Advanced Mathematical Economics

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Lecture 5
4.11.2020
(revised 26.11.2020)

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Chapter 6

Calculus of variations

6.1 Calculus of variations: introduction

Calculus of variations problems consist in finding an extreme of a functional over a function $y : \mathrm{X} \to \mathrm{Y}$, which can be subject to additional requirements. Solving a calculus of variations problem means finding function $y^*(x)$.

The **objective functional** takes the form

$$\mathsf{J}[y] \equiv \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \tag{6.1}$$

in which we call $F(.)$ is known.

We assume that $F_{y'}(x, y, y') = \dfrac{\partial F(x, y, y')}{\partial y'} \neq 0$, except maybe in a subset of measure zero. This is the characteristic of function $F(.)$ which makes the problem a dynamic optimization problem (if time is the independent variable), in the sense that the optimization involves a trade-off between the current state $y(x)$ and the change in the current state, $y'(x)$. If $F_{y'}(x, y(x), y'(x)) = 0$ globally, for any $x \in \mathrm{X}$ the problem will degenerate to a static functional optimization.

To understand the effect of the derivative on the optimum, consider instead the objective functional

$$\mathsf{J}_0[y] \equiv \int_{x_0}^{x_1} F(x, y(x))\, dx.$$

If there are no other conditions, if $y^*(x)$ is the optimum, a necessary condition is

$$\frac{\delta \mathsf{J}_0[y^*]}{\delta y(x)} = F_y(x, y^*(x)) = 0, \text{ for every } x \in \mathrm{X}$$

where $F_y(x, y) = \dfrac{\partial F(x, y)}{\partial y}$. This condition is a point-wise optimality criterium: the optimum $y^*(x)$ is found by finding an extremum for every point in $x \in \mathrm{X}$ independent of any other point $\in \mathrm{X}$. If the objective function depends on the derivative of function $y(\cdot)$, $y'(\cdot)$, this means that the local interaction influences the value of the problem. This has two consequences: first, the optimum cannot be just determined by a point-wise extremum, and, second, any constraint on the value of y will influence the solution.

This also means that the should look for solutions $y \in C^1(\mathrm{X};\mathrm{Y})$, where $C^1(\mathrm{X};\mathrm{Y})$ is the set of continuously differentiable functions mapping X into Y.

Two observations are important referring to the nature of the independent variable, x, and to its domain X.

First, from now on, as in most economic applications, x is a non-negative real number referring to time , i.e. $x = t$ and $\mathrm{X} = \mathrm{T} \subseteq \mathbb{R}_+$. However in some microeconomic problems or static macroeconomic problems with heterogeneity among agents, and, for example, information or searching frictions we are lead to solve optimal control problems in which the independent variable is not time and has a support belonging to a continuum, for instance $\mathrm{X} = [x_0, x_1]$. In time-dependent problems we call $x_0 = t_0$ the initial time and $x_1 = t_1$ the terminal time, or horizon, while for non-time-dependent models the designation depends on the context. For example in models in which x refers to the skill level x_0 refers to the lowest skill in the distribution and x_1 to the highest skill. Therefore, from now on we call x_0 the lower bound and x_1 the upper bound of X.

Second, in time-dependent problems we usually assume that $x_0 = t_0$ and $x_1 = t_1$ may be fixed (v.g., in macroeconomic models) or free (v.g., in microeconomic problems). If x refers to other type of variables x_0 and x_1 may refer to cutoff points which can be free and optimally determined.

At last, another important point to be made, which is particularly important in macroeconomics is related to the boundedness of X. We can consider x_1 to be bounded or unbounded $x_1 = \infty$. In the case in which x refers to time we have to distinguish between **finite or infinite horizon** cases.

In this section, we start with the simplest case, in subsection 6.2.1 the case in which the boundary of X and the values of the state variables at that boundary are also known, i.e, x_0, x_1, $y(x_0)$ and $y(x_1)$ are known. Next we consider the cases in which x_0 are x_1 known but $y(x_0)$ and $y(x_1)$ are free, the cases in which known $y(x_0)$ and $y(x_1)$ are known but x_0 are x_1 are free and the cases in which x_0, x_1, $y(x_0)$ and $y(x_1)$ are all free. Then we deal with two cases which are common to time-dependent models: the existence of terminal constraints and the infinite horizon problem.

6.2 Bounded domains and equality constraints

6.2.1 The simplest CV problem

The simplest CV problem is the following: (i) let the set of independent variables be closed and bounded $\mathrm{X} = [x_0, x_1]$, where the limits x_0 and x_1 are known, (ii) let $y(x_0) = y_0$ and $y(x_1) = y_1$ be also known; (iii) find a function $y : \mathrm{X} \to \mathbb{R}$, that maximizes the **objective functional** (6.1).

Formally, the simplest problem is:

$$\begin{aligned}
&\max_{y(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \\
&\text{subject to} \\
&x_0 \text{ and } x_1 \text{ known} \\
&y(x_0) = y_0 \text{ and } y(x_1) = y_1 \text{ known}
\end{aligned} \tag{P1}$$

We denote by $\varphi = (x_0, x_1, y_0, y_1, .)$ be vector of the data of the problem containing the lower and upper values of the independent variable, the associated values of the state function, and other parameters that might exist in function $F(.)$.

The **value function**, $V(\varphi) = \mathsf{J}[y^*]$, depends on the data of the problem, that is

$$V(x_0, x_1, y_0, y_1, .) = \mathsf{J}[y^*] \equiv \max_{y \in \mathrm{Y}} \int_{x_0}^{x_1} F(x, y^*(x), y^{*'}(x))\, dx,$$

where Y is the **admissibility set**, that is

$$\mathrm{Y} \equiv \left\{ y(x) \in \mathbb{R} : y(x_0) = y_0,\ y(x_1) = y_1 \right\}$$

the set of functions which satisfy the lower and upper boundary data and $\mathrm{X} = [x_0, x_1]$. Therefore, the problem is to find a function $y^* : \mathrm{X} \to \mathrm{Y}$, which maximizes the functional (6.1).

Proposition 1. ***First order necessary conditions for the simplest problem, (P1)*** *: $y^* : [x_0, x_1] \to Y$ is a solution of the simplest CV problem only if it satisfies the* ***Euler-Lagrange equation*** [1]*:*

$$F_y(x, y^*(x), y^{*'}(x)) = \frac{d}{dx}\Big(F_{y'}(x, y^*(x), y^{*'}(x)) \Big), \ \textit{for}\, x \in (x_0, x_1) \tag{6.2}$$

together with the boundary conditions

$$y^*(x_0) = y_0, \textit{and}\, y^*(x_1) = y_1 \tag{6.3}$$

Proof. (Heuristic) Assume we know y^*. Then the maximum value for the functional is

$$\mathsf{J}[y^*] = \int_{x_0}^{x_1} F(x, y^*(x), y^{*'}(x))\, dx.$$

Function y^* is an optimum only if $\mathsf{J}[y^*] \geq \mathsf{J}[y]$ for any other admissible function $y : \mathrm{X} \to \mathrm{Y}$. Take an admissible variation over y^*, $y = y^* + \delta y$ such that the variation is a *parameterized perturbation* of y^*, that is $\delta y = \varepsilon \eta$ where $\eta \mathrm{X} \to \mathrm{Y}$ and ε is a number. A variation to be admissible has to satisfy $y(x_1) = y^*(x_1) = y_1$ and $y(x_0) = y^*(x_0) = y_0$. Therefore, an admissible perturbation has to satisfy $\eta(x_0) = \eta(x_1) = 0$ and it can take arbitrary values $\eta(x) \in \mathrm{Y}$ for x in the interior of the domain X.

[1] We use the notation $F_y\ (x, y, y') = \dfrac{\partial F(x, y, y')}{\partial y}$ and $F_{y'}(x, y, y') = \dfrac{\partial F(x, y, y')}{\partial y'}$.

The value functional for the perturbed function y is

$$\mathsf{J}[y] = \mathsf{J}[y^* + \varepsilon\eta] \; = \int_{x_0}^{x_1} F(x, y^*(x) + \varepsilon\eta(x), y^{*'}(x) + \varepsilon\eta'(x))dx.$$

A first-order expansion of the functional $\mathsf{J}[y]$ in a neighbourhood of y^*,

$$\mathsf{J}[y] \; = \mathsf{J}[y^*] + \delta\mathsf{J}[y^*](\eta)\varepsilon + o(\varepsilon)$$

Then $\mathsf{J}[y^*] \geq \mathsf{J}[y]$ only if the first integral derivative of J is zero: $\delta\mathsf{J}[y^*](\eta) = 0$.

Because the Gâteaux derivative of a functional evaluated at y^* for the perturbation η is

$$\delta\mathsf{J}[y^*](\eta) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathsf{J}[y^* + \varepsilon\eta]$$

we have

$$\delta\mathsf{J}[y^*](\eta) = \int_{x_0}^{x_1} \Big(F_y(x, y^*(x), y^{*'}(x))\, \eta(x) + F_{y'}(x, y^*(x), y^{*'}(x)\Big)\, \eta'(x)\Big)\, dx.$$

Intregrating by parts the second integral yields

$$\int_{x_0}^{x_1} F_{y'}(x, y^*(x), y^{*'}(x))\, \eta'(x)dx = \int_{x_0}^{x_1} F_{y'}(x, y^*(x), y^{*'}(x))\, \eta(x) - \int_{x_0}^{x_1} dF_{y'}(x, y^*(x), y^{*'}(x))\, \eta(x),$$

which implies

$$\begin{aligned} \delta\mathsf{J}[y^*](\eta) = \int_{x_0}^{x_1} \Big(F_y(x, y^*(x), y^{*'}(x)) - \frac{d}{dx}F_{y'}(x, y^*(x), y^{*'}(x))\Big)\, \eta(x)\, dx + \\ + F_{y'}(x_1, y^*(x_1), y^{*'}(x_1))\, \eta(x_1) - F_{y'}(x_0, y^*(x_0), y^{*'}(x_0))\, \eta(x_0). \end{aligned} \tag{6.4}$$

As, in this case with fixed boundary values for the variable y, the admissible perturbation satisfies $\eta(x_1) = \eta(x_0) = 0$ equation (6.4) reduces to

$$\delta\mathsf{J}[y^*](\eta) = \int_{x_0}^{x_1} \Big(F_y(x, y^*(x), y^{*'}(x)) - \frac{d}{dx}F_{y'}(x, y^*(x), y^{*'}(x))\Big)\, \eta(x)\, dx$$

If $F(.)$ is a continuous function we can use the following result: if $h := [x_0, x_1] \to \mathbb{R}$ is a continuous function and $\int_{x_0}^{x_1} h(x)\eta(x)dx$ for all C^1 functions η and if $\eta(x_0) = \eta(x_1) = 0$ then $\int_{x_0}^{x_1} h(x)\eta(x)dx = 0$ if and only if $h(x) = 0$ for all $x \in (x_0, x_1)$.

Therefore $\delta\mathsf{J}[y^*](\eta) = 0$ if an only if $F_y(x, y^*(x), y^{*'}(x)) - \frac{d}{dx}F_{y'}(x, y^*(x), y^{*'}(x)) = 0$ for every $x \in \mathrm{X} = [x_0, x_1]$. □

The Euler-Lagrange equation is a 2nd order ODE (ordinary differential equation) if $F_{y'y'} \neq 0$: if we expand the right-hand-side we find

$$F^*_{y'y'}\, y'' + F^*_{y'y}\, y' \; + F^*_{y'x} - F^*_y = 0, \; x_0 \leq x \leq x_1,$$

where the derivatives of $F(\cdot)$ are evaluated at the optimum $y^*(x)$: for instance $F^*_y = F_y(x, y^*, y^{*'})$.

We can transform it into a system of first order ODE's if we define $y_1 = y$ and $y_2 = y'$ then

$$
\begin{aligned}
y_1' &= y_2 \\
F_{y_2 y_2}(x, y_1, y_2)\, y_2' &= F_{y_1}(x, y_1, y_2) - F_x(x, y_1, y_2) - F_{y_2 y_1}(x, y_1, y_2) y_2.
\end{aligned}
$$

The first order necessary condition only allows for the determination of an extremum. In order to get the a necessary condition for a maximand we need a second order condition:

Proposition 2. ***Second order necessary conditions**: the solution to the CV problem $y^* : X \to Y$ is a maximand only if it satisfies the Legendre-Clebsch condition*

$$F_{y'y'}(x, y^*(x), y^{*'}(x)) \leq 0 \tag{6.5}$$

Proof. (Heuristic but more complicated). Performing a second -order expansion of the functional $\mathsf{J}[x]$ in a neighbourhood of y^*, we obtain

$$\mathsf{J}[y] \ = \mathsf{J}[y^*] + \delta \mathsf{J}[y^*](\eta)\varepsilon + \frac{1}{2}\ \delta^2 \mathsf{J}[y^*](\eta)\varepsilon^2 + o(\varepsilon^2),$$

where

$$\delta^2 \mathsf{J}[y^*](\eta) = \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} \mathsf{J}[y^* \ + \varepsilon\eta].$$

Because at the optimum for any admissible perturbation η we have $\delta \mathsf{J}[y^*](\eta) = 0$, and at a have a maximum $\mathsf{J}[y] \ \leq \mathsf{J}[y^*]$, a necessary condition is $\delta^2 \mathsf{J}[y^*](\eta) \leq 0$.

