Advanced Mathematical Economics

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Lecture 3
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Chapter 3

Linear ODE: planar case

3.1 Introduction

In this chapter we deal with planar linear equations, that is with systems of two independent variables whose behavior is described by two coupled linear ODEs. Two other restrictions are introduced: first, we assume that the independent variable is time and we only consider the autonomous case, that is the case in which the coefficients in the system are constant, i.e., independent of time.

As with the scalar equation, any planar linear ODE has one unique solution. This makes it interesting per se because it allows a complete taxonomy of the types of solution trajectories that we can find. However, as a consequence of the Grobman-Hartmann theorem (see chapter on non-linear ODEs), it provides a qualitative characterization of a large number of non-linear planar systems. It also allows to determine which types of dynamics we can find in non-linear systems which are nor present in linear ones.

Furthermore, a large proportion of dynamic systems in economics are either linear or have a dynamics which is topologically equivalent to a linear ODE. In particular, we will see that most characterizations of the solution to optimal control problems are done by linearization, i.e., by approximating unknown solutions by solutions provided by an equivalent linear ODE.

Planar ODEs feature some new types of dynamics, when compared to the scalar case: first, although asymptotic stability and (global) instability cases can exist, as in the scalar case, the existence of saddle point dynamics (or conditional stability) is a new type of dynamics for the planar case; second in addition to monotonic solution paths, as in the scalar case, several types of non-monotonic solution paths can exist in the planar case. The saddle-point case is a very common type of dynamics in both macroeconomics and growth theory and charaterizes solutions of most optimal control problems.

The general (autonomous) **linear planar ordinary differential equation**, that we will study in this chapter, is defined as

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1$$
$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2.$$

Introducing the real value matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$, the real valued vector $\mathbf{B} \in \mathbb{R}^{2\times 1}$

$$\mathbf{A} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; \mathbf{B} \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

the vector function $\mathbf{y} : \mathrm{T} \to \mathbb{R}^2$ and its gradient $\dot{\mathbf{y}} : \mathrm{T} \to \mathbb{R}^2$

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \; \dot{\mathbf{y}}(t) \equiv \begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix},$$

we write the planar ODE in the equivalent matrix form,

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}, \; \mathbf{y} : \mathrm{T} \subseteq \mathbb{R}_+ \to \mathrm{Y} \subseteq \mathbb{R}^2. \tag{3.1}$$

Solutions to linear planar ODEs exist and are unique, and can be generically written as $\mathbf{y}(t) = \mathbf{\Phi}(t; t_0, \mathbf{y}(t_0); \mathbf{A}, \mathbf{B})$.

We show that, if $\det(\mathbf{A}) \neq 0$ they can be formally written as

$$\mathbf{y}(t) = \bar{\mathbf{y}} \; + \mathbf{e}^{\mathbf{A}(t-t_0)} \left(\mathbf{y}(t_0) - \bar{y}\right) \tag{3.2}$$

where t_0 is an arbitrarily fixed point in time and $\mathbf{y}(t_0) \in Y$ is the unknown value associated with it, belonging to Y which is the range of $\mathbf{y}$, and $\bar{\mathbf{y}} \in Y$ is a steady state (not necessarily unique) of the ODE.

If $\det(\mathbf{A}) = 0$ they can be formally written as

$$\mathbf{y}(t) = \bar{\mathbf{y}} + \mathbf{e}^{\mathbf{A}(t-t_0)} \left(\mathbf{y}(t_0) - \bar{y}\right) + \left(\mathbf{I} - \mathbf{A}^+ \mathbf{A}\right) \mathbf{B}\, t \tag{3.3}$$

where $\bar{\mathbf{y}} \; = -\mathbf{A}^+ \mathbf{B} + (\mathbf{I} - \mathbf{A}^+ \mathbf{A})\, \mathbf{y}(t_0)$ is not unique, because it is a function of an arbitrary $\mathbf{y}(t_0)$, and $\mathbf{A}^+$ is a generalized inverse of $\mathbf{A}$.

The previous equation is also called a **general solution**, and traces out a family of solutions. There are three main elements: first, the type of family of the solutions, which is related to their time behavior, depends on the algebraic properties of matrix $\mathbf{A}$; second, the location, and sometimes the existence, of steady states depends on vector $\mathbf{B}$; and the pair $(t_0, \mathbf{y}(t_0))$ allows for going from an ODE for a model, or a problem, involving an ODE by allowing the introduction of side conditions.

For scalar ODE's we saw that going from general solutions to particular solutions, which are completely specified functions, we have to introduce one side condition. When time is an independent variable, the side condition took the form of an initial or a terminal condition. For planar ODE's obtaining **particular solutions**, or completely specified solutions, we need to introduce **two** side conditions. If the two side conditions involve known values at time $t_0 = 0$, as $\mathbf{y}(t_0) = \mathbf{y}_0$, we say we have an **initial-value problem**, if there is one side condition for the initial value and another for the terminal (if T is finite) or asymptotic (if $\mathrm{T} \to \infty$) the problem can be called **mixed-value problem**, and if the two conditions are on the terminal or asymptotic state we can call it **terminal-value problem**. [1]

[1] If the independent variable is not time the last two cases are usually called **boundary-value problems**.

Solution to the ODE always exists and are unique, and solutions to problems involving ODEs always existe but may not be unique.

In order to find and characterize the solutions for the ODEs, we can follow, separately or jointly, the following types of approaches:

- an algebraic approach by determining explicitly the solutions;
- an analytical and geometrical approach by studying the existence and uniqueness of the steady states and other particular types of solutions (v.g., periodic solutions) and study their stability properties, and by building the **phase diagram**.

This chapter proceeds as follows. In section 3.2 we present some algebraic useful algebraic facts on Jordan canonical forms and on the related matrix exponential function $\mathbf{e}^{\mathbf{A}t}$. In section **??** we obtain the solutions to planar ODE's, first for homogenous equations and next for non-homogenous equations. In section 3.5 we characterize analytically and geometrically the types of solutions for linear planar ODE's. In section 3.7 we provide a present the bifurcation analysis for this type of ODE's which will be useful in the ensuing chapters.

The chapter ends with the solution to problems involving ODE's and with comments on the economic applications.

3.2 Matrix A and the matrix exponential function $\mathbf{e}^{\mathbf{A}t}$

In this section we review some results from linear algebra, in subsection 3.2.1. In subsection 3.2.2 we derive the expressions for $\mathbf{e}^{\mathbf{A}t}$.

3.2.1 The algebraic properties of matrix A and the Jordan canonical forms

Matrix $\mathbf{A}$ fundamentally determines the solution to differential equation (3.1) and allows for the characterization of its dynamics.

A fundamental result is that any matrix $\mathbf{A}$ is similar to one of the following three matrices, called the **Jordan canonical forms**[2]

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix},\ \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},\ \text{or } \mathbf{\Lambda}_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{3.4}$$

Two matrices are similar if they have the same spectrum. The spectrum of matrix $\mathbf{A}$ is a tuple belonging to $\mathbb{C}^2$ (the space of two-dimensional complex numbers)

$$\sigma(\mathbf{A}) = \left\{\lambda \in \mathbb{C}^2 : \det\left(\mathbf{A} - \lambda\,\mathbf{I}\right) = 0\right\}.$$

where $\mathbf{I}$ is the (2×2) identity matrix and

$$\mathbf{A} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ \text{and } \mathbf{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

[2] See the appendix 3.A.1 where we gather some useful results from matrix algebra.

The elements of $\sigma(\mathbf{A})$ are called the **eigenvalues** of $\mathbf{A}$. The characteristic polynomial associated to matrix $\mathbf{A}$ is the square polynomial in λ

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - \text{trace}(\mathbf{A}) \ \lambda + \det(\mathbf{A}),$$

the trace and the determinant are

$$\text{trace}(\mathbf{A}) \ = a_{11} + a_{22}, \text{ and } \det(\mathbf{A}) \ = a_{11}\, a_{22} - a_{12}\, a_{21}.$$

Equation $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ is called characteristic equation. The eigenvalues are the solutions of the characteristic equation:

$$\lambda_- = \frac{\text{trace}(\mathbf{A})}{2} - \sqrt{\Delta(\mathbf{A})}, \ \lambda_+ = \frac{\text{trace}(\mathbf{A})}{2} + \sqrt{\Delta(\mathbf{A})} \tag{3.5}$$

where $\Delta(\mathbf{A}) \equiv \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A})$ is called the discriminant of $\mathbf{A}$.

From equation (3.5), three types of distinctions can be made concerning the properties of the eigenvalues (see Figure 3.1):

First, the two eigenvalues are real if $\Delta(\mathbf{A}) \geq 0$ and they are complex conjugate if $\Delta(\mathbf{A}) < 0$. In particular, if $\Delta(\mathbf{A}) > 0$ the eigenvalues are real and distinct and satisfy $\lambda_- < \lambda_+$, if $\Delta(\mathbf{A}) = 0$ the eigenvalues are real and multiple and satisfy $\lambda = \lambda_- = \lambda_+ = \frac{\text{trace}(\mathbf{A})}{2}$, and if $\Delta(\mathbf{A}) < 0$ they are complex conjugate and satisfy

$$\lambda_\pm = \alpha \pm \beta\, i, \text{ for } i \equiv \sqrt{-1}$$

where $\alpha = \frac{\text{trace}(\mathbf{A})}{2}$ and $\beta = \sqrt{|\Delta(\mathbf{A})|}$.

Second, the eigenvalues are generic in the sense that they will not change their type or sign for small changes in the elements of the coefficient matrix $\mathbf{A}$ if $\Delta(\mathbf{A}) \neq 0$, or $\det(\mathbf{A}) \ \neq 0$, or $\text{trace}(\mathbf{A}) \neq 0$ and $\det(\mathbf{A}) \ \geq 0$, and they are not generic otherwise, that is if $\Delta(\mathbf{A}) = 0$, or $\det(\mathbf{A}) \ = 0$, or $\text{trace}(\mathbf{A}) = 0$ and $\det(\mathbf{A}) \ \geq 0$.

In particular, if $\Delta(\mathbf{A}) > 0$ and $\text{trace}(\mathbf{A}) > 0$ the two eigenvalues have positive real parts, if $\Delta(\mathbf{A}) > 0$ and $\text{trace}(\mathbf{A}) < 0$ the two eigenvalues have negative real parts and if $\Delta(\mathbf{A}) < 0$ the two eigenvalues are real and symmetric signs (that is $\lambda_- < 0 < \lambda_+$). The following non-generic cases include: the case $\Delta(\mathbf{A}) = 0$ in which the two eigenvalues are equal and have the same sign as $\text{trace}(\mathbf{A})$; the case $\det(\mathbf{A}) \ = 0$ in which the eigenvalues are real and at least one of them is equal to zero; the case $\det(\mathbf{A}) \ = 0$ and $\text{trace}(\mathbf{A}) > 0$ in which the eigenvalues are complex conjugate with zero real part, and if $\text{trace}(\mathbf{A}) = \det(\mathbf{A}) \ = 0$ in which the two eigenvalues are both equal to zero.

Figure 3.1, that we call a **bifurcation diagram** [3] for the linear planar ODE shows all the possible relevant cases, in which there are five generic cases (corresponding to two-dimensional subsets) , five non-degenerate cases of co-dimension-one cases (corresponding to lines) and one co-dimension-two case (the origin).

[3] This designation will be made clear in the chapter where we deal with non-linear ODE's

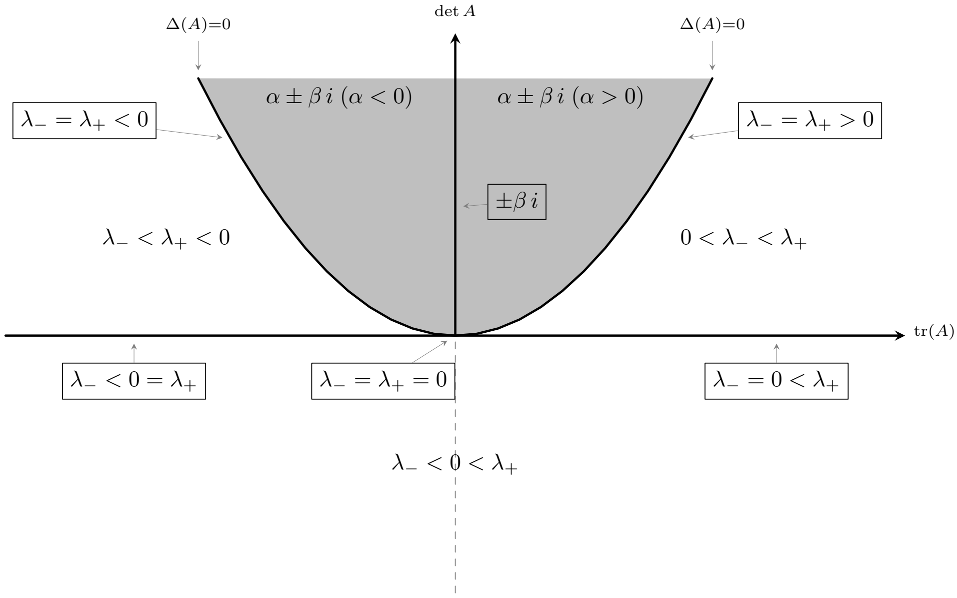

Figure 3.1: Bifurcation diagram for the linear planar ODE in $(\text{trace}(\mathbf{A}), \det(\mathbf{A}))$. The gray area corresponds to the existence of complex conjugate eigenvalues.

There is a useful relating the coefficients of the characteristic equation with elementary operations between the eigenvalues of any matrix $\mathbf{A}$:

$$\lambda_- + \lambda_+ = \text{trace}(\mathbf{A}),\ \lambda_-\, \lambda_+ = \det(\mathbf{A}). \tag{3.6}$$

There is a close relationship between the discriminant of $\mathbf{A}$ and the the Jordan canonical form which is similar to $\mathbf{A}$[4], which can be call the Jordan canonical form of $\mathbf{A}$,

Lemma 1. ***Jordan canonical form of a matrix*** $\mathbf{A}$ *The Jordan canonical form of* $\mathbf{A}$ *is determined by the sign of* $\Delta(\mathbf{A})$*: if* $\Delta(\mathbf{A}) > 0$ *then the Jordan canonical form of* $\mathbf{A}$ *is* $\mathbf{\Lambda}_1$ *if, if* $\Delta(\mathbf{A}) = 0$ *the Jordan canonical of* $\mathbf{A}$ *is* $\mathbf{\Lambda}_2$*, and if* $\Delta(\mathbf{A}) < 0$ *the Jordan canonical form of* $\mathbf{A}$ *is* $\mathbf{\Lambda}_3$*.*

Given any matrix $\mathbf{A}$ and its canonical Jordan form in equation (3.4) fundamental theorem of Algebra proves that there is a (non-singular) linear operator $\mathbf{P} \in \mathbb{R}^{2\times 2}$ such that the following relationship holds

$$\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1} \Leftrightarrow \mathbf{\Lambda} = \mathbf{P}^{-1}\,\mathbf{A}\,\mathbf{P}. \tag{3.7}$$

Matrix $\mathbf{P}$ is called the **eigenvector matrix** associated to matrix $\mathbf{A}$.

The fact that any matrix $\mathbf{A}$ has a one-to-one relationship with one of the Jordan canonical forms allows us to reduce the determination of the general solution of a planar ODE to the cases involving a Jordan canonical form, and then using back the operator $\mathbf{P}$.

[4] See the appendix 3.A.1.

3.2.2 The matrix exponential function

We saw that the (general) solution of the scalar homogeneous equation $\dot{y} = \lambda y$ was $y(t) = y(0)\, e^{\lambda t}$ where $y(0)$ is an arbitrary element of $Y \subseteq \mathbb{R}$ for $t = 0$. Recall that the exponential function has the series representation

$$e^{\lambda t} \equiv \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = 1 + \lambda t + \frac{1}{2}\,(\lambda t)^2 + \frac{1}{6}\,(\lambda t)^3 + ...$$

For the planar problem we can also define a **matrix exponential** function

$$\mathbf{e}^{\mathbf{A}t} \equiv \sum_{n=0}^{+\infty} \frac{1}{n!}\mathbf{A}^n t^n = I + \mathbf{A}t + \frac{1}{2}\,\mathbf{A}^2 t^2 + ... \tag{3.8}$$

which is a mapping $\mathbf{e}^{\mathbf{A}t} : \mathrm{T} \;\to \mathbb{R}^{2\times 2}$ with the following properties:

Lemma 2. Properties of matrix exponentials $\mathbf{e}^{\mathbf{At}}$.

(i) semigroup property: $\mathbf{e}^{\mathbf{A(t+s)}} = \mathbf{e}^{\mathbf{At}}\mathbf{e}^{\mathbf{A}s}$

(ii) inverse of the matrix exponencial is the the exponential of the inverse: $\left(\mathbf{e}^{\mathbf{At}}\right)^{-1} = \mathbf{e}^{-\mathbf{At}}$

(iii) the time derivative commutes: $\frac{d}{dt}\mathbf{e}^{\mathbf{At}} = \mathbf{A}\mathbf{e}^{\mathbf{At}} = \mathbf{e}^{\mathbf{At}}\mathbf{A}$

(iv) if matrices $\mathbf{A}$ *and* $\mathbf{B}$ *commute, (i.e., if* $\mathbf{A}\,\mathbf{B} = \mathbf{B}\,\mathbf{A}$*) then* $\mathbf{e}^{\mathbf{(A+B)t}} = \mathbf{e}^{\mathbf{At}}\mathbf{e}^{\mathbf{Bt}}$

(v) Let $\mathbf{P}$ *be a non-singular and square matrix. Then* $\mathbf{e}^{\mathbf{P^{-1}APt}} = \mathbf{P}^{-1}\mathbf{e}^{\mathbf{At}}\mathbf{P}$*.*

From Lemma 2 *(v)* as $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda}$ then $\mathbf{e}^{\mathbf{\Lambda}t} = \mathbf{e}^{\mathbf{P}^{-1}\mathbf{AP}t} \;= \mathbf{P}^{-1}\mathbf{e}^{\mathbf{A}t}\mathbf{P}$ or, equivalently

$$\mathbf{e}^{\mathbf{At}} = \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}t}\mathbf{P}^{-1},$$

where $\mathbf{\Lambda}$ is the Jordan canonical of $\mathbf{A}$.

Therefore, given any matrix $\mathbf{A}$, the exponential matrix $\mathbf{e}^{\mathbf{At}}$ if a (2×2) dimensional function of t, and the time-dependency is determined a linear transformation of the matrix exponential of Jordan canonical of $\mathbf{A}$, $\mathbf{e}^{\mathbf{\Lambda}t}$.

This is an important result which means that the types of solutions, and the associated phase diagrams, can be completely enumerated.

