{\ y \in \mathrm{Y} : F(y) = 0\}$$

- **stability properties** of the equilibrium point: the steady state is asymptotically stable if for any $y \in \mathrm{Y}$ the flow generated by the ODE (1.11), $(y(t))_{t\in\mathrm{T}}$, has the property

$$\lim_{t\to\infty} \phi(t) = \phi(\infty) = \bar{y};$$

the steady state is **unstable** if for any y in a neighborhood of $\bar{y}$, $y(t)$ does not converge to $\bar{y}$;

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[4] If we redefine the independent variable as $t = \tau$ we can transform the non-autonomous scalar linear ODE into a planar non-linear equation: $\dot{y_1} = \lambda(y_2)y_1 + \beta(y_2)$ $\dot{y}_2 = 1$ where $y_2(t) = t$.

- **invariant subsets** are partitions of set Y containing the whole solution path $(y(t))_{t\in \mathrm{T}}$; we call **attractor set** to the subset of points $y \in \mathrm{Y}$ such that the solution will converge to the steady state and **repelling set** to the set of points $y \in \mathrm{Y}$ such that the solution will not converge to the steady state.

At last, we draw a distinction between an ODE and a **problem** involving an ODE. To get an intuition of the difference, observe that equations (1.9) and (1.10) involve a time variation of $y(t)$ independent of the starting or the terminal point. While the solution of the scalar ODE is called **general solution**, the solution to a problem involving an ODE is called **particular solution**.

There are basically two types of problems. We have an **initial-value problem** if we have an ODE together with a known value for y at the initial time, and we have a **terminal-value problem** if we have an ODE together with a known value for y for the terminal time (or there is a functional constraint over the solution). In the first case, we call the ODE a **forward ODE** because the solution will be obtained from future instants (assuming that the present time is $t = 0$) and in the second case we call the ODE a **backward ODE**.

The evolution described by the ODE can be done forward in time (if we know the initial point) or backward in time (if we know a terminal point). With this additional information we can uniquely determine a forward or a backward path.

As we will see, uniqueness of the solution of an ODE is not the same as uniqueness of a problem involving an ODE. And this distinction has important conceptual differences in economic applications.

1.3.1 Autonomous equations

Next we introduce homogeneous and the non-homogeneous autonomous equations and their related problems.

We assume that $\mathrm{Y} = \mathbb{R}$. This means that, with the exception of the ODE and the initial or terminal constraints, there is no other constraint on the evolution of y.

The **scalar homogeneous ODE** is the linear equation

$$\dot{y} = \lambda y,\ y : \mathrm{T} \to \mathrm{Y} \tag{1.11}$$

where $y = y(t)$ is an unknown function with domain T and range Y. Usually $\mathrm{T} = [0, \infty)$ and Y is the set of real numbers ($\mathbb{R}$) or non-negative real numbers ($\mathbb{R}_+$), and λ is a real-valued parameter ($\lambda \in \mathbb{R}$).

As with any equation (or problem) there are three issues related to its solution: existence of solutions, number of solutions and characterisation of the solution

Proposition 3. *The* ***unique solution of equation*** (1.11) *is a function* $\phi : \mathrm{T} \to \mathrm{Y}$

$$\phi(t, k; \lambda) = ke^{\lambda t} \tag{1.12}$$

where $k \in \mathrm{Y}$, *is an arbitrary member of the range, and* $t \in \mathrm{T}$.

Proof. We use the separation of variables approach. It consists in four steps: first, as $\dot{y} \equiv dy/dt$ we can write equation (1.11) in an equivalent way, by separating y from t

$$\frac{dy}{y} = \lambda dt.$$

Second, we integrate both sides of the equation

$$\int \frac{dy}{y} = \int \lambda dt$$

because λ is a constant

$$\int \frac{dy}{y} = \lambda \int dt$$

third, we find the primitives

$$\ln(y) + C_y = \lambda t + C_t$$

where $C_y \in \mathrm{Y}$ and $C_t \in \mathrm{T}$ are two arbitrary constants of integration; at last, if we take exponentials of the two sides we get

$$e^{\ln(y)} = y = e^{\lambda t + (C_t - C_y)}$$

and we write $k = e^{C_t - C_y}$ □

We see that the solution $y(t) = \phi(t, k; \lambda)$ depends on the parameter λ and on an arbitrary point k in the domain Y. This is the **general solution** to equation (1.11).