The second-order functional derivative is

$$
\begin{aligned}
\delta^2 \mathsf{J}[y^*](\eta) = \int_{x_0}^{x_1} \Big(& F_{yy}(x, y^*(x), y^{*'}(x))\, \eta(x)^2 + \\
& + 2F_{yy'}(x, y^*(x), y^{*'}(x))\, \eta(x)\, \eta'(x) + F_{y'y'}(x, y^*(x), y^{*'}(x))\, (\eta'(x))^2 \Big)\, dx.
\end{aligned}
$$

As

$$
\begin{aligned}
\int_{x_0}^{x_1} 2F^*_{yy'}(x)\, \eta(x)\, \eta'(x) dx &= \int_{x_0}^{x_1} F^*_{yy'}(x) \frac{d}{dx}\ (\eta(x)^2) dx = \\
&= F^*_{yy'}(x)\ \eta(x)^2 \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\ \Big(F^*_{yy'}\Big)(\eta(x)^2) dx \\
&= - \int_{x_0}^{x_1} \frac{d}{dx}\ \Big(F^*_{yy'}\Big)(\eta(x)^2) dx
\end{aligned}
$$

because of the admissibility conditions $\eta(x_0) = \eta(x_1) = 0$. Then

$$\delta^2 \mathsf{J}[y^*](\eta) = \int_{x_0}^{x_1} \left(\Big(F^*_{yy}(x) - \frac{d}{dx} F^*_{yy'}(x) \Big) \eta(x)^2 + F^*_{y'y'}(x) (\eta'(x))^2 \right) dx.$$

Following (Liberzon, 2012, p.59-60)), it can be shown that $\delta^2 \mathsf{J}[y^*](\eta) \leq 0$ only if condition (6.5) holds. □

Proposition 3. ***Sufficient conditions****: let $y^* \in Y$ verify*

$$F_y\ (t, y^*, y^{*'}) = \frac{d}{dx}\ F_{y'}(t, y^*, y^{*'})\ \text{ and } F_{y'y'}(t, y^*, y^{*'}) \leq 0$$

then (under some additional conditions on the trajectory of y) y^* is an optimiser to $\mathsf{J}[y]$.

Proof. See (Liberzon, 2012, p.62-68) □

Proposition 4. ***Necessary and sufficient conditions****: consider the simplest calculus of variations problem and assume that $F_{y'y'}(x, y(x), y'(x)) \leq 0$ for every $x \in [x_0, x_1]$ then equations* (6.3.1) *and* (6.3) *are necessary and sufficient conditions.*

6.2.2 Free boundary values for the state variable

Now we consider the problem: find function y^* among admissible functions $y \in \mathrm{Y}$ having the following properties: $y : \mathrm{X} \to Y \subseteq \mathbb{R}$, where $\mathrm{X} = [x_0, x_1]$ has known boundaries, x_0 and x_1, and such that $y(x_0)$ and/or $y(x_1)$ are free. The objective functional is again (6.1).

Formally, the problem are:

$$\begin{aligned}
&\max_{y(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \\
&\quad \text{subject to} \\
&\quad x_0 \text{ and } x_1 \text{ known} \\
&\quad y(x_0) = y_0 \text{ and } y(x_1) = y_1 \text{ free} && \text{(P2a)} \\
&\quad y(x_0) = y_0 \text{ known, } y(x_1) = y_1 \text{ free} && \text{(P2b)} \\
&\quad y(x_0) = y_0 \text{ free } y(x_1) = y_1 \text{ known} && \text{(P2c)}
\end{aligned}$$

In this case the data of the problem P2a, for instance, is $\varphi = (x_0, x_1, .)$. and the value functional is

$$V(x_0, x_1, .) = \max_{y \in \mathrm{Y}} \int_{x_0}^{x_1} F(x, y(x), y'(x))dx.$$

Proposition 5. ***First order necessary conditions for the free terminal state problem****: $y^* \in Y$ is the solution to one of the CV problem with free boundary values for the state variable and known terminal values for the independent variable, x_0 and x_1, problems P2a, P2b, or P2c, only if it satisfies the Euler equation* (6.3.1) *and the boundary conditions:*

1. *if both boundary values are free (problem P2a)*

$$F_{y'}(x_0, y^*(x_0), y^{*'}(x_0)) = 0,\ \text{ and }\ F_{y'}(x_1, y^*(x_1), y^{*'}(x_1)) = 0 \tag{6.7}$$

2. *if the lower boundary value is given by $y(x_0) = y_0$, and the upper boundary value is free (problem P2b)*

$$y^*(x_0) = y_0,\ \text{ and } F_{y'}(x_1, y^*(x_1), y^{*'}(x_1)) = 0 \tag{6.8}$$

3. *if the upper boundary value is given by $y(x_1) = y_1$, and the lower boundary value is free (problem P2c)*

$$F_{y'}(x_0, y^*(x_0), y^{*'}(x_0)) = 0, \text{ and } y^*(x_1) = y_1. \tag{6.9}$$

Proof. (Heuristic) Now the boundary values for perturbation are $\eta(x_0)$ and $\eta(x_1)$ can take any value, including zero if the associated boundary value $y(x)$, for $x \in \{x_0, x_1\}$ is fixed. The proof follows the same steps as in the proof pf Proposition 1. However, in equation (6.4), in order to get $\delta \mathrm{J}[y^*](\eta) = 0$, and after introducing the Euler-Lagrange condition, we should have

$$F_{y'}(x_j, y^*(x_j), y^{*'}(x_j))\,\eta(x_j) = 0, \text{ for } j = 0, 1. \tag{6.10}$$

Thus we have two cases, concerning the adjoint conditions at boundary x_j, for $j = 0, 1$, for an optimum. First, if the value of the state variable for the boundary x_j is known, i.e., $y(x_j) = y_j$, an admissible perturbation should verify $\eta(x_j) = 0$, implying that condition (6.10) holds automatically. This is the case in Proposition 1. Second, if the value of the state variable for the boundary x_j is free, then the related perturbation value is arbitrary and $\eta(x_j) \neq 0$ in general. The optimally condition (6.10) holds if and only if $F_{y'}(x_j, y^*(x_j), y^{*'}(x_j)) = 0$ which provides one adjoint condition allowing for the determination of the optimal boundary value for the state variable $y^*(x_j)$. This is how we adjoint (6.7) to (6.9) depending on which boundary value for the state variable is free. □

In time-varying models in which the value of the state variable is known at time $t = 0$ and the terminal value of the state variable is endogenous we supplement the Euler-Lagrange with condition (6.8).

However, there are models in which the initial value of the state variable is unknown. This is the case, for instance, in optimal taxation models of the Mirrlees (1971) type in which the independent variable are skill values and the initial condition is related to the cutoff level of skill bellow which taxes should be zero. In this case condition (6.7) can be used.

Observation: as the Euler-Lagrange is a second-order differential equation, in order to fully solve a model we need to have information on the value of y at the two boundaries for $x = x_0$ and $x = x_1$.

6.2.3 Free boundary values for the independent variable

Now we consider the problem: find function $y^* \in \mathrm{Y}$ which is the set of functions $y : \mathrm{X}^* \to \mathbb{R}$, where X^* has at least one unknown boundary, x_0^* and/or x_1^*, but such that the terminal values for the state variable are known. That is $\mathrm{X}^* = [x_0^*, x_1]$ or $\mathrm{X}^* = [x_0, x_1^*]$ or $\mathrm{X}^* = [x_0^*, x_1^*]$ where x_j is known and x_j^* is free. If a boundary value for the independent variable is free the related boundary value for the state variable is known, that is $y(x_j^*) = y_j$. The objective functional is again (6.1).

Formally, the problem are :

$$\max_{y(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx$$

subject to

$$y(x_0) = y_0 \text{ and } y(x_1) = y_1 \text{ known}$$

$$x_0 \text{ and } x_1 \text{ free} \tag{P3a}$$

$$x_0 \text{ known, } x_1 \text{ free} \tag{P3b}$$

$$x_0 \text{ free } x_1 \text{ known} \tag{P3c}$$

In this case the data of the problem is $\varphi = (y_0, y_1, .)$. and the value functional is

$$V(y_0, y_1, .) = \max_{y \in \mathrm{Y}, x_0, x_1} \int_{x_0}^{x_1} F(x, y(x), y'(x))dx = \int_{x_0^*}^{x_1^*} F(x, y^*(x), y^{*'}(x))dx$$

where we x_0^* and/or x_1^* are determined endogenously.

Proposition 6. ***First order necessary conditions for the free boundaries value problem:*** *$y^* \in Y$ is the solution to the CV problem with known boundary values for the state variable, y_0 and y_1, and free terminal values for the independent variable, problems P3a, P3b, or P3c, only if it satisfies the Euler equation* (6.3.1) *and the boundary conditions:*

1. *if both boundary values for the indepdendent variable are free (problem P3a)*

$$F(x_0^*, y_0, y^{*'}(x_0^*)) - F_{y'}(x_0^*, y_0, y^{*'}(x_0^*))\, y^{*'}(x_0^*) = 0$$
$$\text{and } F(x_1^*, y_1, y^{*'}(x_1^*)) - F_{y'}(x_1^*, y_1, y^{*'}(x_1^*))\, y^{*'}(x_1^*) = 0 \tag{6.12}$$

2. *if the lower boundary value for the independent variable is known, $x_0^* = x_0$, and the upper boundary for the independent variable is free (problem P3b)*

$$x_0^* = x_0, \text{ and } F(x_1^*, y_1, y^{*'}(x_1^*)) - F_{y'}(x_1^*, y_1, y^{*'}(x_1^*))y^{*'}(x_1^*) = 0 \tag{6.13}$$

3. *if the upper boundary value for the independent variable is known, $x_1^* = x_1$, and the lower boundary for the independent variable is free (problem P3c)*

$$F(x_0^*, y_0, y^{*'}(x_0^*)) - F_{y'}(x_0^*, y_0, y^{*'}(x_0^*))y^{*'}(x_0^*) = 0, \text{ and } x_1^* = x_1. \tag{6.14}$$

Proof. (Heuristic) Let us assume that we know the solution $y^*(x)$ for $x \in [x_0^*, x_1^*]$, that is for all values of the independent variable contained between the two optimally chosen boundary values. In this case we have to introduce two types of perturbations: a perturbation to the state variable $y(x) = y^*(x) + \varepsilon\eta(x)$ and to the independent variable $x = x^* + \varepsilon\chi(x)$. If we denote $y_j^* = y^*(x_j^*)$, for $j = 0, 1$, the two boundary values for the independent and dependent variables are $P_j^* \equiv (x_j^*, y_j^*)$ for $j = 0, 1$ at the optimum. The related terminal points for the perturbed solution are written as $P_j = (x_j^* + \varepsilon\chi_j, y_j^* + \varepsilon\eta_j)$ for $j = 0, 1$.

At the optimum the objective functional is

$$\mathsf{J}[y^*;x^*]=\int_{x_0^*}^{x_1^*}F(x,y^*(x),y^{*'}(x))dx$$

and

$$\mathsf{J}[y^*+\varepsilon\eta;x^*+\varepsilon\chi]=\int_{x_0^*+\varepsilon\chi_0}^{x_1^*+\varepsilon\chi_1}F(x,y^*(x)+\varepsilon\eta(x),y^{*'}(x)+\varepsilon\eta'\ (x))dx.$$

Then, denoting $\Delta\mathsf{J}(\varepsilon)=\mathsf{J}[y^*+\varepsilon\eta;x^*+\varepsilon\chi]-\mathsf{J}[y^*;x^*]$ we have

$$\begin{aligned}\Delta\mathsf{J}(\varepsilon)=&\int_{x_0^*+\varepsilon\chi_0}^{x_1^*+\varepsilon\chi_1}F(x,y^*(x)+\varepsilon\eta(x),y^{*'}(x)+\varepsilon\,\eta'(x))\,dx-\int_{x_0^*}^{x_1^*}F(x,y^*(x),y^{*'}(x))\,dx\\=&\int_{x_0^*}^{x_1^*}\Big(F(x,y^*(x)+\varepsilon\eta(x),y^{*'}(x)+\varepsilon\eta'\ (x))-F(x,y^*(x),y^{*'}(x))\Big)dx+\\&+\int_{x_1^*}^{x_1^*+\varepsilon\chi_1}F(x,y^*(x)+\varepsilon\eta(x),y^{*'}(x)+\varepsilon\eta'\ (x))dx-\\&-\int_{x_0^*+\varepsilon\chi_0}^{x_0^*}F(x,y^*(x)+\varepsilon\eta(x),y^{*'}(x)+\varepsilon\eta'\ (x))dx\end{aligned}$$

Denoting $F^*(x)=F(t,y^*(x),\dot{y}^*(x))$ and using the mean-value theorem,

$$\Delta\mathsf{J}(\varepsilon)=\varepsilon\int_{x_0^*}^{x_1^*}\Big(F_y^*(x)\eta(x)+F_{y'}^*(x)\eta'(x)\Big)\,dx+F(\tilde{x}_1)\,\varepsilon\,\chi_1-F(\tilde{x}_0)\,\varepsilon\,\chi_0$$

where $\tilde{x}_1\in(x_1^*,x_1^*+\varepsilon\,\chi_1)$ and $\tilde{x}_0\in(x_0^*,x_0^*+\varepsilon\,\chi_0)$. Taking $\delta\mathsf{J}[y^*;x^*]\ (\eta,\chi)=\lim_{\varepsilon\to 0}\dfrac{\Delta\mathsf{J}(\varepsilon)}{\varepsilon}$, the functional derivative becomes

$$\delta\mathsf{J}[y^*;x^*]\ (\eta,\chi)=\int_{x_0^*}^{x_1^*}\Big(F_y^*(x)\eta(x)+F_{y'}^*(x)\,\eta'(x)\Big)dx+\,F^*(x)\Big|_{x=x_1^*}\chi_1-F^*(x)\Big|_{x=x_0^*}\chi_0.$$