The exponential matrices for the Jordan canonical forms are:

Lemma 3. ***Matrix exponentials for the Jordan canonical forms,*** $\mathbf{e}^{\;\mathbf{t}}$
Let $\mathbf{\Lambda}$ *be a matrix in an arbitrary Jordan canonical form as in equation* (3.4) *and let* λ_-*,* λ_+*,* λ*,* α *and* β*. be real numbers. Then,*

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_1$ *then*

$$\mathbf{e}^{\mathbf{\Lambda}t} = \mathbf{e}^{\mathbf{\Lambda}_1 t} = \begin{pmatrix} e^{\lambda_- t} & 0 \\ 0 & e^{\lambda_+ t} \end{pmatrix}. \tag{3.9}$$

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ *then*

$$\mathbf{e}^{\mathbf{\Lambda}t} = \mathbf{e}^{\mathbf{\Lambda}_2 t} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \tag{3.10}$$

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ *then*

$$\mathbf{e}^{\mathbf{\Lambda}t} = \mathbf{e}^{\mathbf{\Lambda}_3 t} = e^{\alpha t} \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}. \tag{3.11}$$

Proof. Consider the definition of matrix exponential, equation (3.8) and the Jordan canonical form matrices in equation (3.4). In the first case, we have

$$\mathbf{e}^{\mathbf{\Lambda}_1 t} = \mathbf{I}_2 + \mathbf{\Lambda}_1 t + \frac{1}{2}\,(\mathbf{\Lambda}_1)^2 t^2 + ... = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda\_t & 0 \\ 0 & \lambda_+ t \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \lambda_-^2 t^2 & 0 \\ 0 & \lambda_+^2 t^2 \end{pmatrix} + ...$$

then, performing the matrix additions,

$$\mathbf{e}^{\mathbf{\Lambda}_1 t} = \begin{pmatrix} 1 + \lambda\_t + \frac{1}{2}\lambda_-^2 t^2 + ... & 0 \\ 0 & 1 + \lambda_+ t + \frac{1}{2}\lambda_+^2 t^2 + ... \end{pmatrix} = \begin{pmatrix} e^{\lambda_- t} & 0 \\ 0 & e^{\lambda_+ t} \end{pmatrix}$$

because $e^y = \sum_{n=0}^{+\infty} \frac{1}{n!} y^n$. That result is straightforward to obtain because the Jordan matrix is diagonal. This is not the case for Jordan matrix $\mathbf{\Lambda}_2$, though. But if we decompose $\mathbf{\Lambda}_2$ as

$$\mathbf{\Lambda}_2 = \mathbf{\Lambda}_{2,1} + \mathbf{\Lambda}_{2,2} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and because the two matrices commute, i.e. $\mathbf{\Lambda}_{2,1}\mathbf{\Lambda}_{2,2} = \mathbf{\Lambda}_{2,2}\mathbf{\Lambda}_{2,1}$, then applying property (iv) of Lemma 2 we obtain

$$\mathbf{e}^{\mathbf{\Lambda}_2 t} = \mathbf{e}^{(\mathbf{\Lambda}_{2,1} + \mathbf{\Lambda}_{2,2})t} = \mathbf{e}^{\mathbf{\Lambda}_{2,1} t}\,\mathbf{e}^{\mathbf{\Lambda}_{2,2} t}$$

where

$$\mathbf{e}^{\mathbf{\Lambda}_{2,1} t} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} = e^{\lambda t}\mathbf{I}_2.$$

Using again formula (3.8) for matrix $\mathbf{\Lambda}_{2,2}$ we get

$$\mathbf{e}^{\mathbf{\Lambda}_{2,2} t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + ... = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

therefore multiplying by matrix $\mathbf{e}^{\mathbf{\Lambda}_{2,1} t}$ yields (3.10).

In the third case, $\mathbf{\Lambda}_3$ is again non-diagonal, but it can also be decomposed into the sum of two matrices, $\mathbf{\Lambda}_{3,1}$ and $\mathbf{\Lambda}_{3,2}$, that commute

$$\mathbf{\Lambda}_3 = \mathbf{\Lambda}_{3,1} + \mathbf{\Lambda}_{3,2} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

Applying again property (iv) of Lemma 2 we get

$$\mathbf{e}^{\mathbf{\Lambda}_3 t} = \mathbf{e}^{\mathbf{\Lambda}_{3,1} t}\, \mathbf{e}^{\mathbf{\Lambda}_{3,2} t},$$

where

$$\mathbf{e}^{\mathbf{\Lambda}_{3,1} t} = e^{\alpha t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using again formula (3.8) for matrix $\mathbf{\Lambda}_{3,2}$ we get

$$\mathbf{e}^{\mathbf{\Lambda}_{3,2} t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \beta t \\ -\beta t & 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} \beta^2 t^2 & 0 \\ 0 & -\beta^2 t^2 \end{pmatrix} + \ldots = \begin{pmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{pmatrix},$$

because $\sin y = \sum_{n=0}^{+\infty} \frac{y^{2n+1}}{(2n+1)}$ and $\cos y = \sum_{n=0}^{+\infty} \frac{y^{2n}}{(2n)}$, we get (3.11). □

In the literature, there are two matrices that can we can call non-canonical:

$$\mathbf{A}_d \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \text{ or } \mathbf{A}_h \equiv \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \tag{3.12}$$

where λ, α and β are real numbers. In the case of $\mathbf{A}_d$ there are multiple eigenvalues, both equal to λ although the matrix is not of the form $\mathbf{\Lambda}_2$, and in the case of $\mathbf{A}_h$ the spectrum is $\sigma(\mathbf{A}_h) = \{\alpha + \beta, \alpha - \beta\}$ which are two real numbers.

Lemma 4. *If* $\mathbf{A}$ *is in the non-canonical form* $\mathbf{A}_d$*, in equation* (3.12)*, then*

$$\mathbf{e}^{\mathbf{A}_d t} = e^{\lambda t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1. *if* $\mathbf{A}$ *is in the non-canonical form* $\mathbf{A}_h$*, in equation* (3.12)*, then* [5]

$$\mathbf{e}^{\mathbf{A}_h t} = e^{\alpha t} \begin{pmatrix} \cosh(\beta t) & \sinh(\beta t) \\ \sinh(\beta t) & \cosh(\beta t) \end{pmatrix} \tag{3.13}$$

Proof. We know that $\mathbf{A} = \mathbf{P\Lambda P}^{-1}$, where $\mathbf{\Lambda}$ is the Jordan form of $\mathbf{A}$. Then $\mathbf{e}^{\mathbf{At}} = \mathbf{e}^{\mathbf{P}\Lambda\mathbf{P}^{-1}\mathbf{t}} = \mathbf{P}\mathbf{e}^{\mathbf{t}}\mathbf{P}^{-1}$ by property (v) of Lemma 2. Matrix $\mathbf{A} = \mathbf{A}_d$ has two equal real eigenvalues equal to λ and, because it is diagonal it satisfies $\mathbf{A}_d \mathbf{P}_d = \mathbf{P}_d \mathbf{A}_d$. Therefore $\mathbf{P}_d = \mathbf{I}$ and

$$\mathbf{e}^{\mathbf{A}_d t} = \mathbf{P}\, e^{\lambda t}\, \mathbf{I}\, \mathbf{P}^{-1} = e^{\lambda t}\, \mathbf{I}.$$

Matrix $\mathbf{A} = \mathbf{A}_h$ has the real spectrum $\sigma = \{\alpha + \beta, \alpha - \beta\}$ and has eigenvector matrix

$$\mathbf{P}_h = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

[5] Recall $\cosh(\beta t) = \frac{1}{2}(e^{\beta t} + e^{-\beta t})$ and $\sinh(\beta t) = \frac{1}{2}(e^{\beta t} - e^{-\beta t})$

Therefore, the exponential matrix is

$$\mathbf{e}^{\mathbf{A}_h t} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(\alpha+\beta)t} & 0 \\ 0 & e^{(\alpha+\beta)t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

which, expanding the matrix multiplication, yields matrix (3.13). □

We start by presenting, in section 3.3, the cases in which $\mathbf{A} = \mathbf{\Lambda}$ and in 3.4 the cases in which matrix $\mathbf{A}$ is not in the Joerdan canonical form. We will see that the first case provides the fundamental types of dynamic systems generated by planar linear ODE's.

3.3 ODE with Jordan coefficients

The most simple type of planar ODEs is one in which the coefficient matrix $\mathbf{A}$ is in a Jordan canonical form

$$\dot{\mathbf{y}} = \mathbf{\Lambda}\mathbf{y} + \mathbf{B} \tag{3.14}$$

which covers the following cases

$$\begin{cases} \dot{y}_1 = \lambda_- \, y_1 + b_1, \\ \dot{y}_2 = \lambda_+ \, y_2 + b_2, \end{cases}, \quad \begin{cases} \dot{y}_1 = \lambda \, y_1 + y_2 + b_1, \\ \dot{y}_2 = \lambda \, y_2 + b_2, \end{cases}, \text{ and } \begin{cases} \dot{y}_1 = \alpha \, y_1 + \beta \, y_2 + b_1, \\ \dot{y}_2 = -\beta \, y_1 + \alpha \, y_2 + b_2 \end{cases},$$

in which all the coefficients are real numbers and b_1 and b_2 can be any real number, including $\mathbf{B} = \mathbf{0}$. We first find the solution for the homogenous ODE which satisfies $\mathbf{B} = \mathbf{0}$:

Proposition 1. ***Solution for the homogenous ODE*** **(3.14)** ***for*** $\mathbf{B} = \mathbf{0}$ *Consider the ODE* (3.14) *with* $\mathbf{B} = \mathbf{0}$. *The solution always exist and is uniquely represented by the mapping* $\mathbf{\Phi} : T \times Y \to Y \subseteq \mathbb{R}^2$,

$$\mathbf{y}(t) = \mathbf{\Phi}(t, \mathbf{y}(0)) \equiv \mathbf{e}^{\mathbf{\Lambda}\, t}\mathbf{y}(0), \text{ for } t \in T = [0, \infty) \tag{3.15}$$

where $\mathbf{y}(0) \in Y$ *is an arbitrary element of the domain of* $\mathbf{y}$ *for* $t = 0$.

Proof. Conjecture that the solution is $\mathbf{y}(t) = \mathbf{e}^{\mathbf{\Lambda}\, t}\mathbf{y}(0)$ for an arbitrary value of $\mathbf{y}(t)$ when $t = 0$. To prove that this function satisfies ODE (3.14) we take a time derivative to find (from Lemma 2 (iii)

$$\frac{d}{dt}\mathbf{y}(t) = \frac{d}{dt}\mathbf{e}^{\mathbf{\Lambda}\, t}\mathbf{y}(0) = \mathbf{\Lambda}\, \mathbf{e}^{\mathbf{\Lambda}\, t}\mathbf{y}(0) = \mathbf{\Lambda}\, \mathbf{y}(t).$$

□

The (general) solution of equation (3.14), $\mathbf{y}(t) = \mathbf{\Phi}(t, \mathbf{y}(0))$ can take one of the following three forms, where $\mathbf{y}(0)$ is an arbitrary value for $\mathbf{y}$ at time $t = 0$:

1. if $\Lambda = \mathbf{\Lambda}_1$ then the solution is similar to the solution for two coupled scalar ODE's

$$\mathbf{y}(t) = \begin{pmatrix} e^{\lambda_- t} & 0 \\ 0 & e^{\lambda_+ t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y_1(0)\, e^{\lambda_- t} \\ y_2(0)\, e^{\lambda_+ t} \end{pmatrix} \tag{3.16}$$

2. if $\Lambda = \mathbf{\Lambda}_2$ then the type of solution is new to planar ODE's

$$\mathbf{y}(t) = \begin{pmatrix} e^{\lambda t} & t \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} y_1(0) + y_2(0)\, t \\ y_2(0) \end{pmatrix} \tag{3.17}$$

3. or, if $\Lambda = \mathbf{\Lambda}_3$ then we have again a new type of solution

$$\mathbf{y}(t) = e^{\alpha t} \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = e^{\alpha t} \begin{pmatrix} y_1(0)\cos\beta t + y_2(0)\sin\beta t \\ -y_1(0)\sin\beta t + y_2(0)\cos\beta t \end{pmatrix} \tag{3.18}$$

Now let $\mathbf{B} \neq \mathbf{0}$ in the planar ODE (3.14), which becomes a non-homogenous ODE. To study this equation it is useful to consider its steady states.

The **set of steady states** of equation (3.14) is the set of elements of the range of $\mathbf{y}$, Y such that

$$\bar{\mathbf{y}} \ = \Big\{ \mathbf{y} \in \mathrm{Y} : \ \mathbf{\Lambda}\mathbf{y} + \mathbf{B} = \mathbf{0} \Big\}.$$

Next we show that this set is non-empty, meaning steady-states always exist, but it may contain several elements, meaning that steady-states may not be unique.

Lemma 5. *A steady state (not necessarily unique) always exists such that*

$$\bar{\mathbf{y}} \ = -\mathbf{\Lambda}^{+}\,\mathbf{B} + \left(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\right)\mathbf{y}(0) \tag{3.19}$$

where $\mathbf{\Lambda}^{+}$ *is the Moore-Penrose inverse of* $\mathbf{\Lambda}$ *and* $\mathbf{y}(0)$ *is an arbitrary element of* Y*.*

Proof. See (Magnus and Neudecker, 1988, p36). □

The following cases are possible.

Non-degenerate case If $\det(\mathbf{\Lambda}) \neq 0$ then the Moore-Penrose inverse is the classical inverse, that is $\mathbf{\Lambda}^{+} = \mathbf{\Lambda}^{-1}$ which satisfies $\mathbf{\Lambda}^{-1}\,\mathbf{\Lambda} = \mathbf{I}$. Thus, from equation (3.19), the steady state is unique and it is

$$\bar{\mathbf{y}} = -\mathbf{\Lambda}^{-1}\,\mathbf{B},$$

is independent of the value of $\mathbf{y}(0)$. If $\mathbf{B} = \mathbf{0}$ then the steady state is $\bar{\mathbf{y}} \ = \mathbf{0}$.

Degenerate cases If $\det(\mathbf{\Lambda}) = 0$ then $\Delta(\mathbf{\Lambda}) > 0$. Then all the eigenvalues are real, which means that the Jordan matrix $\mathbf{\Lambda}$ is diagonal, and it has at least one eigenvalue which is equal to zero. There is one zero eigenvalue if $\mathrm{trace}(\mathbf{A}) \neq 0$ and two zero eigenvalues if $\mathrm{trace}(\mathbf{A}) = 0$. This means that the associated Jordan matrices can be

$$\mathbf{\Lambda} \in \left\{ \begin{pmatrix} \lambda_- & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \lambda_+ \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \tag{3.20}$$

and the associated Moore-Penrose inverses are

$$\mathbf{\Lambda}^{+} \in \left\{ \begin{pmatrix} \frac{1}{\lambda_-} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\lambda_+} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{3.21}$$

Therefore, substituting those matrices in equation (3.19) we find

$$\mathbf{I} - \mathbf{\Lambda}^+ \mathbf{\Lambda} \in \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and there is always an infinite number of steady states depending on the arbitrary element $\mathbf{y}(0)$. If trace$(\mathbf{A}) \neq 0$, for the two first cases, applying equation (3.19), we find the steady states are in infinite number,

$$\bar{\mathbf{y}} \; = \begin{pmatrix} -\dfrac{b_1}{\lambda_-} \\ y_2(0) \end{pmatrix}, \text{ or } \bar{\mathbf{y}} \; = \begin{pmatrix} y_1(0) \\ -\dfrac{b_2}{\lambda_+} \end{pmatrix}. \tag{3.22}$$

In both cases the steady states belong to a one-dimensional manifold in Y: in the first case it traces out a line such that $y_1 = -\dfrac{b_1}{\lambda_-}$ (a vertical line in a Cartesian diagram) and in the second such that $y_2 = -\dfrac{b_2}{\lambda_+}$ (a horizontal line in a Cartesian diagram).

If trace$(\mathbf{\Lambda}) = 0$ there is also an infinite number of steady states

$$\bar{y} = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}, \tag{3.23}$$

which in this case any point in the two-dimensional set (surface) Y is a steady state.

Therefore, a steady state always exists, it is unique if $\det(\mathbf{\Lambda}) \neq 0$ and there is in infinite number if $\det(\mathbf{\Lambda}) = 0$.

Next, we obtain a general form for the solution of ODE (3.14), for any matrices $\mathbf{\Lambda}$ and $\mathbf{B}$.

Proposition 2. ***Solution for the non-homogenous ODE*** (3.14) *Consider the ODE* (3.14) *for an arbitrary real vector $\mathbf{B} \in \mathbf{R}^2$. The solution to the ODE always exist and is uniquely given by*

$$\mathbf{y}(t) = \bar{\mathbf{y}} + \mathbf{e}^{\mathbf{\Lambda}\, t}\left(\mathbf{y}(0) - \bar{\mathbf{y}}\right) + \left(\mathbf{I} - \mathbf{\Lambda}^+ \mathbf{\Lambda}\right) \mathbf{B}\, t, \; for\, t \in T = [0, \infty) \tag{3.24}$$

where $\mathbf{y}(0)$ is an arbitrary element of Y for $t = 0$ and $\bar{\mathbf{y}}$ is the corresponding steady state as in equation (3.19)*.*

Proof. We start with the case in which $\det(\mathbf{\Lambda}) \neq 0$. Then again, matrix $\mathbf{\Lambda}$ has a unique classical inverse, $\mathbf{\Lambda}^+ = \mathbf{\Lambda}^{-1}$, which implies that $\bar{\mathbf{y}} = -\mathbf{\Lambda}^{-1}\, \mathbf{B}$ and $\mathbf{I} - \mathbf{\Lambda}^+ \mathbf{\Lambda} = \mathbf{0}$. Define $\mathbf{z}(t) = \mathbf{y}(t) - \bar{y}$ where $\mathbf{y}$ is given in equation (3.19). Then $\dot{z} = \dot{y} = \mathbf{\Lambda}\, \mathbf{y} + \mathbf{B} = \mathbf{\Lambda}\,(\mathbf{y} - \bar{\mathbf{y}}) = \mathbf{\Lambda}\, \mathbf{z}$, yields a homogenous ODE $\dot{\mathbf{z}} = \mathbf{\Lambda}\, \mathbf{z}$, whose solution is, from equation (3.15), $\mathbf{z}(t) = e^{\mathbf{\Lambda} t}\mathbf{z}(0)$. Going back to the original variables we have

$$\mathbf{y}(t) = \bar{\mathbf{y}} + e^{\mathbf{\Lambda} t}\left(\mathbf{y}(0) - \bar{\mathbf{y}}\right).$$

If $\det(\mathbf{A}) = 0$ the coefficient matrix itakes one of the forms in equation (3.20). Therefore, the ODE's can take one of the following forms

$$\begin{cases} \dot{y}_1 = \lambda_-\, y_1 + b_1 \\ \dot{y}_2 = b_2 \end{cases} \quad \text{or} \quad \begin{cases} \dot{y}_1 = b_1 \\ \dot{y}_2 = \lambda_+\, y_2 + b_2 \end{cases} \quad \text{or} \quad \begin{cases} \dot{y}_1 = b_1 \\ \dot{y}_2 = b_2. \end{cases}$$

Using the results for the scalar ODE, the solutions are

$$\begin{cases} y_1(t) = -\dfrac{b_1}{\lambda_-} + e^{\lambda_- t}\,(y_1(0) + \dfrac{b_1}{\lambda_-}) \\ y_2(t) = y_2(0) + b_2 t \end{cases} \quad \text{or} \quad \begin{cases} y_1(t) = y_1(0) + b_1 t \\ y_2(t) = -\dfrac{b_2}{\lambda_+} + e^{\lambda_+ t}\,(y_2(0) + \dfrac{b_2}{\lambda_+}) \end{cases} \quad \text{or} \quad \begin{cases} y_1(t) = y_1(0) + b_1 t \\ y_2(t) = y_1(0) + b_1 t. \end{cases}$$

If we consider: first, that the steady states in the first and second cases are the same we obtained in equation for the first two cases (3.22) and (3.23) for the third case; second, that the exponential equations are, respectively

$$\begin{pmatrix} e^{\lambda_- t} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda_+ t} \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

and, at last, their Jordan matrices in equation (3.20), their Moore-Penrose inverses in in equation (3.21), we see that equation (3.24) covers all cases. □

3.4 ODE with general coefficients

In this section we solve the general planar ODE

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B} \tag{3.25}$$

where matrix $\mathbf{A}$ is not in the Jordan canonical form and $\mathbf{B}$ can be any real vector. This covers both the homogenous case in which $\mathbf{B} = \mathbf{0}$ and the non-homogeneous case in which $\mathbf{B} \neq \mathbf{0}$.