Characterizing the solution means describing the behavior of the path $(y(t))_{t \in \mathrm{T}}$ from any point k. We readily see that: (1) if $\lambda < 0$ the solution converges to 0 if $t \to \infty$ independently from the value of k; (2) if $\lambda = 0$ the solution becomes $y(t) = k$ for any $t \in \mathrm{T}$, i.e, a constant; and (3) if $\lambda > 0$ the solution depends on the value k; it converges to zero if $k = 0$ and if converges to $+\infty$ if $k > 0$ and to $-\infty$ if $k < 0$.

The **steady states** for equation (1.11) are

$$\bar{y} = \begin{cases} k, & \text{if } \lambda = 0 \\ 0, & \text{if } \lambda \neq 0. \end{cases}$$

In the first case there is an **infinite number** of equilibria, consisting in all points in Y, and in the second there is a **single** equilibria if $0 \in \mathrm{Y}$, or no equilibria if $0 \notin \mathrm{Y}$.

When there is a steady state, that is, when $\lambda \neq 0$ we can characterize its **stability properties**:

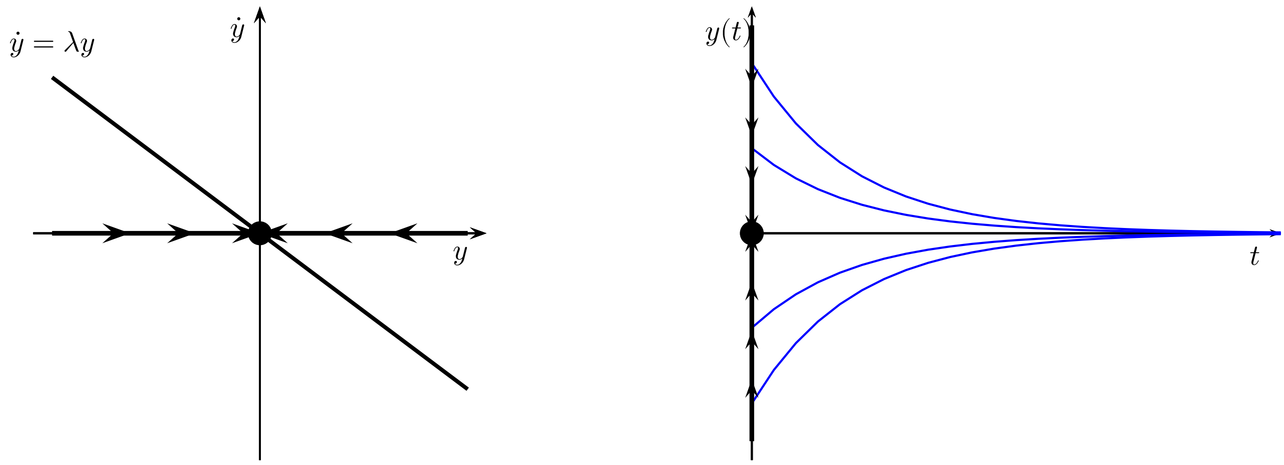

Figure 1.1: Phase diagram and trajectories of equation $\dot{y} = \lambda y$ for $\lambda < 0$

- if $\lambda < 0$ then $\lim_{t\to\infty} \phi(t, k, \lambda) = 0 = \bar{y}$ then the equilibrium point is asymptotically stable;
- if $\lambda > 0$ then

$$\lim_{t\to\infty} \phi(t, k, \lambda) = \begin{cases} \pm\infty & \text{, if } k \neq 0 \\ 0 & \text{, if } k = 0 \end{cases}$$

  and the equilibrium point $\bar{y}$ is unstable. In this case we say the solution can be non-stationary.