Integration by parts yields

$$\begin{aligned}\delta\mathsf{J}[y^*;x^*]\ (\eta,\chi)=&\int_{x_0^*}^{x_1^*}\Big(F_y^*(x)-\frac{d}{dx}F_{y'}^*(x)\Big)\,\eta(x)dx+\\&+\,F_{y'}^*(x)\eta(x)\Big|_{x=x_1^*}-F_{y'}^*(x)\eta(x)\Big|_{x=x_0^*}+F^*\,(x)|_{x=x_1^*}\,\chi_1-F^*(t)|_{x=x_0^*}\,\chi_0.\end{aligned}$$

We only know the perturbations for the state variables at the perturbed boundaries x_0 and x_1 and not at x_0^* and x_1^*, which inhibits the computation of the integral in the last equation. In order to find $\eta(x_j^*)$ the following approximation is introduced

$$\eta(x_j^*)\approx\eta_j-y'(x_j^*)\chi_j,\ \text{for}\ j=0,1.$$

Therefore,

$$\begin{aligned}\delta\mathsf{J}[y^*;x^*]\ (\eta,\chi)=&\int_{x_0^*}^{x_1^*}\Big(F_y^*(x)-\frac{d}{dx}F_{y'}^*(x)\Big)\,\eta(x)dx+F_{y'}^*(x_1^*)\eta_1-F_{y'}^*(x_0^*)\eta_0+\\&+\,\Big(F^*(x)-F_{y'}^*(x)y'(x)\Big)\Big|_{x=x_1^*}\chi_1-\Big(F^*(x)-F_{y'}^*(x)y'(x)\Big)\Big|_{x=_0^*}\chi_0\end{aligned}$$

As the terminal values of the state variables, $y(x_0^*) = y_0$ and $y(x_1^*) = y_1$, are known then the terminal perturbation for the independent variable should satisfy $\eta_0 = \eta_1 = 0$. Therefore, $\delta \mathsf{J}[y^*; x^*]\ (\eta, \chi) = 0$ if and only if the Euler-Lagrange equation holds and $\left(F^*(x) - F^*_{y'}(x)y'(x)\right)\Big|_{x=x_1^*} \chi_1 = 0$ and/or $\left(F^*(x) - F^*_{y'}(x)y'(x)\right)\Big|_{x=_0^*} \chi_0 = 0$. This encompasses the three cases in equations (P5), (6.13) and (6.14). □

6.2.4 Free boundaries for both independent and dependent variables

The most general problem is: find function $y^* \in \mathrm{Y}$ among functions $y : \mathrm{X}^* \to \mathbb{R}$, where X^* has at least one unknown boundary, x_0^* and/or x_1^*, as in the previous subsection, and the terminal values for the state variables, $y(x_0^*)$ and/or $y(x_1^*)$ are also free. The objective functional is again (6.1).

Formally, the problem are :

$$\max_{y(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx$$

$$\text{subject to}$$

$$y(x_0) = y_0 \text{ and } x_0 \text{ free},\ y(x_1) = y_1 \text{ and } x_1 \text{ known} \tag{P4a}$$

$$y(x_0) = y_0 \text{ and } x_0 \text{ known},\ y(x_1) = y_1 \text{ and } x_1 \text{ free} \tag{P4b}$$

$$y(x_0) = y_0,\ x_0,\ y(x_1) = y_1 \text{ and } x_1 \text{ free} \tag{P4c}$$

In this case the data of the problem, $\varphi = (.)$, only involves parameters that may be present in function $F(.)$. The value functional is

$$V(.) = \max_{y \in \mathrm{Y}, x_0, x_1} \int_{x_0}^{x_1} F(x, y(x), y'(x))dx = \int_{x_0^*}^{x_1^*} F(x, y^*(x), y^{*'}(x))dx$$

where we x_0^* and/or x_1^* and $y^*(x_0^*)$ and/or $y^*(x_1^*)$ are determined endogenously.

Proposition 7. ***First order necessary conditions for the free terminal boundary problem:*** *$y^* \in Y$ is the solution to the CV problem with free boundary values for the state variable and for the independent variable, only if it satisfies the Euler equation* (6.3.1) *and the boundary conditions:*

1. *if both values for lower boundary are free (problem P4a)*

$$F_y(x_0^*, y^*(x_0^*), y^{*'}(x_0^*)) = F_{y'}(x_0^*, y^*(x_0^*), y^{*'}(x_0^*)) = 0 \tag{6.16}$$

2. *if both values for upper boundary are free (problem P4b)*

$$F_y(x_1^*, y^*(x_1^*), y^{*'}(x_1^*)) = F_{y'}(x_1^*, y^*(x_1^*), y^{*'}(x_1^*)) = 0 \tag{6.17}$$

3. *if all terminal values for x and $y(x)$ are free (problem P4c)*

$$F_y(x_0^*, y^*(x_0^*), y^{*'}(x_0^*)) = F_{y'}(x_0^*, y^*(x_0^*), y^{*'}(x_0^*)) = 0 \tag{6.18a}$$

$$F_y(x_1^*, y^*(x_1^*), y^{*'}(x_1^*)) = F_{y'}(x_1^*, y^*(x_1^*), y^{*'}(x_1^*)) = 0 \tag{6.18b}$$

Proof. We use the previous proof and, in equation (6.15), we consider $\eta_0 \neq 0$, $\eta_1 \neq 0$, $\chi_0 \neq 0$ and $\chi_1 \neq 0$. □

Table 6.1 assembles all the previous results. Observe that if we consider all the possible combinations of the information on both boundaries we have **16 possible cases**.

Table 6.1: Adjoint conditions for bounded domain CV problems

data		optimum	
x_j	$y(x_j)$	x_j^*	$y^*(x_j^*)$
fixed	fixed	x_j	y_j
fixed	free	x_j	$F_{y'}(x_j, y^*(x_j), y^{*'}(x_j)) = 0$
free	fixed	$F(x_j^*, y_j, y^{*'}(x_j^*)) - y^{*'}(x_j^*)F_{\dot{x}}(x_j^*, y_j, y^{*'}(x_j^*)) = 0$	y_j
free	free	$F\left(x_j^*, y^*(x_j^*), y^{*'}(x_j^*)\right) = 0$	$F_{y'}(x_j^*, y^*(x_j^*), y^{*'}(x_j^*)) = 0$

The index refers to the lower boundary when $j = 0$ and to the upper boundary when $j = 1$

6.2.5 Other constraints

As for the static optimization problem we can consider inequality constraints, for instance inequality constraints on the value of the variable y for some value of the independent variable.

We consider a problem in which the two limits for independent variable are known, i.e, x_0 and x_1 are known, $y(x_0) = y_0$ is known, but we $y(x_1)$ is constrained by the condition $R(x_1, y(x_1)) \geq 0$.

Formally, the problem are :

$$
\begin{aligned}
&\max_{y(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \\
&\text{subject to} \\
&x_0 \text{ and } x_1 \text{ known} \\
&y(x_0) = y_0 \text{ known} \\
&R(x_1, y(x_1)) \geq 0.
\end{aligned}
\tag{P5}
$$

Proposition 8. ***First order necessary conditions for the constrained terminal state problem:*** *$y^* \in Y$ is the solution to the CV problem (P5) only if it satisfies the Euler-Lagrange equation* (6.3.1)*, the initial condition $y^*(x_0) = y_0$ and the boundary condition*

$$
F_{y'}(x_1, y^*(x_1), y^{*'}(x_1))R(x_1, y^*(x_1)) = 0 \tag{6.19}
$$

Proof. In this case we consider the functional we introduce a Lagrange multiplier (a real number) associated to the terminal condition, yielding the Lagrange functional

$$
L[y] = \int_{x_0}^{x_1} F(x, y(x), y'(x))dx + \mu R(x_1, y(x_1)).
$$

We proceed as previously to get the optimality conditions for a perturbation $\eta \in \mathrm{Y}$ over the optimal function y^*. The first order necessary condition is

$$\delta L[y^*](\eta) = \int_{x_0}^{x_1} \Big(F_y^*(x) - \frac{d}{dx}\, F_{y'}^*(x) \Big)\, \eta(x)dx + \Big(F_{y'}^*(x_1) + \mu R_y^*(x_1) \Big)\, \eta(x_1) = 0$$

where $F^*(x) = F(x, y^*(x), y^{*'}(x))$ and $R_y^*(x_1) = \partial_y R(x_1, y^*(x_1))$. Because of the free terminal state, admissible perturbations are such that $\eta(x_1) \neq 0$. Therefore $\delta L[y^*](\eta) = 0$ requires that the adjoint condition $F_{y'}^*(x_1) + \mu R_y^*(x_1) = 0$ holds.

Due to the existence of a static inequality constraint at the boundary x_1, the Karush-Kuhn-Tucker (KKT) complementarity slackness conditions are also necessary:

$$\mu R^*(x_1) = 0,\ \mu \geq 0 \text{ and } \ R^*(x_1) \geq 0$$

where $R^*(x_1) = R(x_1, y^*(x_1))$. Multiplying the adjoint condition by $R^*(x_1)$ we obtain an equivalent condition

$$R^*(x_1)\, F_{y'}^*(x_1) + \mu\, R^*(x_1)\, R_y^*(t_y) = 0,$$

which is equivalent to $F_{y'}^*(x_1) R^*(x_1) = 0$, after considering the KKT condition. Therefore $\delta L[y^*](\eta) = 0$ if the Euler-Lagrange equation (6.3.1) and adjoint boundary condition (6.19) hold. □

6.3 Calculus of variations in time

We can directly apply the previous results for problems in which time is the independent variable. When time is the independent variable the domain of the independent variable it $\mathrm{T} \subseteq \mathbb{R}_+$, if we have a finite interval $\mathrm{T} = [t_0, t_1]$, the dependent variable is $y(t)$, which is a mapping $y : \mathrm{T} \ \to \mathrm{Y} \subseteq \mathbb{R}$, and we denote the time derivative by $\dot{y} = \dfrac{dy(t)}{dt}$.

A particular important problem is the discounted infinite-horizon problem

6.3.1 Discounted infinite horizon

The most common problem in macroeconomics and growth theory has three main common features. First, time is the independent variable, and assumes that the initial time and values are known, usually $x_0 = 0$ and $y(0) = y_0$, and an unbounded value for the terminal time, $x_1 \to \infty$. Second, the objective function is of type $F(t, y, \dot{y}) = f(y, \dot{y})e^{-\rho t}$, where $e^{-\rho t}$ is a discount factor with a time-independent rate of discount $\rho \geq 0$, and the current value objective function $f(y, \dot{y})$ is time -independent. Third, there are two main versions to the problem depending on the terminal value of the state variable, that can be free or constrained.

Free asymptotic state

Find function $y^* \in \mathrm{Y}$ which is the set of functions $y : [0, \infty) \to \mathbb{R}$ such that $y(0) = y_0$ where y_0 are given that maximizes

$$\mathsf{J}[y] \equiv \int_0^\infty f(y(t), \dot{y}(t)) e^{-\rho t} dt, \ \rho \geq 0. \tag{6.20}$$

This can be treated as a problem with a fixed initial time and value for the state variable, a fixed terminal time but a free terminal value for the state variable.

Proposition 9. ***First order necessary conditions for the discounted infinite horizon problem with free terminal state****: $y^* \in Y$ is the solution to the discounted infinite horizon CV problem with a known initial data, $\varphi = (y_0, \rho, .)$, and with a free terminal state only if it satisfies the Euler-Lagrange equation*

$$\frac{d}{dt}\left(f_{\dot{y}}(y^*(t), \dot{y}^*(t)) \right) = f_y(y^*(t), \dot{y}^*(t)) + \rho f_{\dot{y}}(y^*(t), \dot{y}^*(t)), \ \textit{for} \ \ t \in [0, \infty), \tag{6.21}$$

the so-called transversality condition

$$\lim_{t \to \infty} f_{\dot{y}}(y^*(t), \dot{y}^*(t)) e^{-\rho t} = 0 \tag{6.22}$$

and the initial condition $y^*(0) = y_0$

Proof. In the proof for the free boundaries value problem we extend $x_1 \to \infty$ and take it as fixed but let $\lim_{t \to \infty} y^*(t)$ be free. In this discounted problem the Euler-Lagrange equation $F_y^*(t) = \dfrac{d}{dt} F_{\dot{y}}^*(t)$ is equivalent to

$$e^{-\rho t} f_y(y^*, \dot{y}^*) = \frac{d}{dt}\left(e^{-\rho t} f_{\dot{y}}(y^*, \dot{y}^*) \right),$$

and the terminal condition (6.22) is obtained fro the boundary condition $\lim_{t \to \infty} F_{\dot{y}}^*(t) = 0$. □

Observe that the Euler-Lagrange is again a 2nd order non-linear autonomous ODE

$$f_y(y^*, \dot{y}^*) + \rho f_{\dot{y}}(y^*, \dot{y}^*) - f_{\dot{y}y}(y^*, \dot{y}^*) \dot{y} - f_{\dot{y}\dot{y}}(y^*, \dot{y}^*) \ddot{y} = 0.$$

The constrained terminal state problem

In several problems in economics the former condition can lead to an asymptotic state which does not make economic sense (v.g, a negative level for a capital stock).