We start by presenting an useful result

Lemma 6. *Consider the coefficient matrix $\mathbf{A}$ and let $\mathbf{P}$ and $\mathbf{\Lambda}$ be its associated eigenvector matrix and Jordan canonical form. Then, the ODE (3.25) with general coefficient matrix $\mathbf{A}$ can be transformed into an ODE with coefficient matrix mL*

$$\mathbf{y}(t) = \mathbf{P}\,\mathbf{w}(t) \tag{3.26}$$

where $\mathbf{P}$ is the eigenvector matrix associated to $\mathbf{A}$ and $\mathbf{w}(t)$ is the solution of the ODE

$$\dot{\mathbf{w}} = \mathbf{\Lambda}\,\mathbf{w} + \mathbf{P}^{-1}\,\mathbf{B} \tag{3.27}$$

Proof. Recall the transformation $\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ where matrix $\mathbf{P}$ is non-singular. Then we can introduce a unique linear transformation $\mathbf{w}(t) = \mathbf{P}^{-1}\mathbf{y}(t)$. Then

$$\dot{\mathbf{w}} = \mathbf{P}^{-1}\dot{\mathbf{y}} = \mathbf{P}^{-1}\left(\mathbf{A}\mathbf{y} + \mathbf{B}\right) = \mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{y} + \mathbf{P}^{-1}\mathbf{B} = \mathbf{\Lambda}\mathbf{w} + \mathbf{P}^{-1}\mathbf{B}.$$

□

Lemma 7. *The solution to the ODE transformed coordinates* $\mathbf{w}$*, equation* (3.27) *is*

$$\mathbf{w}(t) = \bar{\mathbf{w}} \; + \mathbf{e}^{\mathbf{\Lambda} t}(\mathbf{w}(0) - \bar{\mathbf{w}}) + (\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{P}^{-1}\,\mathbf{B}\,t \tag{3.28}$$

where

$$\bar{\mathbf{w}} \; = -\mathbf{\Lambda}^{+}\,\mathbf{P}^{-1}\,\mathbf{B} + (\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{w}(0)$$

and $\mathbf{w}(0) = \mathbf{P}^{-1}\,\mathbf{y}(0)$*.*

Proof. ODE (3.27) is a non-homogeneous ODE in which the coefficient matrix is in the Jordan canonical form. Comparing with equation (3.14) we find that instead of $\mathbf{B}$ we now have $\mathbf{P}^{-1}\mathbf{B}$. By performing this substitution in the solution to the last ODE, in equation (3.24) we find the solution of the transformed ODE in equation (3.28). □

Proposition 3. ***Steady state for the non-homogenous ODE*** (3.25) *Steady states for equation* (3.25) *exist and are given by*

$$\bar{\mathbf{y}} = -\mathbf{A}^{+}\,\mathbf{B} + (\mathbf{I} - \mathbf{A}^{+}\,\mathbf{A})\,\mathbf{y}(0), \tag{3.29}$$

where $\mathbf{A}^{+}\,\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$*.*

Proof. Multiplying equation (3.26) by $\mathbf{P}$ we get

$$\begin{aligned}
\bar{\mathbf{y}} \; &= \mathbf{P}\,\bar{\mathbf{w}} \\
&= -\mathbf{P}\,\mathbf{\Lambda}^{+}\,\mathbf{P}^{-1}\,\mathbf{B} + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{w}(0) \\
&= -A^{+}\,\mathbf{B} + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{P}^{-1}\; y(0) \\
&= -A^{+}\mathbf{B} + (\mathbf{P}\,\mathbf{P}^{-1} - \mathbf{P}\,\mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}\;)\,\mathbf{y}(0) \\
&= -A^{+}\mathbf{B} + (\mathbf{I} - \mathbf{A}^{+}\,\mathbf{P}\,\mathbf{P}^{-1}\;\;\mathbf{A})\,\mathbf{y}(0) \\
&= -A^{+}\mathbf{B} + (\mathbf{I} - \mathbf{A}^{+}\;\;\mathbf{A})\,\mathbf{y}(0)
\end{aligned}$$

□

The general solution to equation (3.25) exists and is uniquely given by the next result.

Proposition 4. ***Solution for the non-homogenous ODE*** (3.25) *Consider the ODE* (3.25) *for any matrix* $\mathbf{A} \in \mathbb{R}^{2\times 2}$ *and vector* $\mathbf{B} \in \mathbb{R}^{2}$*. The solution to the ODE always exist and is uniquely given by*

$$\mathbf{y}(t) = \bar{\mathbf{y}} + \;\mathbf{e}^{\mathbf{A}\,t}\,(\mathbf{y}(0) - \bar{\mathbf{y}}) + (\mathbf{I} - \mathbf{A}^{+}\,\mathbf{A})\,\mathbf{B}\,t,\; \textit{for}\; t \in T = [0, \infty) \tag{3.30}$$

where the steady state $\bar{\mathbf{y}}$ *is given in equation* (3.29)*, and* $\mathbf{y}(0)$ *is an arbitrary element of* Y *for* $t = 0$*.*

Proof. Multiplying equation (3.26) by $\mathbf{P}$ we get the inverse transformation $\mathbf{y}(t) = \mathbf{P}\,\mathbf{w}(t)$. Using the solution for the transformed variables in equation (3.28) we get

$$\begin{aligned}
y(t) &= \mathbf{P}\,\bar{\mathbf{w}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda} t}(\mathbf{w}(0) - \bar{\mathbf{w}}) + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{P}^{-1}\,\mathbf{B}\,t \\
&= \bar{\mathbf{y}} \; + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda} t}\mathbf{P}^{-1}(\mathbf{y}(0) - \bar{\mathbf{y}}) + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{P}^{-1}\,\mathbf{B}\,t \\
&= \bar{\mathbf{y}} + e^{\mathbf{A} t}(\mathbf{y}(0) - \bar{\mathbf{y}}) + (\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}\;)\,\mathbf{B}\,t
\end{aligned}$$

which gives equation (3.30). □

This general result is consistent with several cases for a general $\mathbf{B}$ matrix.

3.4.1 Non-degenerate ODE's

Next we present the specific forms for the ODE (3.25) in which $\det(\mathbf{A}) \neq 0$

If $\det(\mathbf{A}) \neq 0$ then $\mathbf{A}^{+} = \mathbf{A}^{-1}$ then there is a unique steady state

$$\bar{y} = \mathbf{A}^{-1}\,\mathbf{B}.$$

Expanding the previous formula, we have

$$\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = -\frac{1}{\det(\mathbf{A})} \begin{pmatrix} a_{22}\,b_1 - a_{12}\,b_2 \\ -a_{21}\,b_1 - a_{11}\,b_2 \end{pmatrix}.$$

Remembering that $\mathbf{e}^{\mathbf{A}t} = \mathbf{P}\,\mathbf{e}^{\mathbf{\Lambda}t}\,\mathbf{P}^{-1}$, where $\mathbf{e}^{\mathbf{\Lambda}t}$ is the matrix exponential of the Jordan canonical form which is similar to $\mathbf{A}$, then the solution to the ODE, in equation (3.30), can be written as

$$\mathbf{y}(t) = \bar{\mathbf{y}} + \mathbf{P}\,\mathbf{e}^{\mathbf{\Lambda}t}\,\mathbf{k}$$

where $\mathbf{k} = \mathbf{P}^{-1}(\mathbf{y}(0) - \bar{\mathbf{y}})$, expanding

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{\det(\mathbf{P})} \begin{pmatrix} P_2^{+} & -P_1^{+} \\ -P_2^{-} & P_1^{-} \end{pmatrix} \begin{pmatrix} y_1(0) - \bar{y}_1 \\ y_2(0) - \bar{y}_2 \end{pmatrix}.$$

in which $\mathbf{y}(0)$ is an arbitrary element of Y at time $t = 0$.

Then the solution may be expanded in the following forms, by determining the eigenvalues as in

1. If $\Delta(\mathbf{A}) > 0$ then the Jordan canonical form of matrix $\mathbf{A}$ is $\mathbf{\Lambda}_1$. The general solution is

$$\mathbf{y}(t) = \bar{\mathbf{y}} + k_1 e^{\lambda_- t}\mathbf{P}^{-} + k_2 e^{\lambda_+ t}\mathbf{P}^{+}$$

where $\mathbf{P}^{-}$ ($\mathbf{P}^{+}$) is the simple eigenvector associated with λ_- (λ_+), or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} + k_1 e^{\lambda_- t} \begin{pmatrix} P_1^{-} \\ P_2^{-} \end{pmatrix} + k_2 e^{\lambda_+ t} \begin{pmatrix} P_1^{+} \\ P_2^{+} \end{pmatrix}$$

2. If $\Delta(\mathbf{A}) > 0$ then the Jordan canonical form of matrix $\mathbf{A}$ is $\mathbf{\Lambda}_2$. The general solution is

$$\mathbf{y}(t) = \bar{\mathbf{y}} + e^{\lambda t}\left(\mathbf{P}^{1}(k_1 + k_2 t) + k_2\mathbf{P}^{2}\right)$$

where $\mathbf{P}^{1}$ is a simple eigenvector and $\mathbf{P}^{2}$ is a generalized eigenvector (see the Appendix), or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} + e^{\lambda t}\left((k_1 + k_2 t) \begin{pmatrix} P_1^{-} \\ P_2^{-} \end{pmatrix} + k_2 \begin{pmatrix} P_1^{+} \\ P_2^{+} \end{pmatrix}\right)$$

3. If $\Delta(\mathbf{A}) < 0$ then the Jordan canonical form of matrix $\mathbf{A}$ is $\mathbf{\Lambda}_3$. The general solution is

$$\begin{aligned}\mathbf{y}(t) &= \overline{\mathbf{y}} + e^{\alpha t}\left((k_1 \cos\beta t + k_2 \sin\beta t)\mathbf{P}^1 + (k_2\cos\beta t - k_1 \sin\beta t)\mathbf{P}^2\right) = \\ &= \overline{y} + e^{\alpha t}\left(k_1(\cos\beta t\mathbf{P}^1 - \sin\beta t\mathbf{P}^2) + k_2(\sin\beta t\mathbf{P}^1 + \cos\beta t\mathbf{P}^2)\right).\end{aligned}$$

where $\mathbf{P}$ is a eigenvector (see the Appendix for the determination of the eigenvector matrix in the case in which the eigenvectors are complex) or, equivalently,

$$\begin{pmatrix} y_1(t) \\ y_2(t)\end{pmatrix} = \begin{pmatrix}\overline{y}_1 \\ \overline{y}_2\end{pmatrix} + e^{\alpha t}\left(k_1 \begin{pmatrix} P_1^1\cos\beta t - P_1^2 \sin\beta t \\ P_2^1 \cos\beta t - P_2^2\sin\beta t\end{pmatrix} + k_2 \begin{pmatrix} P_1^1 \sin\beta t + P_1^2\cos\beta t \\ P_2^1\sin\beta t + P_2^2 \cos\beta t\end{pmatrix}\right).$$

3.4.2 Degenerate cases

Degenerate cases occur for $\det(\mathbf{A}) = 0$ implying that $\mathbf{A}^+ \neq \mathbf{A}^{-1}$ and that the Jordan canonical form is diagonal (i.e, of type $\mathbf{\Lambda}_1$ in which one or two of the eigenvalues are equal to zero).

As $\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ then $\mathbf{A}^+ = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}$ and $\mathbf{A}^+\,\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ where $\mathbf{\Lambda}$ is one of the Jordan forms in equation (3.20) and $\mathbf{\Lambda}^+$ is the associated the Moore-Penrose in equation (3.21), depending on the trace being trace$(\mathbf{A}) \neq 0$ or trace$(\mathbf{A}) = 0$.

First observe that (3.30) can be expanded as

$$\mathbf{y}(t) = -\mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{B} + \mathbf{e}^{\mathbf{A}\,t}\left(\mathbf{y}(0) + \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{B}\right) + \left(\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1}\right)\mathbf{B}\,t,$$

where we can see that there are some components which are independent from the particular Jordan form in equation (3.20) and others which depend on the particular Jordan form.

For the first case we have $\tilde{\mathbf{B}} = \mathbf{P}^{-1}\,\mathbf{B}$ and $\mathbf{w}(0) = \mathbf{P}^{-1}\,\mathbf{y}(0)$, and write their expansion as

$$\tilde{\mathbf{B}} = \begin{pmatrix}\tilde{b}_- \\ \tilde{b}_+\end{pmatrix} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} -P_2^-\,b_1 + P_1^-\,b_2 \\ P_2^+\,b_1 - P_1^+\,b_2\end{pmatrix}$$

and

$$\mathbf{w}(0) = \begin{pmatrix} w_-(0) \\ w_+(0)\end{pmatrix} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} -P_2^-\,y_1(0) + P_1^-\,y_2(0) \\ P_2^+\,y_1(0) - P_1^+\,y_2(0)\end{pmatrix}$$

For the second case we have, if $\lambda_- < 0 = \lambda_+$,

$$\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} -P_2^-\,P_1^+ & P_1^-\,P_1^+ \\ -P_2^-\,P_2^+ & P_1^-\,P_2^+\end{pmatrix}$$

for the case in which $\lambda_- = 0 < \lambda_+$ we have

$$\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} P_2^+\,P_1^- & -P_1^+\,P_1^- \\ P_2^+\,P_2^- & -P_1^+\,P_2^-\end{pmatrix}$$

and for $\lambda_- = \lambda_+ = 0$ we have $\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \mathbf{I}$.

Therefore the solutions become

1.
$$\mathbf{y}(t) = \mathbf{P}^{+} w_{+}(0) - \mathbf{P}^{-}\frac{\tilde{b}_{-}}{\lambda_{-}} + \begin{pmatrix} P_1^{-}\, e^{\lambda_{-} t} \\ P_2^{-} \end{pmatrix} (w_{-}(0) + \frac{\tilde{b}_{-}}{\lambda_{-}}) - \mathbf{P}^{+}\,\tilde{b}_{+}$$

2. for $\lambda_{-} = \lambda_{+} = 0$
$$\mathbf{y}(t) = \mathbf{P}^{-} w_{-}(0) - \mathbf{P}^{+}\frac{\tilde{b}_{+}}{\lambda_{+}} + \begin{pmatrix} P_1^{+} \\ P_2^{+}\, e^{\lambda_{+} t} \end{pmatrix} (w_{+}(0) + \frac{\tilde{b}_{+}}{\lambda_{+}}) - \mathbf{P}^{-}\,\tilde{b}_{-}$$

From this point on I will post a revised version soon.

3.5 Characterizing solutions to linear planar ODEs

3.5.1 solutions when A is a Jordan normal form

Characterizing the possible dynamics for coefficient matrix in one of the canonical forms is necessary if matrix $\mathbf{A}$ is in a Jordan canonical form and it is useful if the coefficient matrix is not in a Jordan canonical form. This is because an implication of Lemma 6 is that the time dependency of the solution results from the dynamics of the associated canonical form. Furthermore, this means that the number of cases to analyse can be explicitly enumerated.

We consider next the homogeneous ODE
$$\dot{\mathbf{w}} = \mathbf{\Lambda}\,\mathbf{w}.$$
which can be seen as a homogeneous version of equation (3.14) or of equation (3.27)

We can enumerate the types of solutions along several criteria. We will focus on two criteria: first, the time dependency of the solution, and, second, the asymptotic behavior of the solution, i.e, the path of $\mathbf{w}(t)$ when t tends to infinity.

Time dependency of solutions

From the first perspective we can have the following type of solutions: stationary, monotonic, oscillatory, periodic solutions and hump-shaped.

Stationary solutions We say the solution is stationary if $\mathbf{w}(t)$ is a constant for all $t \in \mathrm{T}$. In this case $\dot{\mathbf{w}}(t) = \mathbf{0}$ for all t.

Monotonic solutions We say the solution is monotonic if $\text{sign}(\dot{\mathbf{w}}(t))$ is the same for all $t \in \mathrm{T}$. This means that the solution is monotonically increasing if $\dot{\mathbf{w}}(t) > \mathbf{0}$ for all t, it is monotonically decreasing if $\dot{\mathbf{w}}(t) < \mathbf{0}$ for all t. A stationary solution can be seen as a particular type of monotonic solution.

Oscillatory solutions A solution is oscillatory if $\mathbf{w}(t) = \mathbf{w}(t+p(t))$ for $t \in \mathrm{T}$ and time-dependent period $p(t) \in \mathrm{T}$: the solution is repeated in increasing intervals if $p^{'}(t) > 0$ or in decreasing intervals if $p^{'}(t) < 0$. For these solutions, there is a sequence of points, increasing or decreasing in time $\tau \in \{t_0, t_1, \dots, t_s, \dots\}$ such that $\dot{\mathbf{w}}(\tau) = 0$. In our case if there are two complex eigenvalues with non-zero real part, that is $\alpha \neq 0$, then the solution is oscillatory

$$\mathbf{w}(t) = e^{\alpha t} \begin{pmatrix} k_1 \cos \beta t + k_2 \sin \beta t \\ k_2 \cos \beta t - k_1 \sin \beta t \end{pmatrix}.$$

Periodic solutions If a solution satisfies $\mathbf{w}(t) = \mathbf{w}(t+p)$ for $t \in \mathrm{T}$ and $p \in T$ it is a periodic solution period p. This is a particular case of an oscillatory solution in which the period is constant. In our case if there are two complex eigenvalues with zero real part then the solution is periodic

$$\mathbf{w}(t) = \begin{pmatrix} k_1 \cos \beta t + k_2 \sin \beta t \\ k_2 \cos \beta t - k_1 \sin \beta t \end{pmatrix}.$$

This case occurs if and only if $\mathrm{trace}(\mathbf{A}) = 2\alpha = 0$. Observe that in this case and if we transform the system into polar coordinates (see appendix 3.A.2 we have $r(t) = r_0$ constant and $\theta(t) = \theta_0 - \beta t$.