Therefore, if $\lambda \neq 0$, and $\bar{y} \in \mathrm{Y}$, there are only two kinds of possible **invariant sets**:

- if $\lambda < 0$ the basin of attraction for $\bar{y}$ is the whole set Y and Y **is the attraction set**. Then we say $\bar{y}$ is **globally asymptotically stable**;
- if $\lambda > 0$ then $\bar{y}$ is repelling and unstable and $\mathrm{Y}/\bar{y}$ is the unstable invariant set.

Figures 1.1 and 1.2 contain the **phase diagram** (left hand panel) and representative **orbits** (right-hand panel) for the asymptotically stable and unstable cases of the homogeneous equation, respectively.

The case $\lambda = 0$, where $\dot{y} = 0$, with solution $\phi(t) = k$ a constant is thus a degenerate case in which the solution is **always** time-invariant, i.e., it is independent from the exogenous variable t. Intuitively we can say that there are no dynamics, or that this corresponds to a boundary case between stability and instability. We can also get time-invariant solutions when $\lambda \neq 0$ but only in one case: when $k = \bar{y}$, i.e., if the arbitrary value of Y happens to be the steady state.

The **scalar linear non-homogenous** ODE is

$$\dot{y} = \beta + \lambda y, \ y \in \mathrm{Y} \subseteq \mathbb{R} \tag{1.13}$$

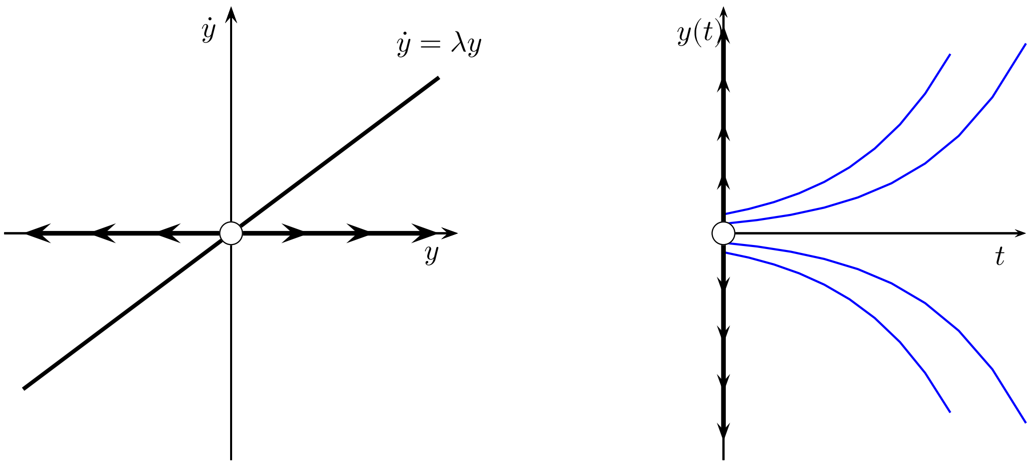

Figure 1.2: Phase diagram and trajectories of equation $\dot{y} = \lambda y$ for $\lambda > 0$

where β and λ are real-valued parameters.

Proposition 4. *The* ***unique solution of equation*** (1.13) *is a function* $\phi : \mathrm{T} \to \mathrm{Y}$

$$\phi(t, k; \beta, \lambda) = \begin{cases} \bar{y} + (k - \bar{y})e^{\lambda t}, & \text{if } \lambda \neq 0 \\ k + \beta t, & \text{if } \lambda = 0 \end{cases} \tag{1.14}$$

where

$$\bar{y} = -\frac{\beta}{\lambda}$$

where $k \in \mathrm{Y}$*, is an arbitrary member of the range, and* $t \in \mathrm{T}$*.*