The most common discounted infinite horizon model is usually the following: find function $y^* \in \mathrm{Y}$ which is the set of functions $y : [0, \infty) \to \mathbb{R}$ such that $y(0) = y_0$ and $\lim_{t \to \infty} R(t, y(t)) \geq 0$, where $x_0 = 0$ and $y(x_0) = y_0$ are given, that maximizes the objective functional (6.20)

Proposition 10. ***First order necessary conditions for the discounted infinite horizon problem with constrained terminal state****: $y^* \in Y$ is the solution to the discounted infinite horizon CV problem with a known initial data, $(x_0, y(x_0)) = (0, y_0)$, and with a terminal state constrained by $\lim_{t \to \infty} R(t, y(t)) \geq 0$ only if it satisfies the Euler-Lagrange equation* (6.21)*, the initial condition $y^*(0) = y_0$, and the (so-called) transversality condition*

$$\lim_{t \to \infty} f_{\dot{y}}(y^*(t), \dot{y}^*(t)) R(t, y^*(t)) e^{-\rho t} = 0 \tag{6.23}$$

Exercise: prove this. Observe that as the terminal constraint is $\lim_{t\to\infty} y(t) \geq 0$ we have to introduce a Lagrange multiplier associated to the terminal time.

6.3.2 Applications

The resource depletion problem

Assume we have a resource W (v.g., a cake) with initial size W_0 and we want to consume it along period $[0, \bar{t}]$. If $C(t)$ denotes the consumption at time x we evaluate the consumption of the resource by the functional $\int_0^{\bar{t}} \ln(C(t))e^{-\rho t}dt$. Several properties: (1) we are impatient (we discount time at a rate $\rho > 0$); (2) the felicity at every point in time is only a function of the instantaneous consumption (preferences are inter temporally additive); (3) more consumption means more felicity but at a decreasing rate (the increase in utility for big bites is smaller than for small bites); and (4) there is no satiation (there is not a bite with a zero or negative marginal utility): consumption is always good.

Cake eating problem with the terminal state given CE problem: find $C^* = (C^*(t))_{0\leq t\leq \bar{t}}$ that

$$\max_C \int_0^{\bar{t}} \ln(C(t))e^{-\rho t}dt$$

subject to

$$\dot{W}(t) = -C(t), \text{ for } t \in [0, T]$$

given $W(0) = W_0$ and $W(\bar{t}) = 0$.

Formulated as a CV problem: find $W^* = (W^*(t))_{0\leq t\leq \bar{t}}$ such that

$$V(W_0, \bar{t}, \rho) = \max_W \mathsf{J}[W] = \max_W \int_0^{\bar{t}} \ln(-\dot{W}(t))e^{-\rho t}dt$$

given $W(0) = W_0$ and $W(\bar{t}) = 0$. The data of the problem is the vector of constants $\varphi = (0, \bar{t}, W_0, 0, \rho)$

The solution of the problem, $(W^*(t))_{t=0}^{\bar{t}}$, is obtained from

$$\begin{cases} \ddot{W}^* + \rho\dot{W}^* = 0, & 0 < t < \bar{t} \\ W^*(0) = W_0, & t = 0 \\ W^*(T) = 0, & t = T \end{cases}$$

The solution of the Euler equation is [2]

$$W(t) = W(0) - \frac{k}{\rho}(1 - e^{-\rho t})$$

[2] Hint: setting $z = \dot{W}$ we get a first-order ODE $\dot{z} = -\rho z$ with solution $\dot{z} = ke^{-\rho t}$. As $dW(t) = z(t)dt$, if we integrate we have $\int_{W(0)}^{W(t)} dW = \int_0^t z(s)ds = \int_0^t ke^{-\rho s}ds$.

where k is an arbitrary constant. Using the adjoint conditions $W^*(\bar{t}) = 0$ and $W^*(0) = W_0$ we find the solution

$$W^*(t) = \frac{e^{-\rho t} - e^{-\rho\,\bar{t}}}{1 - e^{-\rho \bar{t}}}\ W_0, \text{ for } t \in [0, \bar{t}].$$

The value of the cake is

$$\begin{aligned} V(\varphi) &= \int_0^{\bar{t}} \ \ln\left(-\dot{W}^*(t)\right) e^{-\rho t} dt = \\ &= \frac{1}{\rho}\left[\left(1 + \ln\left(\frac{1 - e^{-\rho \bar{t}}}{\rho W_0}\left(e^{-\rho \bar{t}} - 1\right)\right)\right)\right] + \bar{t} e^{-\rho \bar{t}} \end{aligned}$$

if the consumer is rational this should be equal its reservation price for the cake. If $\rho = 0.01$ and the cake lasts for one week and the calorie content is $W_0 = 1000$ then the reservation price for should be $V(10, 0.01, 1/52) \approx 0.12$ per 100 calories.

Cake eating problem: infinite horizon If we assume an infinite horizon and the terminal condition $\lim_{t\to\infty} W(t) \geq 0$, meaning that we cannot have a negative level of resource asymptotically. The first order conditions are:

$$\begin{cases} \rho \dot{W}^*(t) + \ddot{W}^*(t) = 0 \\ W^*(0) = W_0 \\ -\lim_{t\to\infty} e^{-\rho t} \frac{W^*(t)}{\dot{W}^*(t)} \ = 0 \end{cases}$$

We already found

$$W(t) = W_0 - \frac{k}{\rho}\left(1 - e^{-\rho t}\right)$$

then

$$\dot{W}(t) = -k e^{-\rho t}$$

Solution (as $k = \rho W_0$)

$$W^*(t) = W_0 e^{-\rho t},\ t \in \mathbb{R}_+$$

Again $\lim_{t\to\infty} W^*(t) = 0$.

The benchmark representative problem

The benchmark representative consumer problem in macroeconomics is to find optimal consumption and asset holdings (C, A) such that $C : \mathbb{R}_+ \to \mathbb{R}_+$ and $A : \mathbb{R}_+ \to \mathbb{R}$ that maximize the value functional

$$U[C] = \int_0^\infty u(C(t)) e^{-\rho t} dt$$

subject to the instantaneous budget constraint

$$\dot{A} = Y - C + rA$$

given $A(0) = A_0$ and the non-Ponzi game condition $\lim_{t\to} A(t)e^{-rt} \geq 0$. In the above equations Y and r denote, respectively the non-financial income and the interest rate, and are both positive. The following assumptions on utility are standard: $u(0) = 0$, $u^{'}(C) > 0$ and $u^{''}(C) < 0$.

The inverse of the elasticity of intertemporal substitution can be proved to be

$$\theta(C) = -\frac{u^{''}(C)C}{u^{'}(C)} > 0.$$

Assumption: the elasticity of intertemporal substitution $\theta(C) = \theta$ is constant and

$$\theta > \frac{r-\rho}{r} > 0.$$

We can transform it into a CV problem by observing that consumption is a function of the both wealth and savings, $\dot{A}$,

$$C = C(A, \dot{A}) \equiv Y + rA - \dot{A}.$$

Therefore, the problem becomes a CV problem with value functional

$$\mathsf{J}[A] = \int_0^\infty u\left(Y + rA(t) - \dot{A}(t)\right) e^{-\rho t} dt$$

where $f(A(t), \dot{A}(t)) = u\left(Y + rA(t) - \dot{A}(t)\right)$. The optimality conditions (which are necessary and sufficient in this case) are

$$\begin{cases} (r-\rho)u^{'}(C(A,\dot{A})) + (r\dot{A} - \ddot{A})u^{''}(C(A,\dot{A})) = 0 \\ A(0) = A_0 \\ -\lim_{t\to\infty} e^{-\rho t} u^{'}(C(A,\dot{A}))A(t) \;= 0 \end{cases}$$

Observing that $\dot{C} = r\dot{A} - \ddot{A}$ and using the definition of the inverse intertempopral elasticity of substitution we can transform the Euler equation into

$$\dot{C} = \gamma C, \text{ for } \gamma \equiv \frac{r-\rho}{\theta} > 0.$$

This allows us to find a general solution for optimal consumption

$$C(t) = C(0)\, e^{\gamma t},$$

where $C(0)$ is an arbitrary unknown admissible level for consumption, i.e., it should be non-negative. In order to find that value we use the transversality condition. But for this we need to determine admissible values for A. The asset dynamics is then governed by

$$\dot{A} = Y + rA - ke^{\gamma t}, \text{for } t > 0,\; A(0) = A_0, \text{ for } t = 0$$

The solution to this initial value problem is

$$A(t) = -\frac{Y}{r} + \left(A_0 + \frac{Y}{r}\right) e^{rt} + \frac{C(0)}{r-\gamma}\left(e^{rt} - e^{\gamma t}\right).$$

With the previous assumption we have $r > \gamma$. As $u^{'}(C) = C^{-\theta}$ with an isoelastic utility function we find

$$\begin{aligned}\lim_{t\to\infty} u^{'}(C(t))A(t)e^{-\rho t} &= \lim_{t\to\infty} \left(C(0)e^{\gamma t}\right)^{-\theta} A(t)e^{-\rho t} = \\ &= \lim_{t\to\infty} C(0)^{-\theta}e^{-rt}\left[-\frac{Y}{r} + \left(A_0 + \frac{Y}{r}\right)e^{rt} + \frac{C(0)}{r-\gamma}\left(e^{rt} - e^{\gamma t}\right)\right] = \\ &= \lim_{t\to\infty} C(0)^{-\theta}\left[A_0 + \frac{Y}{r} + \frac{C(0)}{r-\gamma} - \frac{C(0)}{r-\gamma}e^{(\gamma-r)t}\right] = \\ &= C(0)^{-\theta}\left[A_0 + \frac{Y}{r} + \frac{C(0)}{r-\gamma}\right] = 0\end{aligned}$$

if and only if $C(0) = (r-\gamma)\left(A_0 + \dfrac{Y}{r}\right)$. Therefore the optimal consumption and asset holdings are

$$C^*(t) = (r-\gamma)\left(A_0 + \frac{Y}{r}\right)e^{\gamma t},\ t \in [0,\infty) \tag{6.24}$$

$$A^*(t) = -\frac{Y}{r} + \left(A_0 + \frac{Y}{r}\right)e^{\gamma t},\ t \in [0,\infty). \tag{6.25}$$

Observations: First, if we define human capital as the present value, at rate r, of the non-financial income

$$H(t) = \int_t^\infty Ye^{r(t-s)}ds$$

we find $H(0) = \dfrac{Y}{r}$. Therefore the solution is a linear function of the total capital, financial and non-financial

$$C^*(t) = (r-\gamma)(A_0 + H(0))e^{\gamma t},\ A^*(t) = -H(0) + (A_0 + H(0))e^{\gamma t}$$

Second, because $\gamma > 0$ then the asymptotic value of the optimal A becomes unbounded. However, it still satisfies that boundary condition $\lim_{t\to\infty} A^*(t)e^{-rt} = 0$ because, by assumption, $r > \gamma$. What matters is not the absolute level of A but its level in present-value terms.

6.4 Bibliography

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Chapter 7

Optimal control: the Pontriyagin's maximum principle

7.1 Introduction

The Pontriagin's maximum principle provides first order necessary conditions for the optimal control problem. We consider the same cases as we did for the calculus of variations problem

We denote again the independent variable by x and assume it has the domain $X \subseteq \mathbb{R}$. We can write $X = [x_0, x_1]$ if it is a closed set, or $X = (x_0, x_1)$ if it is open, where $x_0 < x_1$ are not necessarily known, and x_1 can be bounded or unbounded.

The optimal control problem contains two variables: the **state variable**, denoted by $y(x)$ (or $y(t)$) and the **control variable**, denoted by $u(x)$ (or $u(t)$). As we consider only problems in which the state variable is of dimension one, the state variable is a mapping $y : X \to Y \subseteq R$ and the control variable is a mapping $u : \mathrm{X} \to \mathrm{U} \subseteq R^m$. That is, we may have more m control variables.

The optimal control problem consists in finding functions $y \in \mathrm{Y}$ and $u \in \mathrm{U}$, where $\mathrm{Y} \in C^1(\mathbb{R})$, the set of continous and continuously differentiable functions $y : X \to\subseteq \mathbb{R}$, and $\mathrm{U} \in PC^1(\mathbb{R})$, the set of piecewise continuous functions $u : X \to U \subseteq \mathbb{R}^m$ such that

$$y' = G(y(x), u(x), x), \text{ for } x \in [x_0, x_1] \tag{7.1}$$

that maximize the functional

$$\mathsf{J}[y, u] \equiv \int_{x_0}^{x_1} F(x, y(x), u(x))\, dx \tag{7.2}$$

with additional data is given. The additional data is related to the information concerning the boundary values of the independent variable x_0 and x_1 and the boundary values for the state variable $y(x_0)$ and $y(x_1)$.

In most apoplications in macroeconomics and growth theory the independent variable is time. In this case $x = t$ and $t \in \mathrm{T} \subseteq \mathbb{R}_+$. In this case, the optimal control problem consists in finding functions $y \in \mathrm{Y}$ and $u \in \mathrm{U}$, where $\mathrm{Y} \in C^1(\mathbb{R})$, the set of continous and continuously differentiable functions $y : T \to Y \subseteq \mathbb{R}$, and $\mathrm{U} \in PC^1(\mathbb{R})$, the set of piecewise continuous functions $u : T \to U \subseteq \mathbb{R}^m$ such that

$$\dot{y} = G(y(t), u(t), t), \text{ for } t \in [t_0, t_1] \tag{7.3}$$

that maximize the functional

$$\mathsf{J}[y, u] \equiv \int_{t_0}^{t_1} F(t, y(t), u(t))dt \tag{7.4}$$

with additional data is given. The additional data is related to the information concerning the boundary values of the independent variable t_0 and t_1 and the boundary values for the state variable $y(t_0)$ and $y(t_1)$.

The necessary conditions for an optimum according to the **Pontriyagin's maximum principle** consider the **Hamiltonian** function, defined as

$$H(x, y, u, \lambda) = F(x, y, u) + \lambda G(x, y, u).$$

where λ, called the co-state variable, is a piecewise continuous mapping $\lambda : X \to \mathbb{R}$. When the independent variable is time, i.e, $X = T$, we write

$$H(t, y, u, \lambda) = F(t, y, u) + \lambda G(t, y, u).$$

where λ is a piecewise continuous function $\lambda : T \to \mathbb{R}$.