Hump-shaped solutions If the solution of a planar equation is such that only one variable satisfies $\dot{w}_i(t) = 0$ for a finite $t \in \mathrm{T}$ and the other variable w_{-i} is monotonic, then we say the solution is hump-shaped. This case only occurs for the general homogeneous equation when there are eigenvalues with real parts.

Steady states and stability analysis

The second perspective on equations deals with their convergence as regards steady states.

Steady states

Definition 1. ***Steady states*** *A steady state is a fixed point to equation* **??** *such that* $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} = \mathbf{0}$.

If $\mathbf{A} = \mathbf{\Lambda}$ is in the Jordan form we define the set of steady states

$$\overline{\mathbf{w}} = \{\mathbf{w} \in \mathrm{Y} : \mathbf{\Lambda}\mathbf{w} = 0\}.$$

An important distinction should be made: while a stationary solution is a function of t such that $\mathbf{w}(t)$ is constant, for all $t \in \mathrm{T}$, a steady state is a fixed point of the vector field generated by the differential equation. However, for a planar ODE the solution of the differential equation is stationary if and only if it is a steady state. A stationary solution can only exist for particular values of $\mathbf{k} \in \mathrm{Y}$. We will see that both the number of steady states and the convergence to or divergence from a steady state are determined by the parameters (in vector $\mathbf{\Lambda}$).

Let $\mathbf{0} \in \mathrm{Y}$. Then steady states always exist but need not be unique. We have again three main cases: First if $\mathbf{\Lambda} = \mathbf{\Lambda}_1$ the two eigenvalues are real and distinct and we have four possible cases. i.e.,

$$\overline{\mathbf{w}} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ k_2 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\}$$

where $\mathbf{k} = (k_1, k_2)^\top$ is an arbitrary element of Y. The **steady state is an unique point** $\bar{\mathbf{w}} = \mathbf{0} \in \mathrm{Y}$ If the two eigenvalues are non-zero, $\lambda_+ \neq 0$ and $\lambda_- \neq 0$. The steady state is a **one-dimensional manifold** in Y If $\lambda_+ = 0$, $\lambda_- \neq 0$ and $k_1 \neq 0$ or if $\lambda_+ \neq 0$, $\lambda_- = 0$ and $k_2 \neq 0$. Every point in Y is a steady state if $\lambda_+ = \lambda_- = 0$. If the steady state is unique we call it a **node** and when it is not unique we call it a **degenerate node**.

Second, if $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ then the two eigenvalues are real and equal and we have two possible cases

$$\overline{\mathbf{w}} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k_1 \\ 0 \end{pmatrix} \right\}$$

The steady state is unique if $\overline{\mathbf{w}} = \mathbf{0} \in \mathrm{Y}$ and the eigenvalue if different from zero, and the steady state is one dimensional manifold in Y if the eigenvalue is equal to zero and $k_1 \neq 0$. If the steady state is unique we call it a **node with multiplicity** and if it is not unique it is a **degenerate node with multiplicity**.

Third, if $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ then the steady state is unique

$$\overline{\mathbf{w}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In this case, the steady state is called a **focus**.

Stability properties A solution $\mathbf{w}(t)$ is **asymptotically stable** if $\lim_{t\to\infty} \mathbf{w}(t) = \overline{\mathbf{w}} = \mathbf{0}$ for any $\mathbf{k} \neq \mathbf{0}$, i.e., the solution converges to the steady state.

A solution is **unstable** if for any $\mathbf{k} \neq \overline{\mathbf{w}} = \mathbf{0}$ then $\lim_{t\to\infty} \mathbf{w}(t) = \pm\infty$, i.e., the solution diverges. A solution is **semi-stable** (or conditionally stable) if there is a a subset of values $\mathcal{E}^s \in \mathrm{Y}$ such that if $\mathbf{k} \in \mathcal{E}^s$ then $\lim_{t\to\infty} \mathbf{w}(t) = \overline{\mathbf{w}} = \mathbf{0}$ but if $\mathbf{k} \notin \mathcal{E}^s$ then $\lim_{t\to\infty} \mathbf{w}(t) = \pm\infty$, i.e, the solution is asymptotically stable for some values $\mathbf{k}$ but is unstable for others.

The eigenvalues of Λ not only determine the number of steady states but also their stability properties:

Proposition 5. *The asymptotic dynamic characteristics of the solution of equation* (3.14) *is determined by the real part of the eigenvalues, $Re(\lambda_i)$, $i = 1, 2$ of matrix $\mathbf{\Lambda}$:*

1. *if all the eigenvalues have negative real parts then all solutions of ODE* (3.14) *are asymptotically stable;*

2. *if all eigenvalues have positive real parts then all solutions are unstable;*

3. *if there is one negative and one positive eigenvalue* ($\lambda_1 > 0 > \lambda_-$) *then the solution to ODE* $\dot{\mathbf{w}} = \mathbf{\Lambda w}$ *is semi-stable: it is unstable if* $k_1 \neq 0$ *and it is asymptotically stable if* $k_1 = 0$*;*

4. *if there is one zero eigenvalue the fixed point is a one-dimensional manifold (a center manifold), the solution will converge to it if the other eigenvalue is negative (i.e., in case* $\lambda_+ = 0$ *and* $\lambda_- < 0$*) and will not converge to it if the other eigenvalue is positive (i.e., in case* $\lambda_+ > 0$ *and* $\lambda_- = 0$*). In the first case there is a* ***degenerate stable node*** *and in the second case a* ***degenerate unstable node***

Proof. (1) If we consider the solutions (3.16)-(3.18) such that the real parts of the eigenvalues are negative (i.e, $\lambda_+ < 0$ and $\lambda_- < 0$, or $\lambda < 0$ or $\alpha < 0$) then we see that the solutions tend to the fixed point $\overline{\mathbf{w}} = 0$ for any k_1 and k_2. (2) If there is an eigenvector with a positive real part (i.e, $\lambda_+ > 0$ or $\lambda_- > 0$, or $\lambda > 0$ or $\alpha > 0$) then given any point $\mathbf{k} \neq \overline{\mathbf{w}}$ then the solution will be unbounded. All the other cases can be characterized in an analogous way. □

Eigenspaces The solutions of equation $\dot{\mathbf{y}} = \mathbf{A y}$ is a weighted average of two elementary functions weighted by $\mathbf{h} = (h_1, h_2)$. For example, if $\Lambda = \mathbf{\Lambda}_1$ we could write its solution (**??**) as

$$\mathbf{y}\ (t) = h_1 \mathbf{P}^1 e^{\lambda_+ t} + h_2 \mathbf{P}^2 e^{\lambda_- t}. \tag{3.31}$$

That is, the solution of the ODE is a superposition of two elementary function $e^{\lambda_+ t}$ and $e^{\lambda_- t}$, acting on the directions given by $\mathbf{P}^1$ and $\mathbf{P}^2$, respctively, and weighted by the arbitrary constants h_1 and h_2. In other words, the elementary components of the time behavior of the solutions, $e^{\lambda_+ t}$ and $e^{\lambda_- t}$, are linearly transformed by the eigenvectors $\mathbf{P}^1$ and $\mathbf{P}^2$.

We define the **eigenspaces** as the subsets of space Y which are followed by those two elementary solutions:

$$\begin{aligned} \mathcal{E}^1 &= \{\ \mathbf{w} \in \mathrm{Y} :\ \text{spanned by } \mathbf{P}^1\} \\ \mathcal{E}^2 &= \{\ \mathbf{w}\ \in \mathrm{Y} :\ \text{spanned by } \mathbf{P}^2\} \end{aligned}$$

Clearly the range of $\mathbf{y}$ satisfies $\mathrm{Y} = \mathcal{E}^1 \oplus \mathcal{E}^2$.

In the case of equation $\dot{\mathbf{w}} = \mathbf{\Lambda w}$ because $\mathbf{P} = \mathbf{I}$ we have

$$\begin{aligned} \mathcal{E}^1 &= \{\ \mathbf{w} \in \mathrm{Y} :\ w_2 = 0\} \\ \mathcal{E}^2 &= \{\ \mathbf{w} \in \mathrm{Y} :\ w_1 = 0\} \end{aligned}$$

which is equivalent to setting $k_2 = 0$ in the first case and $k_1 = 0$ in the second.

We call stable, unstable and center eigenspaces to the subsets of Y which are spanned by the eigenspaces associated to the eigenvalues with negative, positive and zero real parts. Formally the **stable eigenspace** is

$$\mathcal{E}^s \equiv \oplus \{\ \mathcal{E}^j : \mathrm{Re}(\lambda_j) < 0\},$$

the **unstable eigenspace** is

$$\mathcal{E}^u \equiv \oplus\{\ \mathcal{E}^j : \mathrm{Re}(\lambda_j) > 0\},$$

and the **center eigenspace** is

$$\mathcal{E}^c \equiv \oplus\{\ \mathcal{E}^j : \mathrm{Re}(\lambda_j) = 0\}.$$

Again we have

$$\mathcal{E}^s \oplus \mathcal{E}^u \oplus \mathcal{E}^c = \mathrm{Y}.$$

Let n_-, n_+ and n_c be respectively the number of eigenvalues with negative, positive and zero real parts. Another way to see the relationship between the eigenspaces and the range of the dynamical system is based on the observation that

$$n_- + n_+ + n_c = 2.$$

and that the dimension of the there eigenspaces are therefore

$$\dim(\mathcal{E}^s) = n_-,\ \dim(\mathcal{E}^u) = n_+,\ \dim(\mathcal{E}^c) = n_c,$$

implying

$$\dim(\mathcal{E}^s) + \dim(\mathcal{E}^u) + \dim(\mathcal{E}^c) = \dim(\mathrm{Y}) = 2.$$

Therefore, for a planar ODE we have:

1. if all eigenvalues have negative real parts, i.e., if $n_- = 2$, then $\mathcal{E}^s = \mathcal{E}^1 \oplus \mathcal{E}^2 = \mathrm{Y}$, and $\mathcal{E}^u$ and $\mathcal{E}^c$ are empty, which means that $\mathcal{E}^s$ is spanned by $\mathcal{E}^1$ and $\mathcal{E}^2$ (i.e, the elements in $\mathcal{E}^s$ are a weighted sum of elements of $\mathcal{E}^1$ and $\mathcal{E}^2$). Then Y is the **attracting set**;

2. if all eigenvalues have positive real parts, i.e., if $n_+ = 2$, then $\mathcal{E}^u = \mathcal{E}^1 \oplus \mathcal{E}^2 = \mathrm{Y}$, and $\mathcal{E}^s$ and $\mathcal{E}^c$ are empty. Then $Y/\bar{y}$ is the **repelling set**

3. if there is a saddle point, i.e., if $n_- = n_+ = 1$, then $\mathcal{E}^s = \mathcal{E}^2$, $\mathcal{E}^u = \mathrm{Y}/E^s$ and , and $\mathcal{E}^c$ is empty. Then $\mathcal{E}^s$ is the **attracting set** and $\mathcal{E}^u$ is the **repelling set**

4. if there is at least one eigenvalue with zero real part, i.e., if $n^c \in \{1, 2\}$, then $\mathcal{E}^c$ is non-empty.

Phase diagrams

The **geometrical** approach for solving ODE consists in drawing a **phase diagram**.

Phase diagrams for planar autonomous ODE are drawn in the space (w_1, w_2) and contain the following elements:

1. **isoclines (or nullclines)** are lines in space (w_1, w_2) such that w_1 or w_2 are constant, that is

$$\mathbb{I}_{w_1} = \{\ (w_1, w_2) \in \mathrm{Y}:\ \dot{w}_1 = 0\},\ \text{and}\ \mathbb{I}_{w_2} = \{\ (w_1, w_2) \in \mathrm{Y}:\ \dot{w}_2 = 0\}.$$

   The steady states are the locus or loci where isoclines intersect;

2. the **eigenspaces** $\mathcal{E}^1$ and $\mathcal{E}^2$ are lines in Y whose slopes are given by those of the eigenvectors $\mathbf{P}^1$ and $\mathbf{P}^2$. They span the stable, unstable and center manifolds, $\mathcal{E}^s$, $\mathcal{E}^u$, and $\mathcal{E}^c$, which are lines or two-dimensional subsets of Y;

3. some representative trajectories, also called **integral curves**, that is parametric curves of the solution to the ODE within space Y. They are usually represented with direction arrows showing the direction of the solution with time.

4. the **vector field** indicating the direction of time evolution for a grid of points in Y.

There are four main types of phase diagrams: **nodes**, if all eigenvalues are real and have the same sign, **saddles** if there is one positive and one negative eigenvalue, **foci** if the two eigenvalues are complex conjugate with non-zero real parts, and **centers** if the two eigenvalues are complex conjugate with zero real parts.

Next we present a complete list **phase diagrams**:

Stable nodes A stable node exists if there is at least one real negative eigenvalue and there are no positive eigenvalues. There are three cases: the non-degenerate stable nodes (figure 3.2), the degenerate stable node (see figure 3.3) and the stable node with multiplicity (see figure 3.4).

In the case of figure 3.2 the phase diagram contains the following elements

- there are two isoclines: the abcissa, associated $\dot{w}_2 = 0$ which is the loci where w_2 is constant, and the ordinate, associated $\dot{w}_1 = 0$ which is the loci where w_1 is constant

- a fixed point where the two isoclines cross at $(w_1, w_2) = (0, 0)$

- the eigenspace $\mathcal{E}^1$ which is coincident with $\dot{w}_2 = 0$ associated to the eigenvector λ_+ and eigenspace $\mathcal{E}^2$ which is coincident with $\dot{w}_1 = 0$ associated to the eigenvector λ_-. This coincidence occurs for decoupled systems where $\mathbf{A}$ has the Jordan form $\mathbf{\Lambda}_1$. The whole space Y (with the exception of the fixed point) corresponds to the stable eigenspace $\mathcal{E}^s$. Both the unstable eigenspace and the center eigenspace are empty.

- four representative trajectories. Observe that the slope of the trajectories is parallel to $\mathcal{E}^2$ for initial points far away from the fixed point and they tend asymptotically to $\mathcal{E}^1$. To prove this we write their slope in the phase diagram, for any t, $s(t)$

$$\frac{w_2(t)}{w_1(t)} = s(t) \equiv \frac{k_2}{k_1} e^{(\lambda_- - \lambda_+)t}.$$

We see that $s(0) = \dfrac{k_2}{k_1}$, $\lim_{t \to -\infty} s(t) = \infty$ and $\lim_{t \to \infty} s(t) = 0$ because $\lambda_- - \lambda_+ < 0$. This means that all trajectories converge to the steady state along trajectories which are tangent to line $w_2 = 0$, that is, to the eigenspace $\mathcal{E}^1$, or to the direction defined by the eigenvector $\mathbf{P}^1$ which is associated with the eigenvalue with **smaller** absolute value.

In the case of the degenerate stable node, such that $0 = \lambda_+ > \lambda_-$, in figure 3.3, we have $\mathcal{E}^c = \mathcal{E}^2$ and $\mathcal{E}^s = \mathrm{Y}/\mathcal{E}^2$. $\mathcal{E}^c$ is also loci of fixed points which are in infinite number.

In the case of multiplicity (see figure 3.4) the trajectories approach P^1 whose slope is given by the simple eigenvector $\mathbf{P}^1 = (1, 0)^\top$.

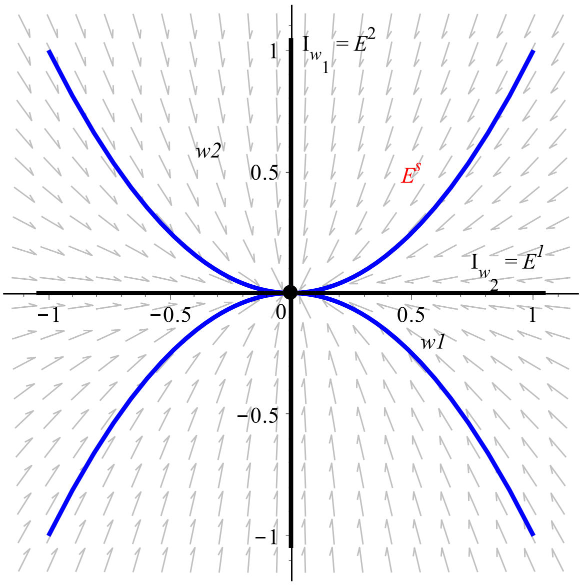

Figure 3.2: Stable node: phase diagram and representative trajectories for the ODE $\dot{w}_1 = -0.5 w_1$, $\dot{w}_2 = -w_2$.

Saddle point A saddle points exists if the two eigenvalues are real and $\lambda_- < 0 < \lambda_+$. Figure 3.5 presents the phase diagram containing the following elements

- there are two isoclines: the abcissa, associated $\dot{w}_2 = 0$ which is the loci where w_2 is constant, and the ordinate, associated $\dot{w}_1 = 0$ which is the loci where w_1 is constant
- a fixed point where the two isoclines cross at $(w_1, w_2) = (0, 0)$
- the unstable eigenspace $\mathcal{E}^1$, which is coincident with $\dot{w}_2 = 0$, associated to the eigenvector $\lambda_+ > 0$ and the stable eigenspace $\mathcal{E}^s = \mathcal{E}^2$, which is coincident with $\dot{w}_1 = 0$, associated to the eigenvector $\lambda_- < 0$. The unstable eigenspace $\mathcal{E}^u$ is almost coincident with all set Y, as $\mathcal{E}^u = \mathrm{Y}/\mathcal{E}^s$This coincidence occurs again for decoupled systems where $\mathbf{A}$ has the Jordan form $\mathbf{\Lambda}_1$

Unstable nodes A unstable node exists if there is at least one real positive eigenvalue and there are no negative eigenvalues. There are three cases: the non-degenerate unstable nodes (figure 3.6), the degenerate unstable node (see figure 3.7).

The interpretation is analogous to the stable nodes, if we introduce a time reversal, and if we substitute the stable eigenspace with unstable eigenspace.

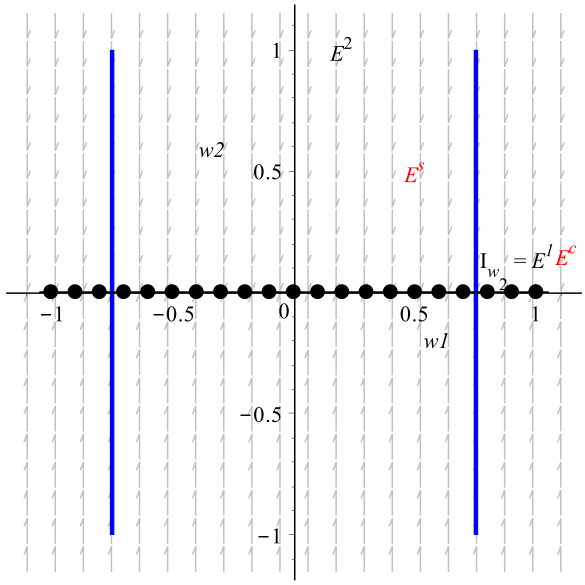

Figure 3.3: Degenerate stable node: $\dot{w}_1 = 0$, $\dot{w}_2 = -w_2$.