Proof. First assume that $\lambda \neq 0$. Introduce a change in variables $z(t) = y(t) + \beta/\lambda$. Then $\dot{z} = \lambda z$, because

$$\dot{z} = \dot{y} = \beta + \lambda y = \beta + \lambda \left(z - \frac{\beta}{\lambda} \right) = \lambda z.$$

But we already know that the solution of $\dot{z} = \lambda z$ is $z(t) = k_z e^{\lambda t}$ where k_z belongs to the domain of z. Mapping back to the domain of y we have

$$y(t) + \frac{\beta}{\lambda} = \left(k + \frac{\beta}{\lambda} \right) e^{\lambda t}.$$

As $\bar{y} = -\beta/\lambda$ is the unique equilibrium point of equation (1.13) yields equation (1.14).

Next, let $\lambda = 0$, then from the l'Hôpital's rule.

$$\begin{aligned}
\phi(t;\lambda=0) &= ke^{\lambda t} - \frac{\beta\left(1-e^{\lambda t}\right)}{\lambda} \mid_{\lambda=0}= \\
&= k - \lim_{\lambda\to 0} \frac{\frac{d\left(\beta\left(1-e^{\lambda t}\right)\right)}{d\lambda}}{\frac{d(\lambda)}{d\lambda}} = \\
&= k - \frac{-\beta t}{1} = \\
&= k + \beta t.
\end{aligned}$$

□

The solutions are qualitatively similar to the homogeneous case (i.e, when $\beta = 0$) when $\lambda \neq 0$. The only (quantitative) difference are:

- the steady state is also unique, although it is shifted from $\bar{y} = 0$, if $\beta = 0$, to $\bar{y} = -\beta/\lambda$, if $\beta \neq 0$;
- the stability behavior is qualitatively the same but now relative to the equilibrium point $\bar{y} = -\beta/\lambda$: it is asymptotically stable if $\lambda < 0$ and it is unstable if $\lambda > 0$.

The solutions are qualitatively different when $\lambda = 0$. While in the homogenous case (i.e., if $\beta = 0$) the solution is stationary and there is an infinite number of steady states (all the elements of Y) in the non-homogeneous case (i.e, if $\beta \neq 0$) **there are no steady states** and the solution of the ODE is always non-stationary.

Figures 1.3 and 1.4 illustrate the phase diagram (left-hand panel) and orbits (right-hand panel) for the asymptotically stable and unstable cases, respectively, for the non-homogeneous equation.

1.3.2 Problems involving scalar ODE

The previous solutions are usually called **general or fundamental solutions**. In most applications we know (or we set) the value, say $y_\tau \in \mathrm{Y}$, at a particular point in time, say $t = \tau$. So a **problem** involving an ODE takes the form

$$\dot{y} = \lambda y,\ y(\tau) = y_\tau \in \mathrm{Y}. \tag{1.15}$$

Proposition 5. *The unique solution for problem* (1.15)*, usually called* ***particular solution****, is*

$$\phi(t, y_\tau; \lambda) = y_\tau e^{\lambda(t-\tau)}, \textit{for}\, t, \tau \in \mathrm{T},\ \textit{and}\, y_\tau \in \mathrm{Y} \tag{1.16}$$