Next we present the optimality conditions for a bounded domain, not necessarily time, in section 7.2 and an example in section 7.3. Then we move to the time domain particular problems in section 7.4 and present several economic applications.

7.2 Bounded domain

In this subsection we assume that the data of the problem includes the boundary values for the independent variable: i.e., x_0 and x_1 are known. The optimal control problem is to find an optimal control $(u^*(x))_{x \in [x_0, x_1]}$ that maximizes the functional (7.2) subject to ODE constraint (**??**) and, possibly additional information for the state variables at the boundary values for the independent variable.

Formally, the problem is

$$\begin{aligned} \max_{u(\cdot)} \int_{x_0}^{x_1} & F(x, y(x), u(x))\, dx \\ & \text{subject to} \\ & y' = G(y(x), u(x), x), \text{ for } x \in [x_0, x_1] \\ & x_0 \text{ and } x_1 \text{ given} \\ & \text{conditions on}\, y(x_0) \text{ and } y(x_1) \end{aligned} \tag{Px}$$

We can consider the following cases:

(P1) both boundary values are known $y(x_0) = y_0$ and $y(x_1) = y_1$;

(P2) the lower boundary values is known $y(x_0) = y_0$ but $y(x_1)$ is free

(P3) the upper boundary values is known $y(x_1) = y_1$ but $y(x_0)$ is free

(P4) both boundary values $y(x_0)$ and $y(x_1)$ are free.

Proposition 1. *[**First order necessary conditions for fixed boundary values of the independent variable**] Let (y^*, u^*) be a solution to the OC problem Px in which one of the conditions (P1), or (P2), or (P3) or (P4) is introduced. Then there is a piecewise continuous function $\lambda : [x_0, x_1] \to \mathbb{R}$, called co-state variable, such that (y^*, u^*, λ) satisfy the following conditions:*

- *the optimality condition:*

$$H_u(x, y^*(x), u^*(x), \lambda(x)) = 0, \quad x \in [x_0, x_1] \tag{7.5}$$

- *the multiplier equation*

$$\lambda' = -H_y(x, y^*(x), u^*(x), \lambda(x)), \quad x \in (x_0, x_1) \tag{7.6}$$

- *the constraint of the problem:*

$$y^{*'} = G(x, y^*(x), u^*(x)), \; x \in (x_0, x_1) \tag{7.7}$$

- *and the adjoint conditions associated to the boundary conditions (P1) to (P4)*
  - *for problem (P1)*

$$y^*(x_0) = y_0 \textit{ for } x = x_0, \textit{ and } \; y^*(x_1) = y_1 \textit{ for } x = x_1, \tag{7.8}$$

  - *for problem (P2)*

$$y^*(x_0) = y_0 \textit{ for } x = x_0, \textit{ and } \; \lambda(x_1) = 0 \textit{ for } x = x_1, \tag{7.9}$$

  - *for problem (P3)*

$$\lambda(x_0) = 0 \textit{ for } x = x_0, \textit{ and } \; y^*(x_1) = y_1 \textit{ for } x = x_1, \tag{7.10}$$

  - *for problem (P4)*

$$\lambda(x_0) = 0 \textit{ for } x = x_0, \textit{ and } \; \lambda(x_1) = 0 \textit{ for } x = x_1. \tag{7.11}$$

Proof. (Heuristic) Let u^* be an optimal control and let y^* be the associated state. The value of the problem.

$$\mathsf{J}[y^*, u^*] = \int_{x_0}^{x_1} F(x, y^*(x), u^*(x))\, dx.$$

it is an optimiser if $\mathsf{J}[y^*, u^*] \geq \mathsf{J}[y, u]$ for any other admissible pair of functions (u, y).

It is convenient to write

$$\begin{aligned}
\mathsf{J}[y^*, u^*] &= \int_{x_0}^{x_1} F(x, y^*(x), u^*(x))dx = \\
&= \int_{x_0}^{x_1} \left[F(x, y^*(x), u^*(x)) + \lambda(x)(G(x, y^*(x), u^*(x)) - y^{*'}(x))\right] dx = \\
&= \int_{x_0}^{x_1} \left(H(x, y^*(x), u^*(x), \lambda(x)) - y^{*'}(x)\lambda(x)\right) \; dx
\end{aligned}$$

Again we introduce a perturbation on the optimal state-control pair $(y, u) = (y^*, u^*) + \varepsilon\;$, where ε is a constant and $\;= (\eta_y, \eta_u)$. The admissible perturbations differ for the different versions of the problem: for (P1) we should have $\eta_y(x_0) = \eta_y(x_1) = 0$, for (P2) we should have $\eta_y(x_0) = 0$ and $\eta_y(x_1) \neq 0$, for (P3) we should have $\eta_y(x_0) \neq 0$ and $\eta_y(x_1) = 0$, and for (P4) we should have $\eta_y(x_0) \neq 0$ and $\eta_y(x_1) \neq 0$.

The first-order Taylor approximation of the functional is

$$\mathsf{J}[y, u] = \mathsf{J}[y^*, u^*] + \delta\mathsf{J}[y^*, u^*](\;)\epsilon + o(\epsilon)$$

where

$$\begin{aligned}
\delta\mathsf{J}[y^*, u^*](\;) &= \int_{x_0}^{x_1} \left(H_u(x, y^*(x), u^*(x), \lambda(x))\eta_u(x) + H_y(x, y^*(x), u^*(x), \lambda(x))\, \eta_y(x) - \lambda(x)\eta_y'(x)\right) dx = \\
&= \int_{x_0}^{x_1} \left(H_u(x, y^*(x), u^*(x), \lambda(x))\, \eta_u(x) + (H_y(x, y^*(x), u^*(x), \lambda(x)) + \lambda'(x))\eta_y(x)\right) dt + \\
&+ \lambda(x_0)\eta_y(x_0) - \lambda(x_1)\eta_y(x_1).
\end{aligned}$$

Then $\mathsf{J}[y, u] \leq \mathsf{J}[y^*, u^*]$ only if $\delta\mathsf{J}[y^*, u^*](\;) = 0$, which, using similar arguments as to the case of the calculus of variations problem, is equivalent to the Pontriyagin's conditions: $H_u(.) = \lambda' - H_y(.) = 0$. The adjoint constraints should verify $\lambda(x_0)\eta_y(x_0) = \lambda(x_1)\eta_y(x_1) = 0$. From this and the admissibility values for $\eta_y(x_0)$ and $\eta_y(x_1)$ then the adjoint constraints are as in equations (7.8) to (7.11) □

7.2.1 Free domain and fixed boundary state variable optimal control problems

In this subsection we consider the case in which one or both bounds of the space of independent variables can be optimally choses, i.e $x \in X^* = [x_0^*, x_1^*]$, where one or both x_j^*, for $j = 0, 1$ are free, but the boundary values for the state variable are fixed: i.e. $y(x_0^*) = y_0$ and/or $y(x_1^*) = y_1$ are fixed. The optimal control problem is to find the optimal cut-off values for the indepdendent variable, x_0^* and/or x_1^* and an optimal control $\left(u^*(x)\right)_{x \in [x_0^*, x_1^*]}$ that maximizes the functional (7.2) subject to ODE constraint (**??**).

We can consider the following cases:

(P5) the lower boundary cut-off x_0 is known but the upper boundary x_1 is free

(P6) the upper boundary cut-off x_1 is known but the lower boundary x_0 is free

(P7) both boundary cut-off values x_0 and x_1 are free.

Proposition 2 (First order necessary conditions for free domain and fixed boundary state variable optimal control problems). *Let (y^*, u^*) be a solution to the OC problem where $y(x_0) = y_0$ and $y(x_1) = y_1$ are fixed. Then there is an optimal domain for the independent variable $x^* = [x_0^*, x_1^*] \subset \mathbb{R}$, a piecewise continuous function $\lambda : x^* \to \mathbb{R}$, called co-state variable, such that (y^*, u^*, λ) satisfy the optimality condition* (7.5)*, the multiplier equation* (7.6) *and the ODE constraint of the problem* (7.7)*, all for $x \in int(x^*)$ and the adjoint conditions associated to the boundary conditions (P5) to (P7)*

- *for problem (P5): $y^*(x_0) = y_0$ and $y^*(x_1^*) = y_1$ and*

$$x_0^* = x_0 \text{ and } H(x_1^*, y_1, u^*(x_1^*)) - y^{*'}(x_1^*)\lambda(x_1^*) = 0, \tag{7.12}$$

- *for problem (P6): $y^*(x_0^*) = y_0$ and $y^*(x_1) = y_1$ and*

$$H(x_0^*, y_0, u^*(x_0^*)) - y^{*'}(x_0^*)\lambda(x_0^*) = 0 \text{ and } x_1^* = x_1, \tag{7.13}$$

- *for problem (P7): $y^*(x_0^*) = y_0$ and $y^*(x_1^*) = y_1$ and*

$$H(x_0^*, y_0, u^*(x_0^*)) - y^{*'}(x_0^*)\lambda(x_0^*) = 0 \text{ and } H(x_1^*, y_1, u^*(x_1^*)) - y^{*'}(x_1^*)\lambda(x_1^*) = 0. \tag{7.14}$$

Proof. Using the same method for finding perturbations we used in the proof of propositions 6 and 1, the obtain the Gâteaux derivative

$$\begin{aligned}\delta \mathsf{J}[y^*, u^*; x^*]\ (\eta, \chi) = &\int_{x_0^*}^{x_1^*} \big(H_u(x, y^*(x), u^*(x), \lambda(x))\eta_u(x) + H_y(x, y^*(x), u^*(x), \lambda(x))\eta_y(x) - \lambda(x)\eta_y'(x)\big)\, dx + \\ &+ H(x, y^*(x), u^*(x), \lambda(x))\big|_{x=x_1^*} \chi_1 - H(x, y^*(x), u^*(x), \lambda(x))\big|_{x=x_0^*} \chi_0\end{aligned}$$

Setting $H^*(x) = H(x, y^*(x), u^*(x), \lambda(x))$, integrating by parts,

$$\begin{aligned}\delta \mathsf{J}[y^*, u^*; x^*]\ (\eta, \chi) = &\int_{x_0^*}^{x_1^*} \big(H_u^*(x)\eta_u(x) + \big(H_y^*(x) + \lambda'(x)\big)\,\eta_y(x)\big)\, dt + \lambda(x_0^*)\eta_y(x_0^*) - \lambda(x_1^*)\eta_y(x_1^*) + \\ &+ H^*(x_1^*)\chi_1 - H^*(x_0^*)\chi_0.\end{aligned}$$

Using the same approximation as in the proof of Proposition 6 yields the analogue to equation (6.15)

$$\begin{aligned}\delta \mathsf{J}[y^*, u^*; x^*]\ (\eta, \chi) = &\int_{x_0^*}^{x_1^*} \big(H_u^*(x)\eta_u(x) + \big(H_y^*(x) + \lambda'(x)\big)\,\eta_y(x)\big)\, dt + \lambda(x_0^*)\eta_0 - \lambda(x_1^*)\eta_1 + \\ &+ \Big(H^*(x_1^*) - y^{*'}(x_1^*)\lambda(x_1^*)\Big)\,\chi_1 - \Big(H^*(x_0^*) - y^{*'}(x_0^*)\lambda(x_0^*)\Big)\,\chi_0\end{aligned} \tag{7.15}$$

The adjoint necessary conditions for the optimum, because $\eta_1 = \eta_0 = 0$, are presented, for the different versions of the problem, in equations (7.12) to (7.14). □

7.2.2 Free domain and boundary state variable optimal control problems

This is a general case that encompasses combinations of all the previous cases: we assume both the domains of the independent variables and the boundary values of the state variables are free. That is x_0 and/or x_1 are unknown and $y(x_0)$ and/or $y(x_1)$ are also unknown and should be optimized. The optimal control problem is to find the optimal cut-off values for the indepdendent variable, x_0^* and/or x_1^* and an optimal control $(u^*(x))_{t\in[x_0^*,x_1^*]}$ that maximizes the functional (7.2) subject to ODE constraint (**??**) and having free boundary values for the state variable.

The necessary conditions include the optimality condition (7.5), the multiplier equation (7.6) and the ODE constraint of the problem (7.7), all for $x \in \text{int}(x^*)$. To get the adjoint condition associated to the terminal values of the state variable, when they need to be optimized, are obtained by setting in equation (7.15), $\eta_0 \neq 0$ and $\eta_1 \neq 0$. Therefore, the adjoint condition associated to $y^*(x_j^*)$ and $\lambda(x_j^*) = 0$, implying that the adjoint condition associated to the optimal boundary value of the independent variable, x_j^* is $H^*(x_j^*) = 0$, for $j = 0, 1$.

The adjoint conditions presented in Table 7.1 cover the same cases as the ones in Table 6.1 for the calculus of variations problem.