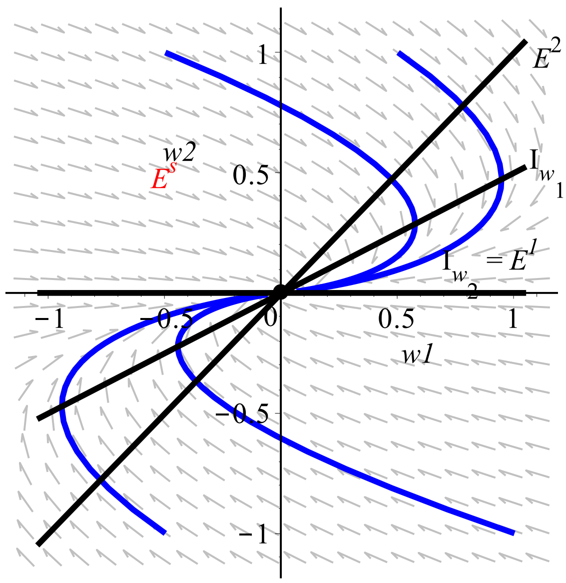

Figure 3.4: Stable node with multiplicity: $\dot{w}_1 = -0.5w_1 + w_2$, $\dot{w}_2 = -0.5w_2$.

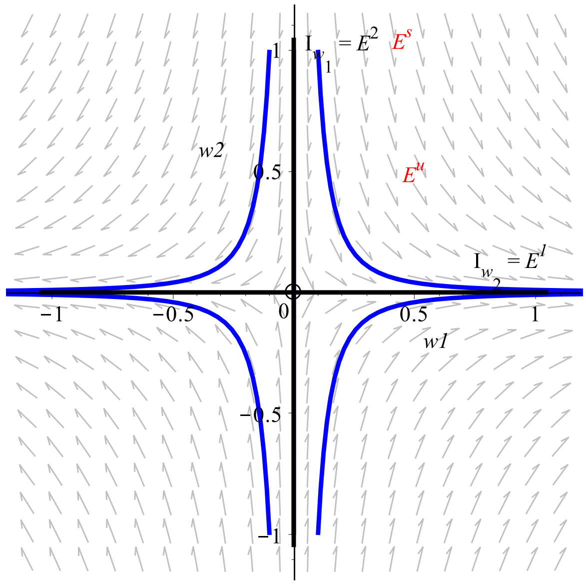

Figure 3.5: Saddle point: $\dot{w}_1 = 0.5w_1$, $\dot{w}_2 = -w_2$.

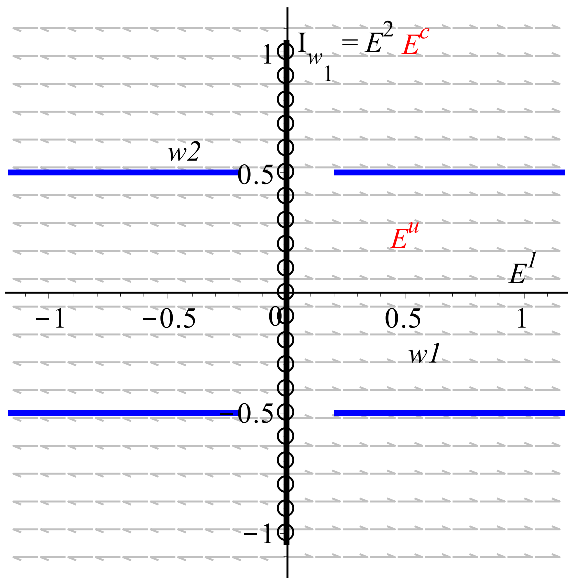

Figure 3.6: Unstable degenerate node: $\dot{w}_1 = 0.5w_1$, $\dot{w}_2 = 0$.

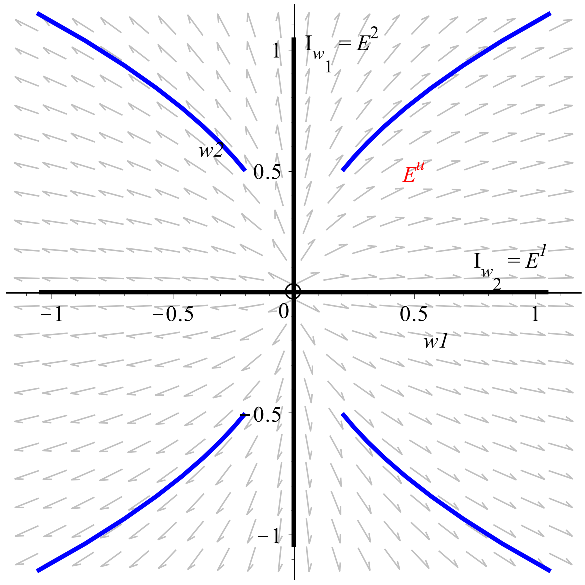

Figure 3.7: Unstable node: $\dot{w}_1 = w_1$, $\dot{w}_2 = 0.5w_2$.

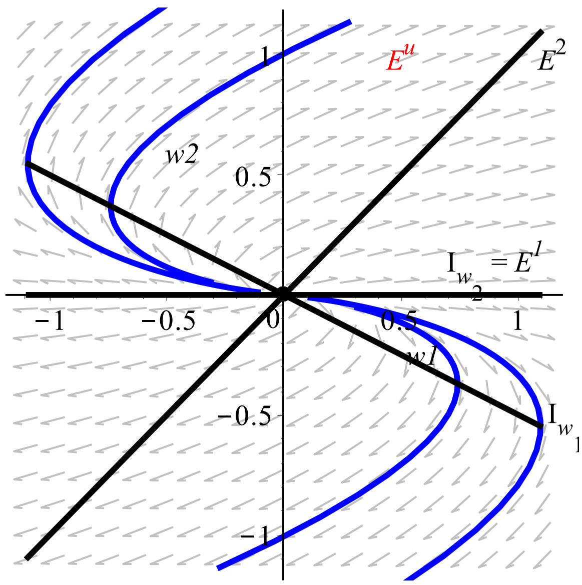

Figure 3.8: Unstable node with multiplicity: $\dot{w}_1 = 0.5w_1 + w_2$, $\dot{w}_2 = 0.5w_2$.

Stable foci

A stable focus exists if there are two complex conjugate eigenvalues with negative real parts (see figures 3.9 for $\beta > 0$ and 3.10 for $\beta < 0$).

In this case we see that there is asymptotic stability, as for the case of the stable node in figure 3.2, but the trajectories are oscillatory. We also see that the stable node with multiplicity 3.4 is a boundary case between stable node and foci. The stable eigenspace is coincident with the whole space Y and the unstable and center eigenspaces are empty.

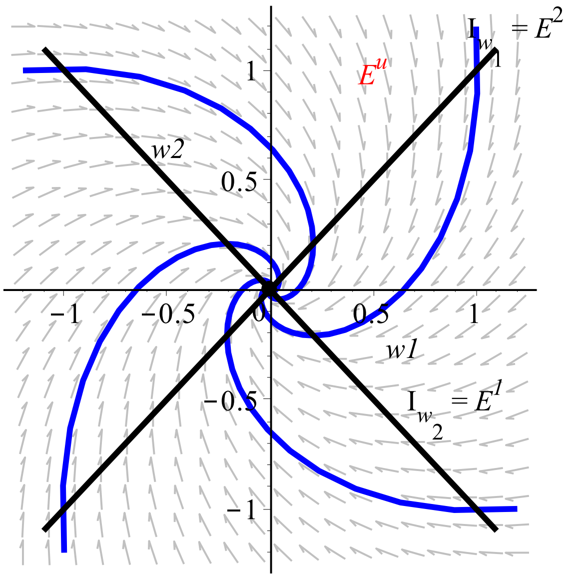

Figure 3.9: Stable focus: $\dot{w}_1 = -0.5w_1 + 0.5w_2$, $\dot{w}_2 = -0.5w_1 - 0.5w_2$ (case $\alpha < 0$ and $\beta > 0$).

Unstable foci

An unstable focus exists if there are two complex conjugate eigenvalues with positive real parts (see figures 3.11 for $\beta > 0$ and 3.12 for $\beta < 0$). The unstable eigenspace is coincident with the whole space Y and the stable and center eigenspaces are empty.

Center A center (see figure 3.13) exists , if eigenvalues are complex conjugate and have zero real parts. The center eigenspace, $\mathcal{E}^c$, is coincident with the whole space Y and the stable and the unstable eigenspaces are empty. If $w \neq 0$ then all the trajectories are periodic.

3.5.2 Characterizing solutions when A is not in the Jordan form

Now we address the general planar linear homogeneous equation $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ already derived in equation (**??**),

$$\mathbf{y}(t) = \mathbf{P}\mathbf{e}^{\mathbf{t}}\ \mathbf{h}$$

where $\mathbf{h} = \mathbf{P}^{-1}\mathbf{k}$. By observing that $\mathbf{y}(t) = \mathbf{P}\mathbf{w}(t)$ we see that the solution in this case is a linear transformation of the solution for the case in which the coefficient matrix is in the Jordan form.

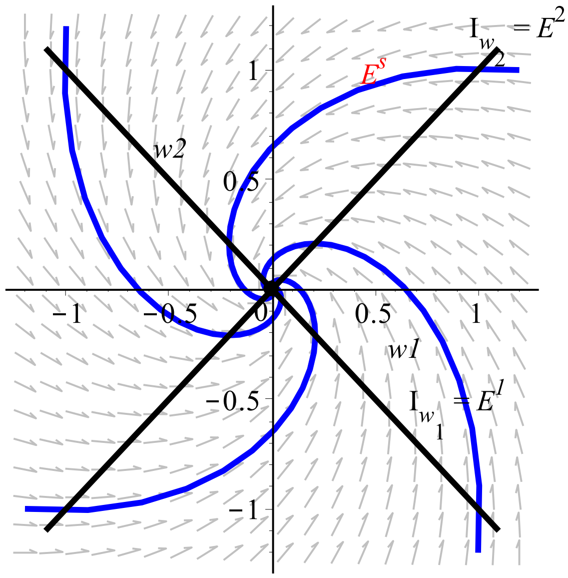

Figure 3.10: Stable focus:$\dot{w}_1 = -0.5w_1 - 0.5w_2$, $\dot{w}_2 = 0.5w_1 - 0.5w_2$ (case $\alpha < 0$ and $\beta < 0$).

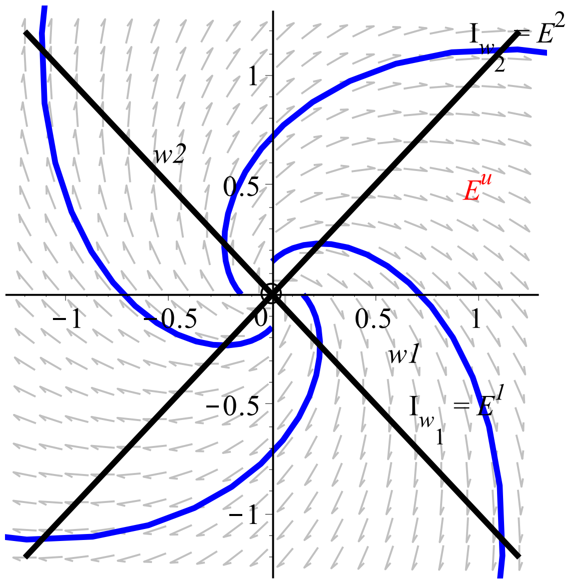

Figure 3.11: Unstable focus: $\dot{w}_1 = 0.5w_1 + 0.5w_2$, $\dot{w}_2 = -0.5w_1 + 0.5w_2$ (case $\alpha > 0$ and $\beta > 0$).

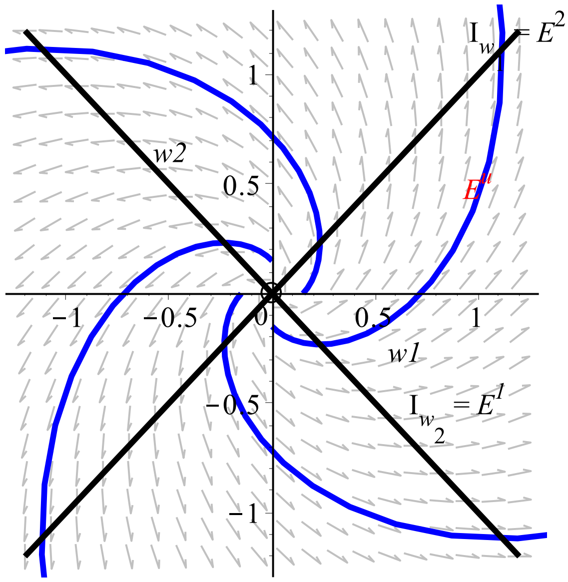

Figure 3.12: Unstable focus: $\dot{w}_1 = 0.5w_1 - 0.5w_2$, $\dot{w}_2 = 0.5w_1 + 0.5w_2$ (case $\alpha < 0$ and $\beta < 0$).

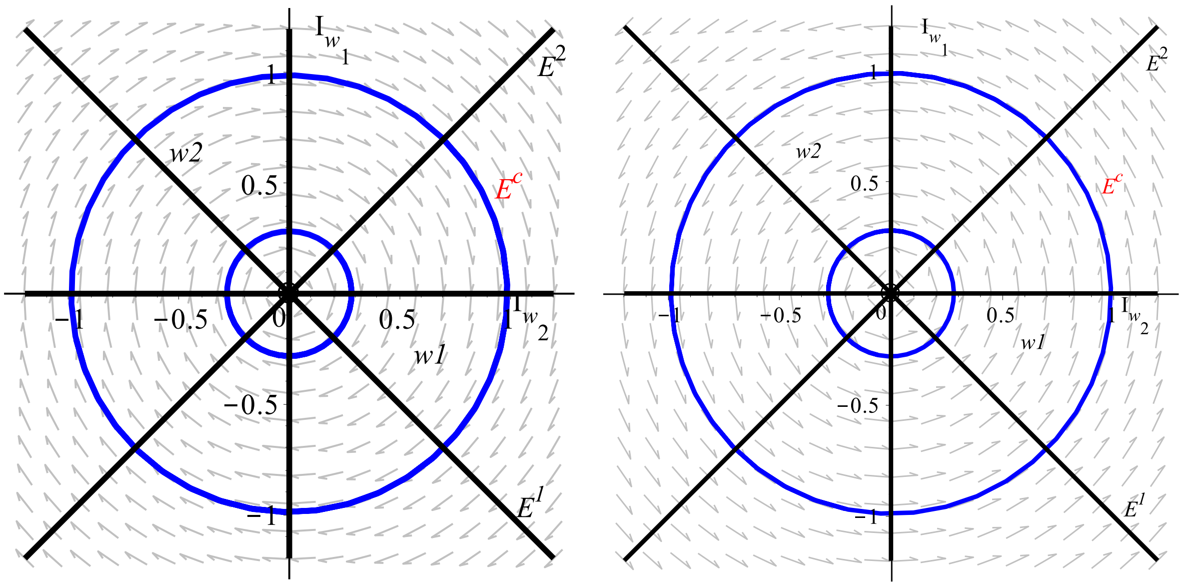

Figure 3.13: Center: $\dot{w}_1 = -0.5w_2$, $\dot{w}_2 = 0.5w_1$ and $\dot{w}_1 = 0.5w_2$, $\dot{w}_2 = -0.5w_1$ (cases $\alpha = 0$ and $\beta > 0$, and $\alpha = 0$ and $\beta < 0$).

This means that

1. the qualitative properties of the dynamics are the same, in particular, the number and stability type of the steady state(s)

2. the dimensions of the stable, unstable and center eigenspaces, partitioning the range Y, is the same

3. the only difference is related to the slopes of the eigenspaces and therefore of the solution trajectories, because the eigenvector matrix $\mathbf{P}$ is different from the identity matrix.

In particular we can have one of the following (general) solutions

1. if $\mathbf{\Lambda} = \mathbf{\Lambda}_1$, the general solution is

$$\mathbf{y}(t) = h_1 e^{\lambda_+ t}\mathbf{P}^1 + h_2 e^{\lambda_- t}\mathbf{P}^2$$

   or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = h_1 e^{\lambda_+ t} \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} + h_2 e^{\lambda_- t} \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix}$$

2. if $\mathbf{\Lambda} = \mathbf{\Lambda}_2$, the general solution is

$$\mathbf{y}(t) = e^{\lambda t}\left(\mathbf{P}^1(h_1 + h_2 t) + h_2\mathbf{P}^2\right)$$

   or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{\lambda t} \left((h_1 + h_2 t) \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} + h_2 \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} \right)$$

3. if $\mathbf{\Lambda} = \mathbf{\Lambda}_3$, the general solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{\alpha t}\left((h_1 \cos\beta t + h_2 \sin\beta t)\mathbf{P}^1 + (h_2\cos\beta t - h_1 \sin\beta t)\mathbf{P}^2\right) = \\ &= e^{\alpha t}\left(h_1(\cos\beta t\mathbf{P}^1 - \sin\beta t\mathbf{P}^2) + h_2(\sin\beta t\mathbf{P}^1 + \cos\beta t\mathbf{P}^2)\right). \end{aligned}$$

   or, equivalently,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{\alpha t} \left(h_1 \begin{pmatrix} P_1^- \cos\beta t - P_1^+ \sin\beta t \\ P_2^- \cos\beta t - P_2^+ \sin\beta t \end{pmatrix} + h_2 \begin{pmatrix} P_1^- \sin\beta t + P_1^+ \cos\beta t \\ P_2^- \sin\beta t + P_2^+ \cos\beta t \end{pmatrix} \right).$$

The next example allows for a comparison between the solutions of an homogeneous problem when the coefficient matrix is a Jordan form with the case in which it is a similar matrix but not in the Jordan normal form

Example 1 Solve the planar ODE assuming that $\mathbf{w} \in \mathbb{R}^2$.

$$\begin{aligned}\dot{w}_1 &= 3w_1\\ \dot{w}_2 &= -3w_2\end{aligned}$$

We readily see that the coefficient matrix in in the Jordan form λ_+

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}.$$

The (general) solution of the ODE is

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} h_1 e^{3t} \\ h_2 e^{-3t} \end{pmatrix} = h_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + h_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

Therefore: (1) there is a unique steady state $\overline{\mathbf{w}} = (\overline{w}_1, \overline{w}_2) = (0, 0)$; (2) the steady state is a saddle point; (3) the eigenvalues of the coefficient matrix are $\lambda_+ = 3$ and $\lambda_- = -3$ and the associated eigenvectors are $\mathbf{P}^1 = (1, 0)^\top$ and $\mathbf{P}^2 = (0, 1)^\top$; (4) the eigenspaces associated to the eigenvalues λ_+ and λ_- are

$$\mathcal{E}^1 = \{\ \mathbf{w} \in \mathbb{R}^2 : w_2 = 0\},\ \mathcal{E}^2 = \{\ \mathbf{w} \in \mathbb{R}^2 : w_1 = 0\};$$

(5) then the center eigenspace $\mathcal{E}^c$ is empty and the stable and unstable eigenspaces are both of dimension 1 and the unstable and stable eigenspaces are

$$\mathcal{E}^s = \mathcal{E}^2,\ \mathcal{E}^u = \mathbb{R}^2 / \mathcal{E}^s$$

meaning that for any $\mathbf{k} \neq (0, k_2)$ the solution is unstable.