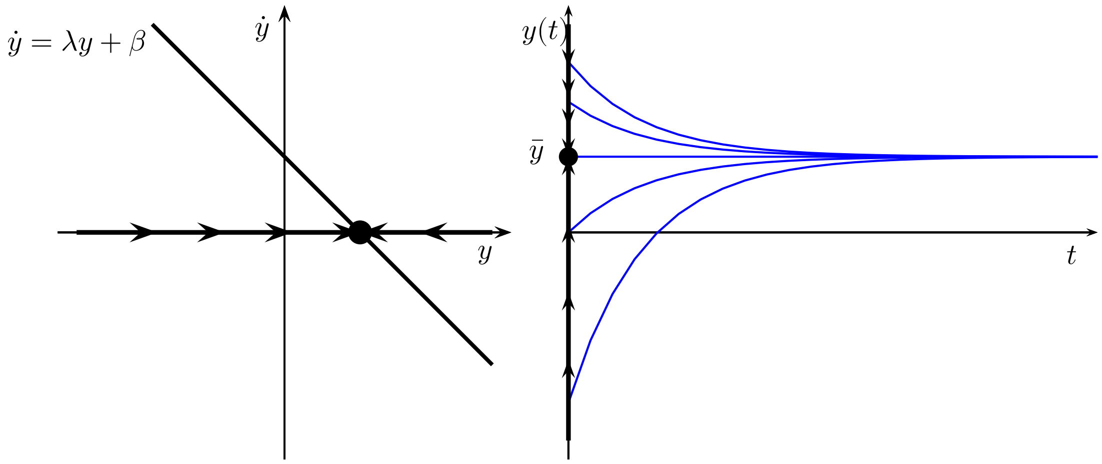

Figure 1.3: Phase diagram and trajectories for $\dot{y} = \lambda y + \beta$ for $\lambda < 0$ and $\beta > 0$

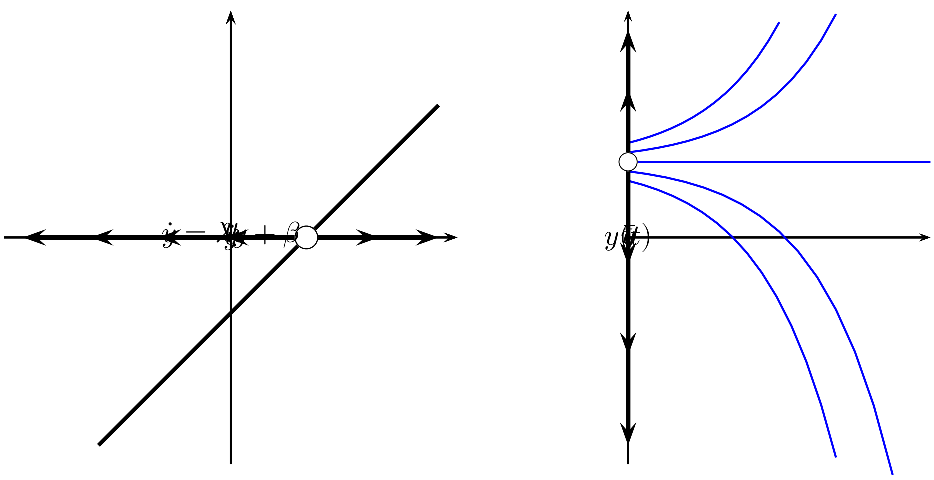

Figure 1.4: Phase diagram and trajectories for $\dot{y} = \lambda y + \beta$ for $\lambda > 0$ and $\beta < 0$

Proof. By using the separation of variables assuming that $t > \tau$ (the method is the same if we consider $t < \tau$) we get

$$\int_{y_\tau}^{y(t)} \frac{dy}{y} = \lambda \int_\tau^t dt$$

for any $t \geq 0$. Finding the definite integrals we get

$$\ln(y(t)) - \ln(y_\tau) = \ln(y(t)/y_\tau) = \lambda(t - \tau).$$

Taking exponentials to both sides we get again (1.16). □

Proof. (Alternative) Take the general solution for the homogenous equation, (1.12), and evaluate it at point $t = \tau$, to get $\phi(\tau, .) = ke^{\lambda\tau}$. But, to solve the problem we should have $\phi(\tau, .) = y_\tau$, a number. Solving for k we get $k = y_\tau e^{-\lambda\tau}$ and substituting in the general solution, we get function (1.16). □

Exercise 1. Find the solution of the problem $\dot{y} = \beta + \lambda y$, where $y(\tau) = y_\tau \in \mathrm{Y}$.