Table 7.1: Adjoint conditions for bounded domain OC problems

data		optimum	
x_j	$y(x_j)$	x_j^*	$y^*(x_j^*)$
fixed	fixed	x_j	y_j
fixed	free	x_j	$\lambda(x_j) = 0$
free	fixed	$H(x_j^*, y_j, u^*(x_j^*)) - y^{*'}(x_j^*)\lambda(x_j^*) = 0$	y_j
free	free	$H(x_j^*, y^*(x_j^*), u^*(x_j^*)) = 0$	$\lambda(x_j^*) = 0$

The index refers to the lower boundary when $j = 0$ and to the upper boundary when $j = 1$

7.3 Economic application: the Mirrlees (1971) model

In Mirrlees (1971) the optimal tax policy problem is addressed when the tax authority has **imperfect information**: it **observes** again both the consumption and the income distributions, $c(\theta)$ and $y(\theta)$, but it **does not observe** the individual productivity, θ, and the effort level of agents $\ell(\theta)$. This creates a problem for policy: a more productive agent may have an interest in reducing the income it reports by reducing its effort. If this is the case, the social welfare will be reduced because the total resources of the economy will be reduced, because, again the resource constraint

$$\int_\Theta \theta\, \tilde{\ell}\,(\theta)d\theta = \int_\Theta c(\theta)d\theta + G$$

should be satisfied, where $\tilde{\ell}(\theta)$ has a distortion generated by the tax policy relative to the perfect information case. This problem creates an **information friction** in the derivation of the optimal tax policy.

The Mirrlees (1971) paper was one of the first papers in the mechanism design literature that adresses principal-agent problems in contexts of imperfect information.

The policy problem is to find

$$\max_{l(\theta)\in(0,1)} \int_{\underline{\theta}}^{\bar{\theta}} W[u(\theta)]\ f(\theta)d\theta \tag{7.16}$$

where the skill domain is $\Theta = [\underline{\theta}, \bar{\theta}]$, subject to the following constraints

$$\int_{\underline{\theta}}^{\bar{\theta}} [\ \theta\ell(\theta) - C(u(\theta), \ell(\theta)]\ f(\theta)d\theta \geq G \tag{7.17a}$$

$$\frac{du}{d\theta} = -\frac{\ell(\theta)u_{\ell}(\theta)}{\theta} \tag{7.17b}$$

$$\underline{\theta}, \bar{\theta} \text{ free} \tag{7.17c}$$

$$u(\underline{\theta}), u(\bar{\theta}) \text{ free.} \tag{7.17d}$$

Equation (7.17a) is the **resource constraint**, equation (7.17b) is the **incentive compatibility constraint**. The constraints (7.17c) and (7.17d) are introduced to account for the fact that the tax authority limits and levels of taxes at both ends of the skill distribution should be optimally derived. This means that there can be upper or lower extremes of the skill distribution that are not taxed.

This is a control problem with state variable $u(\theta)$ and control variable $\ell(\theta)$, and with free boundary values for both the independent and the dependent state variable (see Table 7.1). We have to introduce two types of adjoint variables: λ is skill-independent and is associated to constraint (7.17a), and $h(\theta)$ is skill-dependent and is associated to state variable $u(\theta)$. The Hamiltonian is

$$\begin{aligned} H(\theta) &= H(\theta, \lambda, y(\theta), u(\theta), h(\theta)) \equiv \\ &\equiv \{W[u(\theta)] - \lambda\left(C\left(u(\theta), \ell(\theta)\right) - \theta\,\ell(\theta)\right)\}\ f(\theta) - h(\theta)\frac{\ell(\theta)}{\theta}u_{\ell}\left(C\left(u(\theta), \ell(\theta)\right), \ell(\theta)\right) \end{aligned}$$

Next we present the conditions for an interior solution, i.e., for $0 < \ell^*(\theta) < 1$. The static optimality condition $H^*_{\ell}\ (\theta) = 0$ (see equation (**??**)) yields the optimal distribution of income

$$\lambda\left(C^*_{\ell}(\theta) - \theta\right)\ f(\theta) = \frac{h(\theta)}{\theta}\left[u^*_{\ell}(\theta) + \ell(\theta)\left(u^*_{c\ell}(\theta) + u^*_{\ell\ell}(\theta)\right)\right],\quad \theta \in [\underline{\theta}^*, \bar{\theta}^*]. \tag{7.18}$$

Again, we denote $C^*_j(\theta) \equiv C_j\left(u^*(\theta), \ell^*(\theta)\right)$, for $j = u, \ell$, $u^*_{\ell}(\theta) \equiv u_{\ell}\left(C\left(u^*(\theta), \ell^*(\theta)\right), \ell^*(\theta)\right)$ and analogously for the higher order derivatives of the utility function $u(.)$.

The Euler equation $h^{'}(\theta) + H^*_u(\theta) = 0$ yields the change in the value of the utility along the skill distribution

$$\frac{dh(\theta)}{d\theta} = \left(\lambda C^*_u(\theta) - W^{'}[u^*(\theta)]\right) f(\theta) + \left(\frac{\ell^*(\theta)}{\theta}u^*_{\ell c}(\theta)C^*_u(\theta)\right) h(\theta),\ \theta \in [\underline{\theta}^*, \bar{\theta}^*]. \tag{7.19}$$

The optimal conditions associated to the limit values for households' utility in the two limits of the skill distribution, $u^*(\underline{\theta})$ and $u^*(\bar{\theta})$ satisfy

$$h(\bar{\theta}) = h(\underline{\theta}) = 0 \tag{7.20}$$

and the optimal cutoff-values for skill distribution which is taxable, $\underline{\theta}^*$ and $\bar{\theta}^*$, are

$$H^*(\theta^*) = h(\theta^*)u^{'}(\theta^*), \text{ for } \theta^* = \underline{\theta}^*, \bar{\theta}^* \tag{7.21}$$

The admissibility conditions (7.17a) and (7.17b) should also hold for $\ell(\theta) = \ell^*(\theta)$ and $u(\theta) = u^*(\theta)$.

We see that the information friction introduces a skill-varying change when we compare to the analogous first-order conditions for the perfect information problem :

$$\begin{aligned} C^*_\ell(\theta) - \theta &= \frac{h(\theta)}{\lambda\theta f(\theta)}\left[u^*_\ell(\theta) + \ell(\theta)\left(u^*_{c\ell}(\theta) + u^*_{\ell\ell}(\theta)\right)\right] \\ \lambda C^*_u(\theta) - W^{'}[u^*(\theta)] &= \frac{1}{f(\theta)}\left(\frac{h(\theta)}{d\theta} - \left(\frac{\ell^*(\theta)}{\theta}u^*_{\ell c}(\theta)C^*_u(\theta)\right)h(\theta)\right) \end{aligned}$$

In addition, optimality conditions (7.20) and (7.21) constrain the range of taxable income and the level of taxes at the two extremes of the skill distribution.

A little more intuition on the characterization of the optimal redistribution problem is gained by using the utility function assumed by Diamond (1998): $u(c,\ell) = c + v(1-\ell)$ where $v^{'}(.) > 0$ and $v^{''} < 0$. This utility function simplifies calculations by assuming there are no income effects associated to changes in taxes [1]. With this utility function the elasticity of labor supply, for skill-level θ is

$$\epsilon(\theta) = -\frac{v^{''}(1-\ell(\theta))\,\ell(\theta)}{v^{'}(1-\ell(\theta))}.$$

With this utility function, the first order condition (7.18) becomes

$$\lambda\left(v^{'}(1-\ell^*(\theta)) - \theta\right)f(\theta) = \frac{h(\theta)}{\theta}\left(v^{'}(1-\ell^*(\theta)) \; - \ell^*(\theta)v^{''}(1-\ell^*(\theta))\;\right), \text{ for } \theta \in [\underline{\theta}^*, \bar{\theta}^*], \tag{7.22}$$

and condition (7.19) becomes

$$h^{'}(\theta) \equiv \frac{dh(\theta)}{d\theta} = -\left(W^{'}[u^*(\theta)] - \lambda\right)f(\theta), \text{ for } \theta \in [\underline{\theta}^*, \bar{\theta}^*]. \tag{7.23}$$

This is an ordinary differential equation, which can be solved together with the terminal optimality conditions (7.20). Then, [2], (7.20),

$$h(\theta) = \int_\theta^{\bar{\theta}}\left(W^{'}[u^*(s)] - \lambda\right)f(s)ds = \int_\theta^{\bar{\theta}}\left(W^{'}[u^*(s)] - \lambda\right)dF(s),$$

is a balance equation between the utility of agents of type θ and the net benefit of reducing utility for agents with skill higher than θ.

[1] Saez (2001) proves that introducing income effects do not change qualitatively the results.

[2] From now on we delete the $*$ symbol in functions $\ell^*(\theta)$ and $u^*(\theta)$ and in numbers $\underline{\theta}^*$ and $\bar{\theta}^*$.

Substituting in equation (7.22) yields

$$\lambda\left(\theta - v^{'}(1-\ell(\theta))\right)f(\theta) = \left(\frac{v^{'}(1-\ell(\theta)) - \ell(\theta)v^{''}(1-\ell(\theta))}{\theta}\right)\int_{\theta}^{\bar{\theta}}(\lambda - W^{'}(s))dF(s).$$

Using the definition of the elasticity of labor supply, and rearranging terms we get the well known expression (see Diamond (1998) and (Tuomala, 2016, ch. 4))

$$\frac{\theta - v^{'}(1-\ell(\theta))}{v^{'}(1-\ell(\theta))} = A(\theta)\,B(\theta)\,C(\theta) \tag{7.24}$$

where

$$A(\theta) \equiv 1 + \frac{1}{\epsilon(\theta)}$$

$$B(\theta) \equiv \frac{\int_{\theta}^{\bar{\theta}}\left(\lambda - W^{'}[u(s)]\right)dF(s)}{\lambda(1-F(\theta))}$$

$$C(\theta) \equiv \frac{1-F(\theta)}{\theta f(\theta)}$$

Equation (7.24) basically says that the ratio of the optimal tax policy should equate the marginal rate of substitution between consumption and labor supply, for an agent of skill θ to the product of three terms: the deadweight burden generated by the income tax to people of skill θ ($A(\theta)$), the relative transfer of income from people with higher skills than θ ($B(\theta)$), and the weight of people with higher skills relative to the average skills of people with skill θ ($C(\theta)$).

7.4 Time domain problems

Next we present two problems, usually cast in the time domain, the constrained terminal state problem and the discounted infinite horizon problem.

In this case the objective functional is (7.4) and the constraint is the ordinary differential equation (7.4.1)

7.4.1 Constrained terminal state problem

A common problem in macroeconomics is the following: the set of independent variables is known such as $t_0 = 0$ and $t_1 = \bar{t}$, the initial value of the state value is fixed, $y(0) = y_0$, the structure of the economy given by the ODE (7.4), and value functional , and we assume that the terminal value for the state variable is constrained by $R(\bar{t}, y(\bar{t})) \geq 0$ where $y(\bar{t})$ is free.

Formally the problem is

$$\begin{aligned}
&\max_{u(\cdot)} \int_0^{\bar{t}} F(t, u(t), y(t))\, dt \\
&\text{subject to} \\
&\dot{y} = G(t, u(t), y) \\
&\bar{t} \text{ given} \\
&y(0) = y_0 \text{ given} \\
&R(\bar{t}, y(\bar{t})) \geq 0
\end{aligned} \tag{Pt1}$$

The Hamiltonian function is

$$H(t, u, y, \lambda) = F(t, u, y) + \lambda\, G(t, u, y).$$

Proposition 3. ***1st order necessary conditions for the constrained terminal value problem*** *Let (y^*, u^*) be the solution for problem Pt1. Then it satisfies*

- *the optimality condition*

$$H_u(t, u^*, y^*(t), \lambda(t)) = 0, \text{ for every } t \in [0, \bar{y}];$$

- *the multiplier equation*

$$\dot{\lambda} = \; - H_y(t, u^*, y^*(t), \lambda(t)) = 0, \text{ for every } t \in (0, \bar{y})$$

- *the transversality condition*

$$\lambda(\bar{t}) R(\bar{t}, y^*(\bar{t})) = 0,$$

and $y^*(0) = y_0$.

Proof. In this case the value at the optimum is

$$\mathsf{J}[y^*, u^*] = \int_0^{\bar{t}} \left(H(t, u^*(t), y^*(t), \lambda(t)) - \dot{y}^*(t)\lambda(t)\right)\; dt + \psi R(\bar{t}, y(\bar{t}))$$

where ψ is a Lagrange multiplier. The functional derivative, for an arbitrary perturbation $(\delta y, \delta u) = \varepsilon(\eta_y, \eta_u)$ around (y^*, u^*), is now

$$\begin{aligned}
\delta \mathsf{J}[y^*, u^*](\eta_y, \eta_u) = &\int_0^{\bar{t}} \Big[H_u(t, y^*(t), u^*(t), \lambda(t))\eta_u(t) + (H_y(t, y^*(t), u^*(t), \lambda(t)) + \dot{\lambda}(t))\eta_y(t)\Big]\, dt + \\
&+ \lambda(0)\eta_y(0) + \left(\psi R_y(\bar{t}, y^*(\bar{t})) - \lambda(\bar{t})\right) \eta_y(\bar{t}),
\end{aligned}$$

where admissible perturbations satisfy $\eta_y(0) = 0$ and $\eta_y(\bar{t}) \neq 0$. Given the inequality constraint, the KKT conditions

$$R(\bar{t}, y^*(\bar{t})) \geq 0,\ \psi \geq 0,\ \psi R(\bar{t}, y^*(\bar{t})) = 0,$$

are also necessary for an optimum. Setting $H_u^*(t) = \dot{\lambda}\ (t) - H_y^*(t) = \eta_y(0) = 0$, and because $\eta_y(\bar{t}) \neq 0$, the remaining necessary condition for an optimum is

$$\psi R_y(\bar{t}, y^*(\bar{t})) - \lambda(\bar{t}) = 0,$$

which, multiplying both terms by $R(\bar{t}, y^*(\bar{t}))$ and using the KKT condition yields the remaining adjoint condition $\lambda(\bar{t})R(\bar{t}, y^*(\bar{t})) = 0$. □

7.4.2 Infinite horizon problems

The benchmark problem in macroeconomics and growth theory is the (autonomous) **discounted infinite horizon problem** has the constraint

$$\dot{y} = g(y(t), u(t)),\ \text{for } t \in [0, \infty) \tag{7.25}$$

instead of (7.2), and $y(0) = y_0$ and alternative boundary conditions

$$\lim_{t\to\infty} y(t) \text{ is free or } \lim_{t\to\infty} y(t) \geq 0$$

The value functional is

$$\mathsf{J}[y, u] \equiv \int_0^\infty e^{-\rho t} f(y(t), u(t))dt.$$

Now, we define the **current-value Hamiltonian** function

$$\begin{aligned} h(y(t), u(t), q(t)) &= f(y(t), u(t)) + q(t)g(y(t), u(t)) = \\ &= e^{-\rho t} H(t, y(t), u(t), \lambda(t)). \end{aligned}$$

where $q(t) = e^{\rho t}\lambda(t)$ is the current-value co-state variable. Consistently with the previous definitions we call **discounted Hamiltonian** and **discounted co-state variable** to $H(t, y, u, \lambda)$ and λ.