That is, trajectories belonging to the stable subspace, that is converging to the steady state, should have $k_1 = 0$, that is they are

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ h_2 e^{\lambda_- t} \end{pmatrix}$$

The phase diagram for this equation is very similar to the one depicted in Figure 3.5.

Example 2 Solve the homogeneous ODE over the domain $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$:

$$\begin{aligned}\dot{y}_1 &= -2y_1 + 5y_2,\\ \dot{y}_2 &= y_1 + 2y_2.\end{aligned} \tag{3.32}$$

where

$$\mathbf{A} = \begin{pmatrix} -2 & 5 \\ 1 & 2 \end{pmatrix}.$$

As $\text{trace}(\mathbf{A}) = 0$ and $\det(\mathbf{A}) = -9$ the eigenvalues are $\lambda_+ = 3$ and $\lambda_- = -3$, which means that the coefficient matrix is similar to the previous example. The eigenvector matrix is

$$\mathbf{P} = (\mathbf{P}^1, \mathbf{P}^2) = \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}.$$

This means that the eigenspaces are

$$\mathcal{E}^1 = \{\ \mathbf{y} \in \mathbb{R}^2 : y_1 - y_2 = 0\},\ \mathcal{E}^2 = \{\ (\mathbf{y} \in \mathbb{R}^2 : y_1 + 5\,y_2 = 0\}$$

As $\det(\mathbf{A}) \neq 0$ then the fixed point exists and is unique and is $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2)^\top = (0,0)^\top$.

The (general) solution of the equation, $\mathbf{y}(t) = \mathbf{P}\mathbf{e}^{\mathbf{t}}\,\mathbf{h}$, is

$$y(t) = h_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + h_2 \begin{pmatrix} -5 \\ 1 \end{pmatrix} e^{-3t}. \tag{3.33}$$

The stable and the unstable eigenspaces (the center eigenspace is empty. Why ?) are

$$\mathcal{E}^s = \{\ (y_1, y_2) :\ y_1 + 5y_2 = 0\},\ \mathcal{E}^u = \mathbb{R}^2/(\mathcal{E}^s \cup \{(0,0)\}\ .$$

and the stable subspace is equal to the eigenspace $\mathcal{E}^2$. Then, if the initial point is such that $(y_1(0), y_2(0)) = (-5y_2(0), y_2(0))$ for any choice of $y_2(0)$ the solution converges to the steady state $\bar{y} = (0,0)^\top$. For any other initial point the solution is asymptotically unbounded.

We can prove this in two different but equivalent ways: First, we can consider the general solution in equation of (3.36) and set $h_1 = 0$. Comparing to the case in which the coefficient matrix is in the similar Jordan form we have

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} k_1 + 5k_2 \\ -k_1 + k_2 \end{pmatrix}.$$

we see that this holds if and only if $y_1(0) + 5y_2(0) = 0$, because the constant k_1 and k_2 are arbitrary. The second way (which we can used without having to determine h_1) consider the general solution and the observation, again, that we can only eliminate the unbounded part of the solution if we have h_1. This means that the solution along the stable subspace is

$$y_1(t) = -5h_2 e^{-3t},\ y_2(t) = h_2 e^{-3t}$$

By eliminating $h_2 e^{-3t}$ in the two equations we have $y_1(t) = -5y_2(t)$.

To study the equation geometry we draw the **phase diagram** (see figure 3.14). Given the fact that we have a positive and a negative eigenvalue we know that it is a saddle. However, to determine their configuration in this case, we draw the following elements:

1. the isoclines, that is, the loci for $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$

$$\mathbb{I}_{y_1} = \{\ (y_1, y_2) :\ -2y_1 + 5y_2 = 0\},\ \ \mathbb{I}_{y_2} = \{\ (y_1, y_2) :\ y_1 + 2y_2 = 0\}$$

2. the eigenspaces $\mathcal{E}^1$ and $\mathcal{E}^2$;

3. the vector field;

4. as the model is linear and the vector field should show us that the stable eigenspace is coincident with the eigenspace associated to the eigenvector $\mathbf{P}^2$;

5. all the isoclines and the eigenvectors cross at the steady state $(0, 0)$;

6. if the initial point is not at the origin, then two types of paths are possible: first, if they start at $\mathcal{E}^s$ they will converge to the origin; second, if they do not start at the origin they will be parallel to $\mathcal{E}^2$ at the beginning and will converge to $\mathcal{E}^1$ asymptotically. Observe that when they cross any isocline they should change direction as regards the variable associated to the isocline. For instance, if they cross $\mathbb{I}_{y_1}$ ($\mathbb{I}_{y_2}$) they should be tangent to a vertical (horizontal) line.

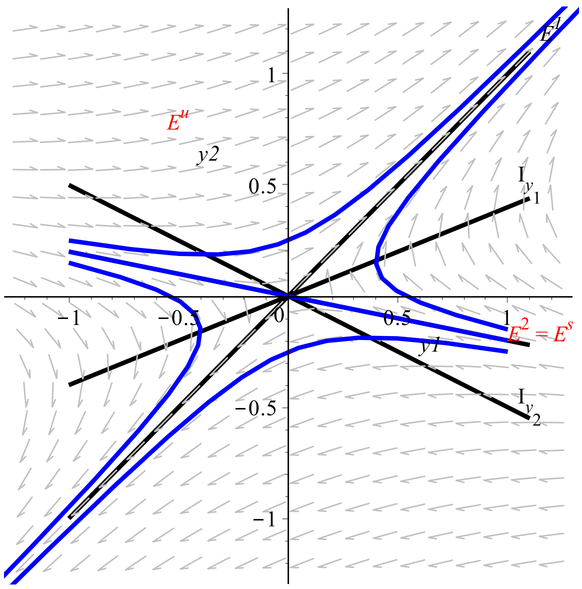

Figure 3.14: Saddle: $\dot{y}_1 = -2y_1 + 5y_2$, $\dot{y}_2 = y_1 + 2y_2$.

3.6 The non-homogeneous equation

In this section we solve the non-homogeneous equation (3.1), $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, where $\mathbf{A}$ is similar to one of the Jordan forms already presented or is equal to a new matrix

$$\mathbf{\Lambda}_4 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

It is convenient to start by addressing the existence and number of steady states, or stationary solutions.

Steady states are defined as the elements of the set

$$\overline{\mathbf{y}} \ = \{ \ \mathbf{y} \in \mathrm{Y} : \ \mathbf{A}\mathbf{y} + \mathbf{B} = 0\}$$

Again we write the eigenvector matrix associated to coefficient matrix $\mathbf{A}$

$$\mathbf{P} = \begin{pmatrix} P_1^- & P_1^+ \\ P_2^- & P_2^+ \end{pmatrix}.$$

Proposition 6. *(Existence and number of fixed points)*

1. *If $\mathbf{A}$ has no zero eigenvalues then a steady state exists and is unique and it is*

$$\bar{\mathbf{y}} = -\mathbf{A}^{-1}\mathbf{B}.$$

2. *If $\Delta(\mathbf{A}) > 0$, and $\lambda_+ = 0$, $\lambda_- < 0$, and $P_2^+ b_2 = P_1^+ b_1$ then there is an infinite number of equilibrium points belonging to a one-dimensional manifold (a line)*

$$\bar{y} \in \{\ (y_1, y_2) \in Y :\ P_1^-(\lambda_- y_2 - b_2) = P_2^-(\lambda_+ y_1 - b_1)\}.$$

3. *If $\Delta(\mathbf{A}) > 0$, and $\lambda_+ > 0$, $\lambda_- = 0$, and $P_1^- b_2 = P_2^- b_1$ then there is an infinite number of equilibrium points belonging to a one-dimensional manifold*

$$\bar{y} \in \{\ (y_1, y_2) \in Y :\ P_2^+(\lambda_+ y_1 - b_1) = P_1^+(\lambda_- y_2 - b_2)\}\ .$$

4. *If $\Delta(\mathbf{A}) = 0$, $\lambda = 0$, and $P_2^+ b_1 = P_1^+ b_2$ then there is an infinite number of equilibrium points belonging to a one-dimensional manifold*

$$\bar{y} \in \{\ (y_1, y_2) \in Y :\ P_2^-(y_1 - b_1) = P_1^-(y_2 - b_2)\}\ .$$

5. *if $\mathbf{A} = 0$ and $P_2^+ b_2 - P_1^+ b_1 = P_1^- b_2 - P_2^- b_1 = 0$ then we have an infinity of equilibrium points belonging to a two-dimensional manifold (i.e., the whole space Y).*

6. *If none of the former conditions hold there are no steady states.*

Proof. A steady state is a point $\mathbf{y}$ such that $\mathbf{A}\mathbf{y} = -\mathbf{B}$. If $\det(\mathbf{A}) \neq 0$ then a there is a unique inverse matrix $\mathbf{A}^{-1}$ and therefore a unique fixed point exits $\bar{\mathbf{y}} = -\mathbf{A}^{-1}\mathbf{B}$. If matrix $\mathbf{A}$ is singular, that is $\det(\mathbf{A}) = 0$, then a classical inverse does not exist. In this case, observe that $\mathbf{A}\mathbf{y} = -\mathbf{B}$ is equivalent to $\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}\mathbf{y} = -\mathbf{B}$ and also $\boldsymbol{\Lambda}\mathbf{P}^{-1}\mathbf{y} = -\mathbf{P}^{-1}\ \mathbf{B}$. Because in this case there only real eigenvalues, there are two forms for expanding this equation. The first form is

$$\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} P_2^+ & -P_1^+ \\ -P_2^- & P_1^- \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_2^+ & -P_1^+ \\ -P_2^- & P_1^- \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

for $\lambda_+ = 0$ and $\lambda_- \neq 0$, $\lambda_+ \neq 0$ and $\lambda_- = 0$ or $\lambda_+ = \lambda_- = 0$. Then, in the first case,

$$P_2^+ b_2 = P_1^+ b_1, \text{ and } P_1^-(\lambda_- y_2 - b_2) = P_2^-(\lambda_- y_1 - b_1)$$

in the second case

$$P_1^- b_2 = P_2^- b_1, \text{ and } P_2^+(\lambda_+ y_1 - b_1) = P_1^+(\lambda_+ y_2 - b_2)$$

in the third case, we have $P_2^+ b_2 - P_1^+ b_1 = P_1^- b_2 - P_2^- b_1 = 0$ which is a condition for existence.

The second form is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_2^+ & -P_1^+ \\ -P_2^- & P_1^- \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_2^+ & -P_1^+ \\ -P_2^- & P_1^- \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

which is equivalent to

$$P_2^+ b_1 = P_1^+ b_2, \text{ and } P_2^-(y_1 - b_1) = P_1^-(y_2 - b_2)$$

In all other cases, fixed points will not exist. □

Next we derive the general solution for the case in which there is a steady state

Proposition 7. *Consider the planar ode* (3.1)*, and assume that an equilibrium point* $\overline{\mathbf{y}} \in Y$ *exists. Then, the unique solution is*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{t}}\ \mathbf{P}^{-1}(\mathbf{h} - \overline{\mathbf{y}}) \tag{3.34}$$

where $\mathbf{h} \in Y$ *is an arbitrary element of the range of* $\mathbf{y}$*.*

Proof. Assume that a fixed point $\overline{\mathbf{y}}$ exists. Let $\mathbf{y}(t) - \overline{\mathbf{y}} = \mathbf{P}\mathbf{w}(t)$. Then $\mathbf{w}\ (t) = \mathbf{P}^{-1}(\mathbf{y}(t) - \overline{\mathbf{y}})$ and $\dot{\mathbf{w}} = \mathbf{P}^{-1}\dot{\mathbf{y}} = \mathbf{P}^{-1}(\mathbf{A}\mathbf{y} + \mathbf{B}) = \mathbf{P}^{-1}\mathbf{A}\left((\mathbf{P}\mathbf{w} + \overline{\mathbf{y}}) + \mathbf{B}\right) = \mathbf{\Lambda}\mathbf{w} + \mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{A}\overline{\mathbf{y}}\ + \mathbf{P}^{-1}\mathbf{B} = \mathbf{\Lambda}\mathbf{w} - \mathbf{P}^{-1}\mathbf{B} + \mathbf{P}^{-1}\mathbf{B} = \mathbf{\Lambda}\mathbf{w}$ for any matrix $\mathbf{\Lambda}$. Then, we get equivalently $\dot{\mathbf{w}} = \mathbf{\Lambda}\mathbf{w}$, which has solution $\mathbf{w}(t) = \mathbf{e}^{\mathbf{t}}\mathbf{k}$, where $\mathbf{k}$ is a vector of arbitrary constants. Therefore, the solution for $\mathbf{y}$ is $\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{w}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{t}}\ \mathbf{P}^{-1}(\mathbf{h} - \overline{\mathbf{y}})$ where $\mathbf{h}$ is a vector of arbitrary constants, in the units of $\mathbf{y}$. □

The solution (3.34) can be written as

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{t}}\ \mathbf{k}$$

where

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{\det(\mathbf{P})} \begin{pmatrix} P_2^+ & -P_1^+ \\ -P_2^- & P_1^- \end{pmatrix} \begin{pmatrix} h_1 - \overline{y}_1 \\ h_2 - \overline{y}_2 \end{pmatrix}.$$

Then, recalling what we have learned from solving equation (**??**), it can take one of the following three forms

1. if $\mathbf{\Lambda} = \mathbf{\Lambda}_1$, the general solution is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + k_1 e^{\lambda_+ t}\mathbf{P}^1 + k_2 e^{\lambda_- t}\mathbf{P}^2$$

   or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + k_1 e^{\lambda_+ t} \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} + k_2 e^{\lambda_- t} \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix}$$

2. if $\mathbf{\Lambda} = \mathbf{\Lambda}_2$, the general solution is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + e^{\lambda t}\left(\mathbf{P}^1(k_1 + k_2 t) + k_2\mathbf{P}^2\right)$$

or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + e^{\lambda t}\left((k_1 + k_2 t)\begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} + k_2\begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix}\right)$$

3. if $\Lambda = \Lambda_3$, the general solution is

$$\begin{aligned} \mathbf{y}(t) &= \overline{\mathbf{y}} + e^{\alpha t}\left((k_1 \cos\beta t + k_2 \sin\beta t)\mathbf{P}^1 + (k_2 \cos\beta t - k_1 \sin\beta t)\mathbf{P}^2\right) = \\ &= \overline{y} + e^{\alpha t}\left(k_1(\cos\beta t\mathbf{P}^1 - \sin\beta t\mathbf{P}^2) + k_2(\sin\beta t\mathbf{P}^1 + \cos\beta t\mathbf{P}^2)\right). \end{aligned}$$

or, equivalently,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + e^{\alpha t}\left(k_1\begin{pmatrix} P_1^- \cos\beta t - P_1^+ \sin\beta t \\ P_2^- \cos\beta t - P_2^+ \sin\beta t \end{pmatrix} + k_2\begin{pmatrix} P_1^- \sin\beta t + P_1^+ \cos\beta t \\ P_2^- \sin\beta t + P_2^+ \cos\beta t \end{pmatrix}\right).$$

Eigenspaces and stability analysis Let $\mathbf{\Lambda} = \mathbf{\Lambda}_1$. We can determine again the eigenspaces, spanned by eigenvectors $\mathbf{P}^1$ and $\mathbf{P}^2$, by making $k_2 = 0$ and $k_1 = 0$, respectively. Then [6]

$$\mathcal{E}^1 = \{\mathbf{y} \in \mathrm{Y} : \ P_1^-(y_2 - \overline{y}_2) = P_2^-(y_1 - \overline{y}_1)\}$$

and

$$\mathcal{E}^2 = \{\mathbf{y} \in \mathrm{Y} : \ P_1^+(y_2 - \overline{y}_2) = P_2^+(y_1 - \overline{y}_1)\}$$

We can also partition the state space according to the stability properties of the solutions belonging to them. We define the **stable eigenspace** as

$$\mathcal{E}^s = \{\mathbf{h} \neq \overline{\mathbf{y}} \in \mathrm{Y} : \ \lim_{t\to\infty} \mathbf{y}(t, \mathbf{h}) = \overline{\mathbf{y}}\}$$

we define the **unstable eigenspace** as

$$\mathcal{E}^u = \{\mathbf{h} \neq \overline{\mathbf{y}} \in \mathrm{Y} : \ \lim_{t\to-\infty} \mathbf{y}(t, \mathbf{h}) = \overline{\mathbf{y}}\}$$

and the **center eigenspace** as

$$\mathcal{E}^c = \{\mathbf{h} \in \mathrm{Y} : \ \mathbf{y}(t, \mathbf{h}) = \text{const }\}$$

if there is at least one eigenvalue with zero real part.

If all eigenvalues have negative real parts then we have **asymptotic stability**, and $\mathcal{E}^s = \mathcal{E}^1 \oplus \mathcal{E}^2 = \mathrm{Y}$, and $\mathcal{E}^u$ and $\mathcal{E}^c$ are empty. If all eigenvalues have positive real parts then we have **instability** and $\mathcal{E}^u = \mathcal{E}^1 \oplus \mathcal{E}^2 = \mathrm{Y}$, and $\mathcal{E}^s$ and $\mathcal{E}^c$ are empty. If there is one negative and one positive eigenvalue then we have a **saddle point** and $\mathcal{E}^s = \mathcal{E}^2$ and $\mathcal{E}^u = \mathrm{Y}/\mathcal{E}^2$.

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[6] If we set $k_2 = 0$ we have $h_1 e^{\lambda_+ t} P_1^1 = y_1(t) - \overline{y}_1$ and $h_1 e^{\lambda_+ t} P_2^1 = y_2(t) - \overline{y}_2$. Thus $h_1 e^{\lambda_+ t} = \dfrac{y_1(t) - \overline{y}_1}{P_1^1} = \dfrac{y_2(t) - \overline{y}_2}{P_2^1}$. We proceed in an analogous way for $\mathcal{E}^2$.

Changes in the phase diagrams Next we extend the case in Example 2 to show that adding a vector **B** only changes the value of the steady state but not its stability properties, as regards the associated homogeneous case.