We have an **initial-value problem** when we have information on the value of the variable y at time $t = 0$, $y(0) = y_0$, and $\mathrm{T} = [0, T]$ for $T > 0$. In this case the solution exists and is unique and is given by

$$\phi(t, y_0; \lambda) = y_0\, e^{\lambda t}, \text{for } t \in [0, T]\ .$$

We call **terminal value problem** (sometimes called boundary value problem) if we know (or want to set) he solution for a terminal time T as $y(T) = y_T$. In this case the solution exists and is unique

$$\phi(t, y_T; \lambda, T) = y_T\, e^{-\lambda(T-t)}, \text{for } t \in [0, T]\ .$$

1.3.3 Non-autonomous equations

The scalar non-autonomous homogeneous equation

$$\dot{y} = a(t)y \tag{1.17}$$

where $a(t)$ is any function defined over T.

Proposition 6. *Equation* (1.24) *has the unique solution*

$$y(t) = ke^{\int a(t)dt} \tag{1.18}$$

where $k \in \mathrm{Y}$.

Proof. Separating variables and integrating we get

$$\int \frac{dy}{y} = \int a(t)dt,$$

which is equivalent to

$$\ln y(t) + C_y = \int a(t)dt$$

where C_y is a constant of integration, thus leading to equation (1.18) if we set $k = e^{C_y}$. □

The **initial value problem**

$$\begin{cases} \dot{y} = a(t)y, \\ y(t_0) = y_0 \end{cases} \tag{1.19}$$

for $\mathrm{T} = [t_0, t_1]$, has the unique solution

$$y(t) = y_0 e^{\int_{t_0}^{t} a(s)ds}, \text{ for } t \in [t_0, t_1]. \tag{1.20}$$

Exercise: prove this.

The initial value problem

$$\begin{cases} \dot{y} = a(t)y + b(t) \\ y(t_0) = y_0 \end{cases} \tag{1.21}$$

for $\mathrm{T} = [t_0, t_1]$.

Proposition 7. *The initial value problem* (1.19) *has the unique (particular) solution*

$$y(t) = y_0 e^{\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} b(s) e^{\int_{s}^{t} a(z)dz} ds, \text{ for } t \in [t_0, t_1] \tag{1.22}$$

Proof. We apply the variation of constant method[5]. First, we consider the solution for the homogeneous equation, such that $b(t) = 0$ for all $t \in \mathrm{T}$. Its solution is, using equation (1.20)

$$y_h(t, y_C) = y_C e^{\int_{t_0}^{t} a(s)ds}.$$

We expect the solution to problem (1.21) to be

$$y(t) = y_h(t, y_C(t)) = y_C(t) e^{\int_{t_0}^{t} a(s)ds}. \tag{1.23}$$

[5] Due to Lagrange (1811).

Taking time derivatives of the last equation we get

$$\dot{y} = \dot{y}_C e^{\int_{t_0}^t a(s)ds} + y_C(t)a(t)e^{\int_{t_0}^t a(s)ds}$$

which should be equal to equation (1.21). Equating the right-hand sides of both equations we get the ODE

$$\dot{y}_C = b(t)e^{-\int_{t_0}^t a(s)ds}.$$

Applying the separation of variables to solve this equation we find

$$y_C(t) = y_C(t_0) + \int_{t_0}^t b(s)e^{-\int_{t_0}^s a(z)dz} ds.$$

Substituting in equation (1.23) and because $y_C(t_0) = y(t_0) = y_0$ we finally get solution (1.22) □

In economics the following models are of interest:

Example 1 Consider the initial value problem

$$\dot{y} = ay + b(t), \text{ for } t \in [0, \infty)$$

where $a \neq 0$ and

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^* \\ b_1 & \text{if } t^* \leq t < \infty \end{cases}$$

and $y(0) = y_0$ is given.