Observe that the current-value Hamiltonian is time-independent.

Proposition 4 (**First order necessary conditions: Pontriyagin maximum principle**). *Let (y^*, u^*) be the optimal state and control pair. Then there is a PC^1 continuous co-state variable q such that the following conditions hold:*

- *the optimality condition*

$$h_u(y^*(t), u^*(t), q(t)) = 0,\ t \in [0, \infty)$$

- *the multipliers equation for the current co-state variable (also called canonical equation)*

$$\dot{q} = \rho q - h_y(y^*(t), u^*(t), q(t)),\ t \in [0, \infty)$$

- *the transversality condition for the free or the constrained terminal state*

$$\lim_{t\to\infty} e^{-\rho t} q(t) = 0,\ \textit{or} \lim_{t\to\infty} e^{-\rho t} q(t) y^*(t) = 0$$

- *and the admissibility conditions*

$$\begin{cases} \dot{y}^* = g(y^*(t), u^*(t)), & t \in [0, \infty) \\ y^*(0) = y_0 & t = 0 \end{cases}$$

7.4.3 The Modified Hamiltonian Dynamic System

In regular cases we can have a geometric interpretation for the solution of an optimal control problem. The necessary conditions for the infinite-horizon discounted optimal control problem, feature a differential-algebraic system:

$$\begin{aligned} \dot{y} &= g(y, u) \\ \dot{q} &= \rho q - h_y(u, y, q). \\ 0 &= h_u(u, y, q) \end{aligned} \tag{7.26}$$

where $h(u, y, q) = f(u, y) + q\, g(u, y)$. Therefore, $h_q(u, y, q) = g(u, y)$.

If functions $f(.)$ and $g(.)$ are sufficiently smooth we may qualitative characterize the optimal path for (y, q) (or for (u, y)).

If $\partial^2 h/\partial u^2 \neq 0$, the implicit function theorem allows for obtaining from the optimality condition for u, $h_u(u, y, q) = 0$, an implicit representation of the control as a function of the state and co-state variables $u = u(y, q)$. If we substitute this control representation in the differential equations of (7.26) we obtain the **modified Hamiltonian dynamic system** (MHDS) as a non-linear planar ODE,

$$\begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \mathbf{M}\ (y, q) \equiv \begin{pmatrix} g(u(y, q), y) \\ \rho q - h_y(u(y, q), y, q). \end{pmatrix} \tag{7.27}$$

Assume there is one steady state for the MHDS, $(\bar{y}, \bar{q}) = \{(y, q) : \dot{y} = \dot{q} = 0\}$. In the neighbourhood of $(\bar{y}, \bar{q})$ we can approximate the non-linear MHDS (7.27) by the linear system

$$\begin{pmatrix} \dot{y}(t) \\ \dot{q}(t) \end{pmatrix} = D_{(y,q)}\mathbf{M}(\bar{y}, \bar{q}) \begin{pmatrix} y(t) - \bar{y} \\ q(t) - \bar{q} \end{pmatrix}$$

where the Jacobian, evaluated at the steady state $(\bar{y}, \bar{q})$ is the matrix of constants

$$D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q}) = \begin{pmatrix} \frac{\partial \dot{y}(\bar{y},\bar{q})}{\partial y} & \frac{\partial \dot{y}(\bar{y},\bar{q})}{\partial q} \\ \frac{\partial \dot{q}(\bar{y},\bar{q})}{\partial y} & \frac{\partial \dot{q}(\bar{y},\bar{q})}{\partial q} \end{pmatrix}.$$

If functions $f(.)$ and $g(.)$ have no singularities we can obtain a generic characterization of the dynamics of the MHDS, and, therefore, of the solution to the optimal control problem.

Proposition 5. *Let there be a steady state for the MHDS system. It can never be locally a stable node or focus, and if there is transitional dynamics it can only be a saddle-point.*

Proof. The differential of the current value Hamiltonian, $h = f(u, y) + qg(u, y)$, is

$$dh = (f_u + qg_u)du + (f_y + qg_y)dy + gdq.$$

At the optimum $h_u(u, y, q) = f_u(u, y) + qg_u(u, y) = 0$. Taking the differential to this static optimality condition, we have

$$h_{uu}du + h_{uy}dy + g_u dq = 0$$

and if $h_{uu} \neq 0$, by the implicit function theorem, function $u = u(y, q)$ has derivatives

$$u_y = -\frac{h_{uy}}{h_{uu}} \ , \ u_q = -\frac{g_u}{h_{uu}}.$$

Now, we can determine the Jacobian for matrix $\mathbf{M}$, evaluated at any optimum pair (y, q). The differential of the first row of $\mathbf{M}$ is

$$dg(y, u(y, q)) = \left(g_y - g_u \frac{h_{uy}}{h_{uu}}\right) dy - g_u \frac{g_u}{h_{uu}} dq$$

and the differential of the second row is

$$\rho dq - dh_y(y, u(y, q)) = -\left(h_{yy} - h_{yu}\frac{h_{uy}}{h_{uu}}\right) dy + \left(\rho - g_y + h_{yu}\frac{g_u}{h_{uu}} +\right) dq.$$

Evaluating the derivatives at the steady state $(\bar{y}, \bar{q})$, with $\bar{u} = u(\bar{y}, \bar{q})$, we find that the Jacobian matrix has the following structure

$$D_{(y,q)}\mathbf{M}\ (\bar{y}, \bar{q}) = \begin{pmatrix} \bar{g}_y - \frac{\bar{g}_u \bar{h}_{uy}}{\bar{h}_{uu}} & -\frac{(\bar{g}_u)^2}{\bar{h}_{uu}} \\ -\bar{h}_{yy} + \frac{(\bar{h}_{uy})^2}{\bar{h}_{uu}} & \rho - \bar{g}_y + \frac{\bar{g}_u \bar{h}_{uy}}{\bar{h}_{uu}} \end{pmatrix}$$

where $\bar{g}_y = g(u(\bar{y}, \bar{q}), \bar{y})$, etc[3]. Observe that the Jacobian matrix has a particular structure

$$D_{(y,q)}\mathbf{M}(\bar{y}, \bar{q}) = \begin{pmatrix} a & b \\ c & \rho - a \end{pmatrix}.$$

[3]because if $h(.)$ is continuous then $h_{uy}(.) = h_{yu}(.)$.

impiying that the trace is equal to the rate of time preference,

$$\text{trace}\,(D_{(q,y)}\mathbf{M}(\bar{y},\bar{q})) = \rho > 0$$

and is always positive and

$$\det\,(D_{(q,y)}\mathbf{M}(\bar{y},\bar{q})) = a(\rho - a) - bc.$$

This implies that, if there is a steady state, it can never a stable node or focus. Therefore, it can be an unstable node or focus if $\det\,(D_{(q,y)}\mathbf{M}(\bar{y},\bar{q})) > 0$, a saddle-point if $\det\,(D_{(q,y)}\mathbf{M}(\bar{y},\bar{q})) < 0$ or a degenerate saddle node if $\det\,(D_{(q,y)}\mathbf{M}(\bar{y},\bar{q})) = 0$. There can only be transitional dynamics if it is a saddle-point. □

Then we can conclude the following:

1. in generic cases the equilibrium point $(\bar{y},\bar{q})$ is a saddle point. The stable manifold associated with $(\bar{y},\bar{q})$
$$W^s = \{\ (y,q) : \lim_{t\to}(y(t),q(t)) = (\bar{y},\bar{q})\}$$
passing through point $y(0) = y_0$ **is the solution set of the OC problem**;

2. this means that the solution to the OC problem is (at least locally) unique;

3. the optimal trajectory is asymptotically tangent to the stable eigenspace E^s associated to Jacobian $D_{(q,y)}\mathbf{M}(\bar{y},\bar{q})$

7.5 Economic applications

We consider the same problems as in the calculus of variations section.

7.5.1 Two simple problems

Example 1: Resource depletion problem

The (non-renewable) resource depletion problem can now be solved by using the Pontriyagin's principle. Recall that the problem is

$$\max_{C} \int_0^{\infty} e^{-\rho t} \ln\,(C(t))dt,\ \rho > 0$$

subject to

$$\begin{cases} \dot{W}(t) = -C(t),\ t \in [0,\infty) \\ W(0) = W_0,\ \text{given} \\ \lim_{t\to\infty} W(t) \geq 0. \end{cases}$$

In this problem, the control variable is consumption, C, and the state variable is the remaining level of the resource, W. What is the best path for consumption-depletion ?

For applying the Pontryiagin maximum principle we write the current-value Hamiltonian

$$h = ln(C) - qC.$$

The first order conditions are

$$\begin{aligned} C(t) &= 1/q(t) \\ \dot{q} &= \rho q(t) \\ \lim_{t\to\infty} e^{-\rho t} q(t) W(t) &= 0 \\ \dot{W} &= -C(t) \\ W(0) &= W_0 : \end{aligned}$$

and can be written as a planar differential equation in (W, C), together with the initial and the transversality condition is

$$\begin{aligned} \dot{W} &= -C(t) \\ \dot{C} &= -\rho C(t) \\ W(0) &= W_0 \\ \lim_{t\to\infty} e^{-\rho t} \frac{W(t)}{C(t)} &= 0 \end{aligned}$$

If we want to find the solution we must solve the system, together with the conditions on time.

There are several ways to solve it. Here is a simple one. First, define $z(t) \equiv W(t)/C(t)$. Time-differenting and substituting, we get the scalar terminal-value problem

$$\begin{cases} \dot{z} = -1 + \rho z \\ \lim_{t\to\infty} e^{-\rho t} z(t) = 0 \end{cases}$$

which has a constant solution $z(t) = \frac{1}{\rho}$ for every $t \in [0, \infty)$. Second, substitute $C(t) = W(t)/z(t) = \rho W(t)$. therefore,

$$\begin{cases} \dot{W} = -C(t) = -\rho W(t) \\ W(0) = W_0 \end{cases}$$

Then $W^*(t) = W_0 e^{-\rho t}$ for $t \in [0, \infty)$ and $C^*(t) = \rho W^*(t)$.

Characterization of the solution: there is asymptotic extinction

$$\lim_{t\to\infty} W^*(t) = 0,$$

at a speed given by the half-life of the process

$$\tau \equiv \left\{ t : \ W^*(t) = \frac{W(0) - W^*(\infty)}{2} \right\} = -\frac{\ln(1/2)}{\rho}$$

if $\rho = 0.02$ then $\tau \approx 34.6574$ years.

Example 2: the consumption-savings problem

Problem: find the (A, C) pair that maximizes the functional

$$\max_C \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta} - 1}{1-\theta} dt,\ \rho > 0$$

subject to

$$\begin{cases} \dot{A}(t) = Y - C(t) + rA,\ t \in [0, \infty) \\ A(0) = A_0,\ \text{given} \\ \lim_{t\to\infty} A(t)^{-rt} \geq 0. \end{cases}$$

In this problem, the control variable is consumption, C, and the state variable is the level of net wealth, A. The current value Hamiltonian is

$$h(A, C, Q) = \frac{C^{1-\theta}}{1-\theta} + Q(Y - C + rA)$$

and the first order conditions according to the Pontiyagin's principle are

$$\begin{cases} C(t)^{-\theta} = Q(t) \\ \dot{Q} = Q(\rho - r) \\ \dot{A} = Y - C + rA \\ A(0) = A_0 \\ \lim_{t\to\infty} Q(t)A(t)e^{-\rho t} = 0 \end{cases}$$

As

$$\frac{\dot{Q}}{Q} = -\theta \frac{\dot{C}}{C}$$

we can obtain the solution by solving the mixed initial-terminal value problem for ODE's

$$\begin{cases} \dot{A} = Y - C + rA \\ \dot{C} = \gamma C \\ A(0) = A_0 \\ \lim_{t\to\infty} C(t)^{-\theta} A(t)e^{-\rho t} = 0 \end{cases}$$

where again $\gamma \equiv \frac{r-\rho}{\theta}$. We present and discuss next the solution to this problem.

7.5.2 Qualitatively specified problems

Next we present a general Ramsey (1928) model in which the behavioral functions are qualitatively specified. This allows us to study the qualitative solution to the optimal control problem.

The Ramsey problem is:

$$\max_C \int_0^\infty e^{-\rho t} U(C(t))dt,\ \rho > 0,$$

subject to

$$\dot{K}(t) = F(K(t)) - C(t),\ t \in [0, \infty)$$

$K(0) = K_0$ given and $\lim_{t\to\infty} e^{-\rho t} K(t) \geq 0$. We also assume that $(K, C) : \mathbb{R}_+ \to \mathbb{R}_+^2$. In this problem the control variable is C and the state variable is the stock of capital K.