Example 3 Consider the ODE, where $y \in \mathbb{R}^2$, which is slightly modification of equation (3.32):

$$\begin{aligned}\dot{y}_1 &= -2y_1 + 5y_2 - 1/5,\\ \dot{y}_2 &= y_1 + 2y_2 - 4/5.\end{aligned} \tag{3.35}$$

This is a non-homogenous equation of type $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, where matrix $\mathbf{A}$ is as in example (3.32). As $\text{trace}(\mathbf{A}) = 0$ and $\det(A) = -9$ then the eigenvalues are $\lambda \in \{-3, 3\}$. The steady state is

$$\bar{y} = -A^{-1}B = \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix}.$$

In this case the general solution is

$$y(t) = \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} + h_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + h_2 \begin{pmatrix} -5 \\ 1 \end{pmatrix} e^{-3t}. \tag{3.36}$$

where

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} k_1 - \bar{y}_1 \\ k_2 - \bar{y}_2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} k_1 + 5h_2 - 7/5 \\ -k_1 + k_2 + 1/5 \end{pmatrix}.$$

Therefore, the eigenspaces are

$$\mathcal{E}^1 = \{(y_1, y_2) : y_1 + 5y_2 - 7/5 = 0\},\ \mathcal{E}^2 = \{(y_1, y_2) : -y_1 + y_2 + 1/5 = 0\}$$

The fixed point is again a saddle point and the stable eigenspace is again $\mathcal{E}^s = \mathcal{E}^1$

The phase diagram is in figure 3.15. If we compare with figure 3.14 we see that they have the same shape (i.e, the isoclines and the eigenspaces have the same slopes) with the fixed point shifted from the origin to the new steady state $\bar{y} = (2/5, 1/5)^\top$.

Comparing examples 1, 2, and 3, with phase diagrams in 3.5, 3.14 and 3.15 lead to the following observations:

1. as we already saw, when the equation is homogeneous, i.e., $\mathbf{B} = \mathbf{0}$, but matrix $\mathbf{A}$ is not in a Jordan normal form, the steady state is still in the origin but the isoclines and the eigenspaces are rotated (compare figures 3.5 and 3.14);

2. when the equation is non-homogeneous, i.e., vector $\mathbf{B} \neq \mathbf{0}$, the steady state is shifted out of the origin but the isoclines and the eigenspaces are the same as for the similar homogeneous equation (compare figures 3.14 and 3.15).

Next we present a case in which we have a **stable node** and show that **hump-shaped** trajectories can occur for a non-homogeneous ODE.

Example 4 Consider the ODE, where $y \in \mathbb{R}^2$:

$$\begin{aligned} \dot{y}_1 &= -2y_1 + y_2 + 1/5, \\ \dot{y}_2 &= y_1 - 2y_2 + 4/5. \end{aligned} \tag{3.37}$$

Prove that the solution for the initial value problem, for any $\mathbf{y}(0) = (y_1(0), y_2(0))$ is

$$y(t) = \begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \end{pmatrix} + \frac{1}{2}\left(-y_1(0) + y_2(0) - \frac{1}{5}\right)\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3\,t} + \frac{1}{2}\left(y_1(0) + y_2(0) - 1\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\,t}$$

The phase diagram is in figure 3.16. We see that it is a stable node. In addition, the unstable and the centre subspaces are empty and the stable subspace is the whole set minus the fixed point $\bar{y} = (\frac{2}{5}, \frac{3}{5})^\top$. In addition observe that the trajectories at $t = 0$ tend to be parallel to the eigenspace associated to the negative eigenvalue larger in absolute value $\mathcal{E}^1 = \{(y_1, y_2) :\ y_1 + y_2 = 0\}$ and they become asymptoticaly tangent to the eigenspace associated to the negative eigenvalue smaller in absolute value $\mathcal{E}^2 = \{(y_1, y_2) :\ y_1 - y_2 = 0\}$. This means that are trajectories that cross the isoclines and which are, therefore, non-monotonous.

Non-monotonous trajectories The previous example displays another difference between homogeneous and similar non-homogeneous equations (see phase diagram in Figure 3.15). Consider a planar ODE (homogeneous or not) in which the coefficient matrix $\mathbf{A}$ is such that there is a unique steady state which is a stable nodes (i.e., there are two real eigenvalues with negative and distinct real parts). Now compare the cases with similar matrices for the case in $\mathbf{A}$ is a Jordan form, as in figure 3.2, and it is not, as in figure 3.16). We observe that in the last case we see that there are **hump-shaped trajectories**, that is, trajectories that converge to the steady state after crossing an isocline, but affecting only one variable which.

This also allows for a partition of space Y. The isoclines $\mathbb{I}_{y_1}$ and $\mathbb{I}_{y_2}$ allow for a partition of Y into for subsets, say

$$\begin{aligned} \mathrm{Y}^{++} &= \{\ \mathbf{y} \in \mathrm{Y} : \dot{y}_1 > 0,\ \dot{y}_2 > 0\} \\ \mathrm{Y}^{-+} &= \{\ \mathbf{y} \in \mathrm{Y} : \dot{y}_1 < 0,\ \dot{y}_2 > 0\} \\ \mathrm{Y}^{+-} &= \{\ \mathbf{y} \in \mathrm{Y} : \dot{y}_1 > 0,\ \dot{y}_2 < 0\} \\ \mathrm{Y}^{--} &= \{\ \mathbf{y} \in \mathrm{Y} : \dot{y}_1 < 0,\ \dot{y}_2 < 0\} \end{aligned}$$

As we saw, for stable nodes, the solution path will be asymptotically attracted to the direction defined by the eigenspace associated to the smaller eigenvalue in absolute value. In our case it is $\mathcal{E}^1$. This eigenspace will be contained in the union of two of the subsets defined by the isoclines. It can be proved that if $\mathbf{h}$ does not belong to union of those subsets then one of the variables will be hump-shaped.

For example, let $\mathcal{E}^1 \subset \mathrm{Y}^{++} \cup \mathrm{Y}^{--}$. If $\mathbf{h} \in \mathrm{Y}^{-+} \cup \mathrm{Y}^{+-}$ then one of the solution paths will be hump-shaped, depending which isocline is crossed: if it crosses $\mathbb{I}_{y_1}$ then $y_1(t)$ will be hump-shaped and $y_2(t)$ will be monotonous and if it crosses $\mathbb{I}_{y_2}$ then $y_2(t)$ will be hump-shaped and $y_1(t)$ will be monotonous

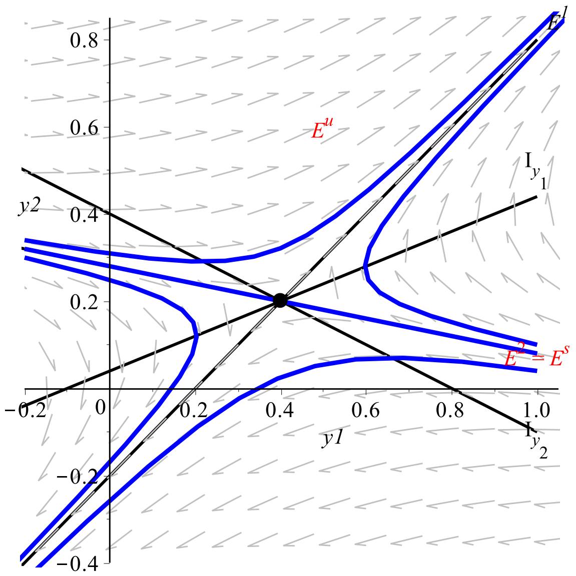

Figure 3.15: Saddle: $\dot{y}_1 = -2y_1 + 5y_2 - 0.2$, $\dot{y}_2 = y_1 + 2y_2 - 0.8$.

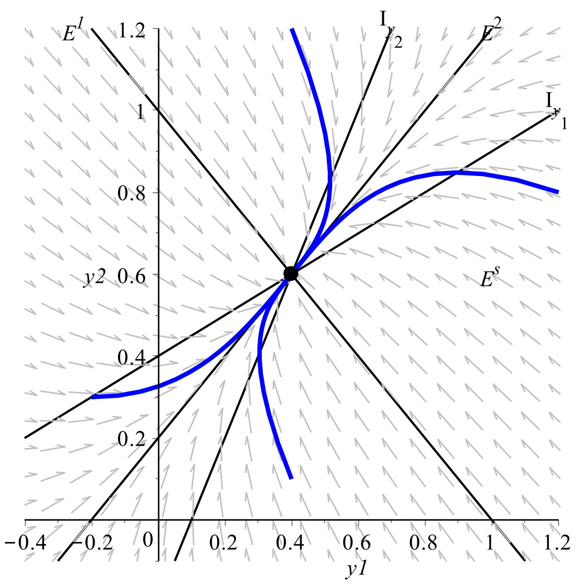

Figure 3.16: A sink or stable node: $\dot{y}_1 = -2y_1 + y_2 + 0.2$, $\dot{y}_2 = y_1 - 2y_2 + 0.8$.

Summing up, we may have three types of trajectories, independently from the uniqueness and stability properties of steady states:

1. **monotonous trajectories** both stable and unstable: if the steady state is a saddle point, or a node and the coefficient matrix $\mathbf{A}$ is in the Jordan form or if it is not the arbitrary constant does not involve trajectories crossing isoclines;

2. **oscillatory trajectories** both stable and unstable: when there is a focus

3. **hump-shaped trajectories**: when there is a node, the coefficient matrix is not in the Jordan form and trajectories cross isoclines.

3.7 Main result on stability theory

The dynamic behavior of the solution for equation (3.1) is similar to that of equation (**??**), but relative to a fixed point which is not necessarily coincident with the origin .

Theorem 1. *Consider the planar ODE* (3.1). *Assume that a fixed point* $\overline{\mathbf{y}} \in Y$ *exists if* $\det(\mathbf{A}) \neq 0$ *or that an infinite number of fixed points exist if* $\det(\mathbf{A}) = 0$. *The asymptotic properties of the solution as a function of the trace and determinant of* $\mathbf{A}$ *are:*

1. *asymptotic stability if and only if trace*$(\mathbf{A}) < 0$ *and* $\det(\mathbf{A}) \geq 0$*;*

2. *saddle path (or conditional) stability if and only if* $\det(\mathbf{A}) < 0$*;*

3. *instability if and only if trace*$(\mathbf{A}) > 0$ *and* $\det(\mathbf{A}) \geq 0$*;*

4. *stability but not asymptotic stability if trace*$(\mathbf{A}) = 0$ *and* $\det(\mathbf{A}) \geq 0$.

In figure 3.17 we present a bifurcation diagram where the phase diagrams associated to the different values of the trace and determinant of $\mathbf{A}$ ar epresented

3.8 Problems involving planar ODE's

As we saw all the solutions involve a vector of arbitrary elements of Y, $\mathbf{k}$ or $\mathbf{h}$. This means that we have existence but not uniqueness for **general** solutions.

In applications we introduce further information on the system. The type of **problem** involving planar ODE's depends on this additional information. We can define the following types of problems:

- if we know the initial point $\mathbf{y}(0) = \mathbf{y}_0 = (y_{1,0}, y_{2,0})$ and want to solve the problem forward in time, we say we have an **initial-value problem**;

- if we know the value of at least one variable at a point in time $T > 0$, $\mathbf{y}(T) = \mathbf{y}_T$, or $y_1(T) = y_{1,T}$, $y_2(T) = y_{2,T}$, we say we have a **boundary-value problem**;

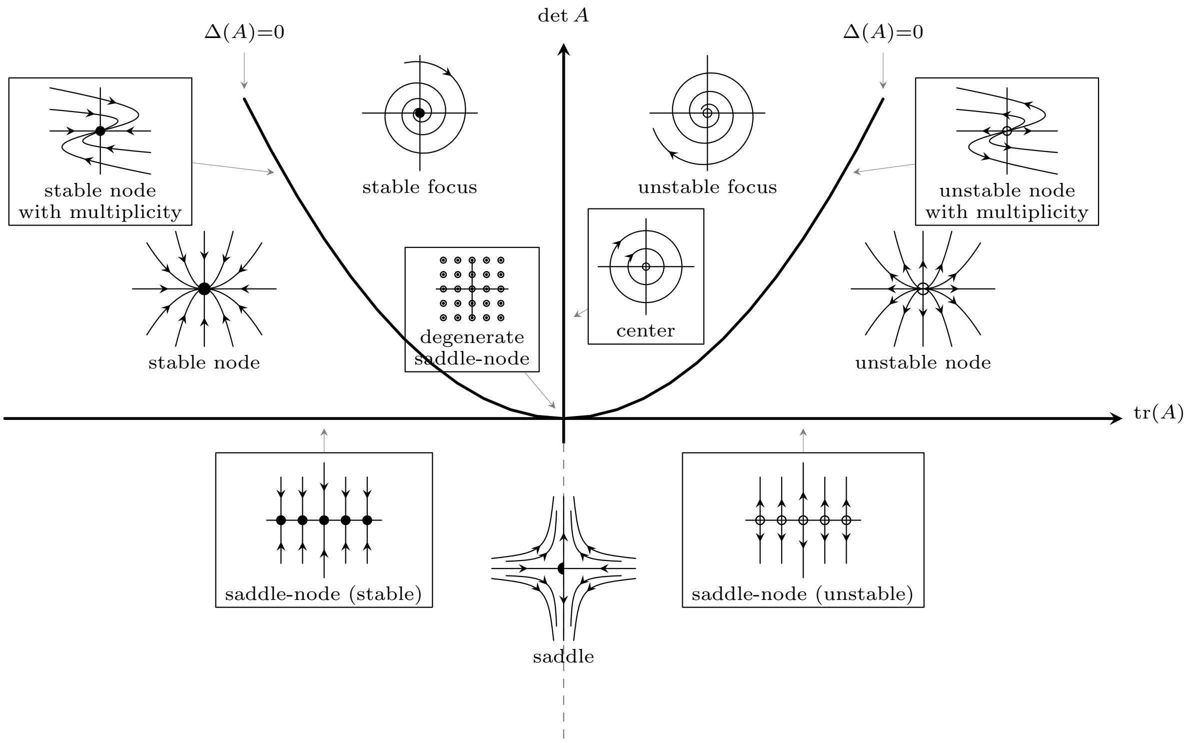

Figure 3.17: Bifurcation diagram in the $(\text{trace}A, \det A)$ space

- in economics a common problem is a mixes initial-terminal value problem, where we know the initial value for one variable and a boundary condition for the asymptotic value of another. Example: $y_1(0) = y_{1,0}$ and $\lim_{t\to\infty} e^{-\mu t} y_2(t) = 0$, where μ is a non-negative constant.

When the initial, boundary or terminal conditions are imposed we say we have **particular** solutions. Off course, the issues of existence, uniqueness and characterization still hold.

In economics it has been standard to refer to problems having an unique solution as **determinate** and to problems having multiple solutions as **indeterminate**.

3.8.1 Initial-value problems

Proposition 8. *Let* $\mathbf{y}(0) = \mathbf{y}_0$ *then the solution for the initial-value problem is unique*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}t}\mathbf{P}^{-1}(\mathbf{y}_0 - \overline{\mathbf{y}})$$

Proof. The general solution for a planar non-homogeneous equation is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}t}\ \mathbf{k}.$$

As $\mathbf{e}^{\mathbf{t}}\ |_{t=0} = \mathbf{I}$ then evaluating the solution at time $t = 0$, we have

$$\mathbf{y}(0) - \overline{\mathbf{y}} = \mathbf{P}\mathbf{k}$$

then, because $\mathbf{P}$ is non-singular

$$\mathbf{k} = \mathbf{P}^{-1}(\mathbf{y}(0) - \overline{\mathbf{y}})$$

Plugging the initial condition we have a particular value for $\mathbf{k}$

$$\mathbf{h} = \mathbf{P}^{-1}(\mathbf{y}_0 - \overline{\mathbf{y}}).$$

□

3.8.2 Terminal value problems

Proposition 9. *Consider the problem defined by planar non-homogeneous equation and the limiting constraint*

$$\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}} \in Y.$$

Then:

(1) if $\overline{\mathbf{y}}$ *is a stable node or a stable focus then the solution is indeterminate*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}t}\ \mathbf{k}$$

for any $\mathbf{k} = \mathbf{P}^{-1}(\mathbf{h} - \overline{\mathbf{y}}$ *with* $\mathbf{h} \in Y$*;*

(2) if $\overline{\mathbf{y}}$ is an unstable node or an unstable focus then the solution is determinate

$$\mathbf{y}(t) = \overline{\mathbf{y}}, \text{ for all } t \in T$$

(3) if $\overline{\mathbf{y}}$ is a saddle-point then the solution is indeterminate

$$\mathbf{y}(t) = \overline{\mathbf{y}} + k_2 \mathbf{P}^2 e^{\lambda_- t}.$$

Proof. (1) If all the eigenvalues of $\mathbf{A}$ have negative real parts then

$$\lim_{t\to\infty} \mathbf{e}^{\mathbf{\Lambda} t} = \mathbf{I}_{2\times 2}$$

which implies $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{y}$ independently of the value of $\mathbf{h}$. (2) if all the eigenvalues of $\mathbf{A}$ have positive real parts then all the exponential functions $e^{\lambda_+ t}$, $e^{\lambda_- t}$, $e^{\lambda t}$ or $e^{\alpha t}$ become unbounded, which means that we can only have $\lim_{t\to\infty} \ \mathbf{P}\mathbf{e}^{\mathbf{t}}\ \mathbf{k} = \mathbf{0}$ if and only if $\mathbf{k} = \mathbf{0}$. Then as $\mathbf{k}$ is uniquely determined, the solution is unique. (3) If the steady state is a saddle point we know that the Jacobian form of $\mathbf{A}$ is $\mathbf{\Lambda}_1$, the solution takes the form

$$\mathbf{y}(t) = \overline{\mathbf{y}} + k_1 \mathbf{P}^1 e^{\lambda_+ t} + k_2 \mathbf{P}^2 e^{\lambda_- t}$$

and $\lim_{t\to\infty} e^{\lambda_+ t} = +\infty$ and $\lim_{t\to\infty} e^{\lambda_- t} = 0$. Therefore $\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}}$ if and only if $k_1 = 0$, and the solution is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + k_2 \mathbf{P}^2 e^{\lambda_- t}.$$

□

3.8.3 Initial-terminal value problems

Proposition 10. *Consider the problem defined by planar non-homogeneous equation in which the steady state is a saddle point, the limiting constraint*

$$\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}}$$

and the initial value $y_1(0) = y_{10}$ hold. Then the solution exists and is unique

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \frac{(y_{1,0} - \overline{y}_1)}{P_1^2} \mathbf{P}^2 e^{\lambda_- t}.$$

Proof. We can take the solution of case (3) of the terminal-valure problem and evaluate it at time $t = 0$ to get

$$\mathbf{y}(0) = \overline{\mathbf{y}} + k_2 \mathbf{P}^2 \Leftrightarrow k_2 \mathbf{P}^2 + \overline{\mathbf{y}} - \mathbf{y}(0) = \mathbf{0},$$

or, expanding and substituting the initial condition

$$\begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix} k_2 + \begin{pmatrix} \overline{y}_1 - y_{1,0} \\ \overline{y}_2 - y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As we want to solve this system for for $y_2(0) - \overline{y}_2$ and k_2 it is convenient to re-arrange it as

$$\begin{pmatrix} P_1^2 & 0 \\ P_2^2 & 1 \end{pmatrix} \begin{pmatrix} k_2 \\ \overline{y}_2 - y_2(0) \end{pmatrix} = \begin{pmatrix} y_{1,0} - \overline{y}_1 \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} k_2 \\ \overline{y}_2 - y_2(0) \end{pmatrix} &= \begin{pmatrix} P_1^2 & 0 \\ P_2^2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1(0) - \overline{y}_1 \\ 0 \end{pmatrix} = \\ &= \frac{1}{P_1^2} \begin{pmatrix} 1 & 0 \\ -P_2^2 & P_1^2 \end{pmatrix} \begin{pmatrix} y_{1,0} - \overline{y}_1 \\ 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 \\ -P_2^2 \end{pmatrix} \frac{(y_{1,0} - \overline{y}_1)}{P_1^2}. \end{aligned}$$

□

In this case the initial value for $y_2(0)$ is determined

$$y_2(0) = \overline{y}_2 + \frac{P_2^2}{P_1^2}(y_{1,0} - \overline{y}_1)$$

where $\dfrac{P_2^2}{P_1^2}$ is the slope of $\mathcal{E}^2$ which is co-incident with the stable eigenspace $\mathcal{E}^s$.