The solution is

$$y(t) = \begin{cases} y_0 e^{at} + \frac{b_0}{a}\left(e^{at} - 1\right) & \text{if } 0 \leq t < t^* \\ y_0 e^{at} + \frac{b_0}{a}e^{at} + \left(\frac{b_1 - b_0}{a}\right)e^{a(t-t^*)} - \frac{b_1}{a} & \text{if } t^* \leq t < \infty \end{cases}$$

Observe that the solution, at any point in time, is capitalizing on the past changes of the variable $b(t)$.

Example 2 Consider the terminal value problem

$$\dot{y} = ay + b(t), \text{ for } t \in [0, \infty)$$

where $a > 0$ and

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^* \\ b_1 & \text{if } t^* \leq t < \infty \end{cases}$$

and $\lim_{t\to\infty} y(t)e^{-at} = 0$.

The solution for the problem is

$$y(t) = \begin{cases} -\dfrac{b_0}{a} - \left(\dfrac{b_1 - b_0}{a}\right) e^{a(t-t^*)} & \text{if } 0 \le t < t^* \\ -\dfrac{b_1}{a} & \text{if } t^* \le t < \infty \end{cases}$$

Comparing to the initial-value problem we see that the solution has an anticipating feature: for $0 < t < t^*$ the solution depends on the value of the variable $b(t)$ **after** its change, b_1, and after the change, for $t \ge t^*$, it is not influenced by the value before the change, b_0.

1.3.4 Economic applications

In economics, the following problem is common

$$\dot{y} = \lambda y, \ \lim_{t\to\infty} y(t)e^{-\mu t} = 0 \tag{1.24}$$

where $\mathrm{T} = [0, \infty)$, $0 \in \mathrm{Y}$ and $\mu > 0$.

To get the solution observe that

$$\lim_{t\to\infty} y(t)e^{-\mu t} = \lim_{t\to\infty} ke^{(\lambda-\mu)t}$$

then we have the following cases: (1) if $\lambda \ge \mu > 0$ then there is only **one** solution if $k = 0$; if $\lambda < \mu$ then there is an **infinity** of solutions, that is the terminal condition is met for any $k \in \mathrm{Y}$.

In economic models we use the following classification of variables and economic equilibrium

- **pre-determined** and **non-pre-determined** variables: the first are observed and the second are anticipated, that is, we have information for $t = 0$ for the first type of variables and we have asymptotic information on the second type of variables;
- **stationary** or **non-stationary** variables if they converge to a constant or are unbounded asymptotically (i.e., when $t \to \infty$);
- **determinacy** or **indeterminacy** if an equilibrium or a state of the economy modelled by a differential equation is unique or not

The relationship between them depends on the existence or not of a steady state and on their stability properties, for states within set Y.

For instance

- if a variable is pre-determined the trajectory described by the solution is always determinate, however, it can be stationary (if $\lambda < 0$) or non-stationary (if $\lambda > 0$). The first case is common in models with adaptative expectations, v.g. $\dot{p} = \lambda(\bar{p} - p)$, for $\lambda > 0$ and p is the log of price. The second case is common in endogenous growth models in which the GDP dynamics is given by $\dot{y} = Ay$, where y is GDP per capita;

- if a variable is non-predetermined the trajectory can be determinate if k is determined uniquely and is indeterminate if k can be any value within set Y. For scalar models the solutions are usually stationary if the terminal condition is of the type $\lim_{t\to\infty} y(t)e^{-\mu t} = 0$ for $\mu > 0$.

Table 1.1 summarizes this concepts, used in dynamic general equilibrium models (DGE).

Table 1.1: Classification of equilibrium paths in DGE models

y	$\lambda < 0$	$\lambda = 0$	$\lambda > 0$
pre-determined	determined and stable	determined and stationary	determined and non-stationary
non- pre-determined	indeterminate	ambiguous	determined

1.4 References

Mathematics: there is a huge literature on scalar linear ODE, but (Hale and Koçak, 1991, ch 1) is a great modern.

Applications to economics: Gandolfo (1997).

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