Sometimes if we assume we know the initial value for variable y_2, $y_2(0) = y_{2,0}$ the difference $y_{2,0} - \left(\overline{y}_2 + \dfrac{P_2^2}{P_1^2}(y_{1,0} - \overline{y}_1) \right)$ is interpreted as the initial "jump" to the saddle path.

3.9 Applications in Economics

Types of variables and types of problems

1. macroeconomics pre rational expectations models: are usually initial-value problems in which the dynamic system is a for stable node or stable focus;

2. post rational expectations and DGE (dynamic general equilibrium) models: are usually initial-terminal value problems in which the dynamic system is a saddle point. This structure allows for the both forward (pre-determined) and backward (non-predetermined or expected) dynamics and for existence and uniqueness of DGE paths;

3. neo-Keynesian DGE models: are interested in cases in which for initial-terminal value problem in which the dynamic system can be a stable node or stable focus. This structure allows for

the both forward (pre-determined) and backward (non-predetermined or expected) dynamics, for the existence of DGE paths but non necessarily for their uniqueness. If DGE paths are not unique the dynamics is said to be indeterminate, meaning that self-fulfilling prophecies are possible, and these are related with the existence of imperfections in the markets (externalities, incompleteness of contracts, policy rules, etc);

4. growth theory models: are usually initial or initial-terminal value problems in which there are no positively valued steady states or steady states are a degenerate node (with a zero and a positive eigenvalue). Two-dimensional endogenous growth models usually feature dynamic systems with a zero and a positive real eigenvalue which is associated with the existence of a balanced-growth path.

3.10 Bibliographic references

Mathematical textbooks: Hirsch and Smale (1974), (Hale and Koçak, 1991, ch 8) and Perko (1996)

Economics textbooks: on dynamical systems applied to economics (Gandolfo (1997), Tu (1994)), general mathematical economics textbooks with chapters on dynamic systems (Simon and Blume, 1994, ch. 24,25), de la Fuente (2000).

3.A Appendix

3.A.1 Review of matrix algebra

Consider matrix $\mathbf{A}$ of order 2 with real entries

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

that is $\mathbf{A} \in \mathbb{R}^{2\times 2}$. The **trace** and the **determinant** of $\mathbf{A}$ are, respectively,

$$\text{trace}(\mathbf{A}) = a_{11} + a_{22},\ \det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

The kernel (or null space) of matrix $\mathbf{A}$ is a vector $\mathbf{v}$ defined as

$$\text{kern}(\mathbf{A}) = \{\ \mathbf{v} : \mathbf{A}\mathbf{v} = \mathbf{0}\}$$

The dimension of the kernel gives a measure of the linear independence between the rows of $\mathbf{A}$.

The characteristic polynomial of matrix $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda \mathbf{I}_2) = \lambda^2 - trace(\mathbf{A})\lambda + \det(\mathbf{A}) \tag{3.38}$$

where $\lambda \in \mathbb{C}$ is an eigenvalue, which is complex valued.

The **spectrum** of $\mathbf{A}$ is the **set of eigenvalues**

$$\sigma(\mathbf{A}) \equiv \{\ \ \lambda \in \mathbb{C} : \det(\mathbf{A} - \lambda \mathbf{I}_2)\ \ = 0\}$$

The **eigenvalues** of any 2×2 matrix $\mathbf{A}$ are

$$\lambda_+ = \frac{\text{trace}(\mathbf{A})}{2} + \Delta(\mathbf{A})^{\frac{1}{2}},\ \lambda_- = \frac{\text{trace}(\mathbf{A})}{2} - \Delta(\mathbf{A})^{\frac{1}{2}} \tag{3.39}$$

where the discriminant is

$$\Delta(\mathbf{A}) \equiv \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A}).$$

A useful result on the relationship between the eigenvalues and the trace and the determinant of $\mathbf{A}$:

Lemma 8. *Let λ_+ and λ_- be the eigenvalues of a 2×2 matrix $\mathbf{A}$. Then they are verify:*

$$\begin{aligned} \lambda_+ + \lambda_- &= trace(\mathbf{A}) \\ \lambda_+ \lambda_- &= \det(\mathbf{A}). \end{aligned}$$

Three cases can occur:

1. if $\Delta(\mathbf{A}) > 0$ then λ_+ and λ_- are real and distinct and $\lambda_+ > \lambda_-$
2. if $\Delta(\mathbf{A}) = 0$ then $\lambda_+ = \lambda_- = \lambda = \text{trace}(\mathbf{A})/2$ are real and multiple,

3. if $\Delta(\mathbf{A}) < 0$ then λ_+ and λ_- are complex conjugate $\lambda_+ = \alpha + \beta i$ and $\lambda_- = \alpha - \beta i$ where $\alpha = \frac{tr(A)}{2}$ and $\beta = \sqrt{|\Delta(\mathbf{A})|}$ and $i = \sqrt{-1}$.

In the last case, we can write the eigenvalues in polar coordinates as

$$\lambda_+ = r(\cos\theta + \sin\theta i), \ \lambda_- = r(\cos\theta - \sin\theta i)$$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\tan\theta = \beta/\alpha$, or

$$\alpha = r\cos\theta, \ \beta = r\sin\theta$$

Jordan canonical forms Two matrices $\mathbf{A}$ and $\mathbf{A}'$ with the equal eigenvalues are called **similar**. This allows for classifying matrices according to their eigenvalues.

The Jordan canonical forms for 2×2 matrices are

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \ \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \ \mathbf{\Lambda}_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{3.40}$$

Lemma 9 (Jordan canonical from of matrix $\mathbf{A}$)**.** *Consider any* 2×2 *matrix with real entries and its discriminant* $\Delta(\mathbf{A})$*. Then*

1. *If* $\Delta(\mathbf{A}) > 0$ *then the Jordan canonical form associated to* $\mathbf{A}$ *is* $\mathbf{\Lambda}_1$*.*

2. *If* $\Delta(\mathbf{A}) = 0$ *then the Jordan canonical form associated to* $\mathbf{A}$ *is* $\mathbf{\Lambda}_2$*.*

3. *If* $\Delta(\mathbf{A}) < 0$ *then the Jordan canonical form associated to* $\mathbf{A}$ *is* $\mathbf{\Lambda}_3$*.*

The Jordan canonical form $\mathbf{\Lambda}_3$ can also be represented by a diagonal matrix with complex entries

$$\mathbf{\Lambda}_3 = \begin{pmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{pmatrix}.$$

In this sense, if $\Delta(\mathbf{A}) \neq 0$ then matrix $\mathbf{A}$ is diagonalizable and it is not diagonalizable if $\Delta(\mathbf{A}) = 0$.

Figure 3.18 presents the different cases in a $(\text{trace}(\mathbf{A}), \det(\mathbf{A}))$ diagram. It has the following information:

- Jordan canonical forms are associated to the following areas: $\mathbf{\Lambda}_1$ is outside the parabola; $\mathbf{\Lambda}_3$ is inside the parabola, and $\mathbf{\Lambda}_2$ is represented by the parabola;

- in the positive orthant the two eigenvalues have positive real parts, in the negative orthant they have negative real parts and bellow the abcissa there are two real eigenvalues with opposite signs;

- the abcissa corresponds to the locus of points in which there is at least one zero-valued eigenvalue, the upper part of the ordinate corresponds to complex eigenvalues with zero real part, and the origin to the case in which there are two eigenvalues equal to zero.

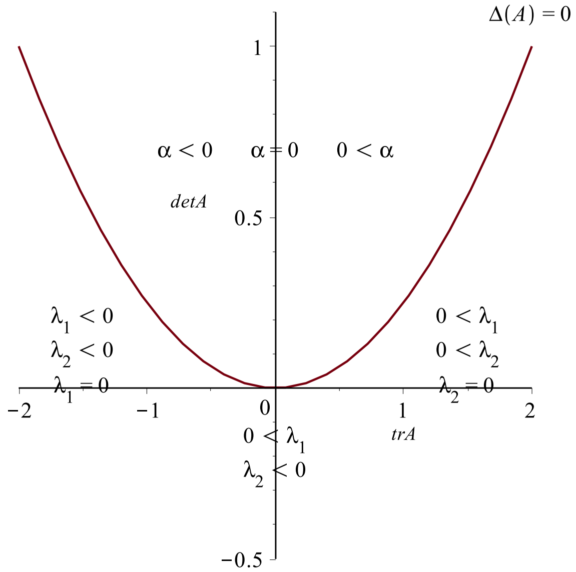

Figure 3.18: Eigenvalues of $\mathbf{A}$ in the trace-determinant space:

Eigenvectors of A

Lemma 10. *Let* $\mathbf{A}$ *be a* 2×2 *matrix with real entries. Then, there exists a non-singular matrix* $\mathbf{P}$ *such that*

$$\mathbf{A} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}$$

where $\boldsymbol{\Lambda}$ *is the Jordan canonical form of* $\mathbf{A}$*, and matrix* $\mathbf{P}$ *is a* 2×2 ***eigenvector*** *matrix associated to* $\mathbf{A}$*.*

There are two types of eigenvectors:

1. **simple eigenvectors** if $\Delta(\mathbf{A}) \neq 0$. In this case the eigenvector is $\mathbf{P} = (\mathbf{P}^-, \mathbf{P}^+)$ concatenating the eigenvectors $\mathbf{P}^-$ and $\mathbf{P}^+$ associated to the eigenvalues λ_+ and λ_-, which are obtained from solving the homogeneous system

$$(\mathbf{A} - \lambda_j \mathbf{I}_2)\mathbf{P}^j = 0, \; j = 1, 2$$

   where $\mathbf{I}_2$ is the identity matrix of order 2. Observe that $\mathbf{P}^j = \text{kern}(\mathbf{A} - \lambda_j \mathbf{I}_2)$, i.e, it is the null space of matrix $(\mathbf{A} - \lambda_j \mathbf{I}_2)$;

2. **generalized eigenvectors** if $\Delta(\mathbf{A}) = 0$, that is, when we have multiple eigenvalues $\lambda_+ = \lambda_- = \lambda$. In this case we determine $\mathbf{P} = (\mathbf{P}^1, \mathbf{P}^2)$ where $\mathbf{P}^1$ is a simple eigenvalue and $\mathbf{P}^2$ is a generalized eigenvalue. They are obtained in the following way: first, $\mathbf{P}^1$ solves $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P}^1 = 0$, where $\mathbf{I} = \mathbf{I}_2$; second, (a) if $(\mathbf{A} - \lambda \mathbf{I})^2 \neq \mathbf{0}$ we determine $\mathbf{P}^2$ from $(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{P}^2 = 0$; however, (b) if $(\mathbf{A} - \lambda \mathbf{I})^2 = \mathbf{0}$ then we determine $\mathbf{P}^2$ from $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P}^2 = \mathbf{P}^1$.

When $\Delta(\mathbf{A}) < 0$ we can use one of the following two approaches:

1. either we write the Jordan matrix as a complex-valued matrix

$$\Lambda_3 = \begin{pmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{pmatrix}$$

   and compute $\mathbf{P}^j$ as a complex-valued vector from

$$(\mathbf{A} - \lambda_j \mathbf{I}_2)\mathbf{P}^j = 0,$$

2. or we write the Jordan matrix as a real-valued matrix as in equation (3.40) and compute $\mathbf{P}$ as a real-valued matrix by setting $\mathbf{P} = (\mathbf{u}, \mathbf{v})$ where $\mathbf{Q} = \mathbf{u} + \mathbf{v}i$ is the solution of the homogeneous system

$$(\mathbf{A} - (\alpha + \beta i)\mathbf{I}_2)\mathbf{Q} = 0$$

Conclusion: given a matrix $\mathbf{A}$, we can find matrices $\mathbf{\Lambda}$ and $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\Lambda\mathbf{P}^{-1}$ where $\mathbf{P}$ is invertible. Equivalently $\Lambda = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Proposition 11. *The eigenvector matrices associated to the Jordan canonical forms are:*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{3.41}$$

for $\Lambda = \mathbf{\Lambda}_1$, $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ *and* $\mathbf{\Lambda} = \mathbf{\Lambda}_3$, *respectively*

Proof. For $\mathbf{\Lambda} = \mathbf{\Lambda}_1$, because $(\mathbf{\Lambda}_1 - \lambda_+ \mathbf{I})\mathbf{P}^- = 0$ and $(\mathbf{\Lambda}_1 - \lambda\_\mathbf{I})\mathbf{P}^+ = 0$ are

$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda_- - \lambda_+ \end{pmatrix} \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda_+ - \lambda_- & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then we get $\mathbf{P} = (\mathbf{P}^- \mathbf{P}^+) = \mathbf{I}$, because $\lambda_+ \neq \lambda_-$. For $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ we determine the simple eigenvector from $(\mathbf{\Lambda}_2 - \lambda\mathbf{I})\mathbf{P}^- = 0$. To determine the second eigenvector as $(\lambda_- - \lambda\mathbf{I})^2 = \mathbf{0}$, because

$$(\lambda_- - \lambda\mathbf{I})^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then we use $(\mathbf{\Lambda}_2 - \lambda\mathbf{I})\mathbf{P}^2 = \mathbf{P}^1$,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} = \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix},$$

to get $\mathbf{P}^1 = (1, 0)$ and $\mathbf{P}^2 = (1, 1)$.

For $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ consider eigenvalue $\lambda_+ = \alpha + \beta i$ and assume that there is a complex vector

$$\mathbf{z} = \begin{pmatrix} u_1 + v_1 i \\ u_2 + v_2 i \end{pmatrix}$$

that solves $(\Lambda_3 - (\alpha + \beta i)I)\mathbf{z} = 0$, that is [7]

$$\begin{cases} \beta\,(u_2 + v_1 + (v_2 - u_1)i) & = 0 \\ \beta\,((v_2 - u_1) - (u_2 + v_1)i) & = 0 \end{cases}$$

then we should have $u_1 = v_2$ and $u_2 = -v_1$. We can arbitrarily set $u_1 = 1$ and $v_1 = 1$, in $\mathbf{P}^1 = (u_1, u_2)^\top$ and $\mathbf{P}_2 = (v_1, v_2)^\top$, to get the third eigenvector matrix. □

Eigenspaces As matrix $\mathbf{P}$ is non singular it forms a basis for vector space $\mathbf{A}$. Then vector space $\mathbf{A}$ can be seen as a direct sum $\mathbf{A} = \mathcal{E}^1 \oplus \mathcal{E}^2$ where

$$\begin{aligned} \mathcal{E}^1 &= \{\text{eigenspace associated with } \lambda_+\} \\ \mathcal{E}^2 &= \{\text{ eigenspace associated with } \lambda\_\}. \end{aligned}$$

3.A.2 Polar coordinates

When the eigenvalues are complex (or the model is non-linear) sometimes we can simplify the solution and get a better geometrical intuition of it, if we transform the ODE from cartesian coordinates $(y_1, y_2) \in \mathbb{R}$ into polar coordinates (r, θ) by using the transformation:

$$y_1(t) = r(t)\cos(\theta(t)),\ y_2(t) = r(t)\sin(\theta(t)).$$

where r measures the distance from a reference point (the radius) and θ the angular coordinate.

The following relationships hold $r^2 = y_1^2 + y_2^2$, because $\cos(\theta)^2 + \sin(\theta)^2 = 1$ and $\tan(\theta) = \sin(\theta)/\cos(\theta) = y_2/y_1$. If we take time derivatives of this two relationships we find

$$\begin{aligned} r' &= \frac{y_1\dot{y}_1 + y_2\dot{y}_2}{r} \\ \theta' &= \frac{y_1\dot{y}_2 - y_2\dot{y}_1}{r^2} \end{aligned}$$

Exercise: provide a proof (hint $d(\tan(\theta(t))/dt = (1 + \tan(\theta)^2)\theta' = (1 + (y_2/y_1)^2)\theta'$.

In order to apply this transformation, consider the ODE

$$\begin{aligned} \dot{y}_1 &= \alpha y_1 + \beta y_2 \\ \dot{y}_2 &= -\beta y_1 + \alpha y_2 \end{aligned}$$

The ODE in polar coordinates becomes

$$\begin{aligned} r' &= \alpha r \\ \theta' &= -\beta \end{aligned}$$

[7]We use the rules for sums and multiplications of complex numbers: if $x_1 = a_1 + b_1 i$ and $x_2 = a_2 + b_2 i$, then $x_1 + x_2 = (a_1 + a_2) + (b_1 + b_2)i$ and $x_1 x_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$ because $i^2 = -1$.

which has the general solution

$$r(t) = r_0 e^{\alpha t}$$
$$\theta(t) = \theta_0 - \beta t$$

If $\alpha < 0$ the radius converges to zero (meaning that the the dynamics is stable) and if $\theta > 0$ the movement is clockwise.

3.A.3 Second order equations

Consider a general second order equation.

$$\ddot{y} - a_1 \dot{y} + a_0 y = 0$$

If we define $y_1 = y$ and $y_2 = \dot{y} = \dot{y}_1$, then, we can transform the equation into the system

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = a_0 y_1 + a_1 y_2.$$

Bibliography

de la Fuente, A. (2000). *Mathematical Methods and Models for Economists.* Cambridge University Press.

Gandolfo, G. (1997). *Economic Dynamics.* Springer-Verlag.

Hale, J. and Koçak, H. (1991). *Dynamics and Bifurcations.* Springer-Verlag.

Hirsch, M. and Smale, S. (1974). *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press.

Magnus, J. R. and Neudecker, H. (1988). *Matrix Differential Calculus with Applications in Statistics and Econometrics.* Wiley.

Perko, L. (1996). *Differential Equations and Dynamical Systems, 2nd Ed.* Springer-Verlag.

Simon, C. P. and Blume, L. (1994). *Mathematics for Economists.* W. W. Norton & Co., New York.

Tu, P. (1994). *Dynamical Systems, 2nd ed.* Springer-Verlag.