The utility and the production functions, $u(C)$ and $F(K)$, are usually assumed to have the following properties: Increasing, concave and Inada :

$$U'(.) > 0,\ U''(.) < 0, F'(.) > 0,\ F''(.) < 0$$

$$U'(0) = \infty,\ U'(\infty) = 0,\ F'(0) = \infty,\ F'(\infty) = 0.$$

Although we do not have explicit utility and production functions we can still characterize the optimal consumption-accumulation process (we are using the Grobman-Hartmann theorem).

The current-value Hamiltonian is

$$h(C, K, Q) = U(C) + Q(F(K) - C)$$

The necessary (and sufficient) conditions according to Pontriyagin's maximum principle are

$$\begin{aligned}
U^{'}(C(t)) &= Q(t) \\
\dot{Q} &= Q(t)\left(\rho - F^{'}(K(t))\right) \\
\lim_{t\to\infty} e^{-\rho t} Q(t) K(t) &= 0 \\
\dot{K} &= F(K(t)) - C(t) \\
K(0) &= K_0
\end{aligned}$$

The MHDS and the initial and transversality conditions become

$$\begin{aligned}
\dot{K} &= F(K(t)) - C(t) \\
\dot{C} &= \frac{C(t)}{\theta(C(t))}\left(F^{'}(K(t)) - \rho\right) \\
K(0) &= K_0 > 0 \\
0 &= \lim_{t\to\infty} e^{-\rho t} U^{'}(C(t)) K(t)
\end{aligned}$$

where $\theta(C) = -\frac{U^{''}(C)C}{U^{'}(C)} > 0$ is the inverse of the elasticity of intertemporal substitution.

The MHDS has no explicit solution (it is not even explicitly defined) : we can only use **qualitative methods**. They consist in:

- determining the steady state(s): $(\bar{C}, \bar{K})$
- characterizing the linearised dynamics (it is useful to build a phase diagram).

The steady state (if $K > 0$) is

$$\begin{aligned} F^{'}(\bar{K}) &= \rho \Rightarrow \bar{K} = (F^{'})^{-1}(\rho) \\ \bar{C} &= F(\bar{K}) \end{aligned}$$

The linearized MHDS is

$$\begin{pmatrix} \dot{K} \\ \dot{C} \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \frac{\bar{C}}{\theta(\bar{C})} F^{''}(\bar{K}) & 0 \end{pmatrix} \begin{pmatrix} K(t) - \bar{K} \\ C(t) - \bar{C} \end{pmatrix}$$

where we denote $D\mathbf{M}$ the Jacobian matrix. The jacobian J has trace and determinant:

$$\text{tr}(D\mathbf{M}) = \rho, \; \det{(D\mathbf{M})} = \frac{\bar{C}}{\theta(\bar{C})} F^{''}(\bar{K}) < 0$$

the steady state $(\bar{C}, \bar{K})$ is a saddle point. The eigenvalues of $D\mathbf{M}$ are

$$\lambda_s = \frac{\rho}{2} - \sqrt{\Delta} \; < 0, \; \lambda_u = \frac{\rho}{2} + \sqrt{\delta} \; > \rho > 0$$

where the discriminant is

$$\Delta = \; \left(\frac{\rho}{2}\right)^2 - \frac{\bar{C}}{\theta(\bar{C})} F^{''}(\bar{K}) > \left(\frac{\rho}{2}\right)^2.$$

and the eigenvector matrix of $D\mathbf{M}$ is

$$\mathbf{P} \; = (\mathbf{P}^s \mathbf{P}^u) = \begin{pmatrix} 1 & 1 \\ \lambda_u & \lambda_s \end{pmatrix}$$

Then the approximate solution for the Ramsey problem, in the neighbourhood of the steady state, is

$$\begin{pmatrix} K^*(t) \\ C^*(t) \end{pmatrix} = \begin{pmatrix} \bar{K} \\ \bar{C} \end{pmatrix} + K_0 \begin{pmatrix} 1 \\ \lambda_u \end{pmatrix} e^{\lambda_s t}, \; t \in [0, \infty) \tag{7.28}$$

Then the local stable manifold has slope higher than the isocline $\dot{K}(C, K) = 0$

$$\left.\frac{dC}{dK}\right|_{W^s} = \lambda_u > \left.\frac{dC}{dK}\right|_{\dot{K}} = F^{'}(\bar{K}) = \rho$$

Geometrically (see figure 7.1) the **approximate** solution (7.28) belongs to the stable sub space E^s

$$E^s = \{ \; (K, C) : (C - \bar{C}) = \lambda_u (K - \bar{K})\}$$

while the **exact** solution belongs to the stable manifold W^s (which cannot be determined explicitly). Observe that while the slope of the isocline in the neighborhood of the steady is flatter then the slope of the stable manifold

$$\left.\frac{dC}{dK}\right|_{\dot{k}=0} = F^{'}(\bar{K}) = \rho < \left.\frac{dC}{dK}\right|_{W^s} = \lambda_u$$

meaning that the solution approaches the steady state by accumulating (reducing) capital is the initial capital level is smaller (bigger) than the steady state level.

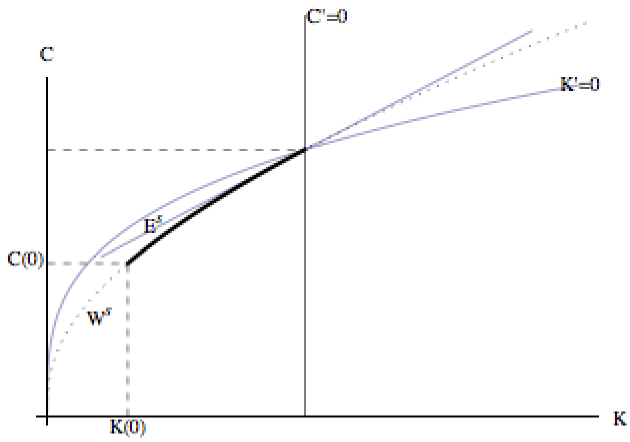

Figure 7.1: The phase diagram for the Ramsey model: it depicts the isoclines $\dot{C} = 0$ and $\dot{K} = 0$, the stable manifold W^s and the stable eigenspace, E^s, which is tangent asymptotically to the stable manifold. The exact solution follows along the stable manifold, but we have determined just the approximation along the stable eigenspace.

7.5.3 Unbounded solutions

In the previous section we saw that if the solution converges to a steady state we can have a qualitative characterization of the solution appealing to the Grobman-Hartman theorem. However, in some cases, in particular in endogenous growth theory models, solutions may not converge to a steady state, or the solution which interests us can be unbounded in time.

In particular, the consumer-saver problem may have an unbounded solution. In the next chapter we will study the AK model.

If we write the MHDS in the (A, Q) space, we have

$$\begin{cases} \dot{A} = Y - Q^{-\frac{1}{\theta}} + rA \\ \dot{Q} = Q(\rho - r) \end{cases}$$

the solution of the optimal control problem are the solutions of that MHDS together with the inicial and transversality conditions

$$A(0) = a_0, \ \lim_{t\to\infty} Q(t)A(t)e^{-\rho t} = 0.$$

There are two interesting cases. First, if $r = \rho$ then there is an infinity of stationary solutions satisfying $Q^{-\frac{1}{\theta}} = Y + rA$. Second, if $r \neq \rho$ it has no steady state in $\mathbb{R}$. To see this note that, $\dot{Q} = 0$ if and only if $Q = 0$ but then $\dot{A} = 0$ can only be reached asymptotically when $A \to \infty$.

We can have a clearer characterization if we recast the problem in the (A, C) spac. Recall that in this case we have the MHDS

$$\begin{cases} \dot{A} = Y - C + rA \\ \dot{C} = \gamma C, \end{cases}$$

where

$$\gamma \equiv \frac{r-\rho}{\theta},$$

which, for the moment, we assume has an ambiguous sign.

The solution of the optimal control problem are the solutions of that MHDS together with the inicial and transversality conditions

$$\begin{cases} A(0) = a_0, \\ \lim_{t\to\infty} C(t)^{-\frac{1}{\theta}} A(t) e^{-\rho t} = 0. \end{cases}$$

The MHDS is linear planar ODE with coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} r & -1 \\ 0 & \gamma \end{pmatrix}$$

that has eigenvalues

$$\lambda_- = \gamma, \ \lambda_+ = r > 0.$$

and has eigenvector matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ r-\gamma & 0 \end{pmatrix}$$

The solution to the MHDS is, for $\gamma \neq 0$

$$\begin{pmatrix} A(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} -\frac{Y}{r} \\ 0 \end{pmatrix} + h_- \begin{pmatrix} 1 \\ r-\gamma \end{pmatrix} e^{\gamma t} + h_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{rt}.$$

For later use, observe that the trajectories starting from $A(0) = a_0$ and travelling along the eigenspace associated to eigenvalue λ_- are

$$\begin{pmatrix} A(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} -\frac{Y}{r} \\ 0 \end{pmatrix} + (A_0 + \frac{Y}{r}) \begin{pmatrix} 1 \\ r-\gamma \end{pmatrix} e^{\gamma t}.$$

that is

$$\mathbb{E}^- = \left\{ (A, C) \in \mathbb{R} \times \mathbb{R}_+ : \ C = (r-\gamma)\left(A + \frac{Y}{r}\right)\right\}.$$

We saw that the only requirement for the transversality condition to be me, and therefore for the optimal control problem to have a solution was $r > \gamma$. Even if we keep this assumption, three cases are possible

1. if $r < \rho$ then $\lambda_- = \gamma < 0$ and and the steady state $(\bar{A}, \bar{C}) = (-Y/r, 0)$ is a saddle-point. The solution of the optimal control problem, which lies along the stable manifold converges to $C^*(\infty) = 0$ and $A^*(\infty) = -Y/r < 0$. The steady state is a saddle point. The intuition is: the consumer is more impatient than the market and therefore will be asymptotically a debtor to a point in which it can collateralize the debt by its human capital $A(\infty) + H(0) = 0$;

2. if $\gamma < r = \rho$ then $\lambda_{\_} = 0$ and the solution is constant $C^*(t) = Y + rA_0$ and $A^*(t) = A_0$ for all $t \in [0, \infty)$. This was the case corresponding to the existence of an infinite number of equilibria when the characterization is conducted in the (A, Q) space;

3. if $r > \rho$ then $\lambda_{\_} = \gamma > 0$ and the steady state $(\bar{A}, \bar{C}) = (-Y/r, 0)$ is an unstable node. In this case, there are admissible solutions only if $A_0 \geq -Y/r$, otherwise consumption would be negative. However, if $A_0 > -Y/r$ there is an admissible solution to the optimal control problem but it is unbounded.

The question the last case poses is the following. First, if we look at the MHDS as a dynamical system we would say that it is unstable but most of the qualitative theory of ODE characterizes the dynamics close to a steady state. But we already found that this case is indeed a solution to the optimal control problem. How can we reconcile the two points ?

A way to deal with the last type of behavior is to consider convergence of the solution to a kind of invariant structure and to consider convergence to that structure. An approach which is used in the economic growth literature (see Acemoglu (2009)) is to consider convergence to an exponential solution, called **balanced growth path**, such that the initial and the transversality conditions hold.

The method proceeds along five steps.

First, define the variables using multiplicative deviations along an exponential trends with proportional growth rates. In our case we try the case in which the rates of growth are equal

$$A(t) = a(t)e^{gt}, \ C(t) = c(t)e^{gt}$$

Second, obtain the dynamic system for the detrended variables (a, c). If we observe that

$$\frac{\dot{a}}{a} = \frac{\dot{A}}{A} - g, \ \frac{\dot{c}}{c} = \frac{\dot{C}}{C} - g,$$

we get

$$\begin{cases} \dot{a} = Ye^{-gt} - c + (r - g)a \\ \dot{c} = (\gamma - g)c \end{cases}$$

Third, obtain g from a stationary solution to the system in detrended variables. In our case setting $g = \gamma$ transforms the previous system to

$$\begin{cases} \dot{a} = Ye^{-\gamma t} - c + (r - \gamma)a \\ \dot{c} = 0 \end{cases}$$

which implies that $c(t) = \bar{c}$ which is an unknown constant. Setting $a(0) = A_0$ and $c(t) = \bar{c}$ we can solve the equation for the detrended asset holdings

$$a(t) = \left(A_0 - \frac{\bar{c}}{r - \gamma} + \frac{Y}{r}\left(1 - e^{-rt}\right)\right) e^{(r-\gamma)t} + \frac{\bar{c}}{r - \gamma}.$$

Fourth, we can determine $\bar{c}$ from the transversality condition

$$\begin{aligned}
\lim_{t\to\infty}(C(t))^{-\theta}A(t)e^{-\rho t} &= \lim_{t\to\infty}\bar{c}^{-\theta}e^{(\gamma(1-\theta)-\rho)t}a(t) = \\
&= \lim_{t\to\infty}\bar{c}^{-\theta}e^{(\gamma(\theta-1)-\rho+r-\gamma)t}\left(A_0-\frac{\bar{c}}{r-\gamma}+\frac{Y}{r}\left(1-e^{-rt}\right)+\frac{\bar{c}}{r-\gamma}e^{-(r-\gamma)t}\right) = \\
&= \lim_{t\to\infty}\bar{c}^{-\theta}\left(A_0-\frac{\bar{c}}{r-\gamma}+\frac{Y}{r}\left(1-e^{-rt}\right)+\frac{\bar{c}}{r-\gamma}e^{-(r-\gamma)t}\right) = \\
&= \bar{c}^{-\theta}\left(A_0-\frac{\bar{c}}{r-\gamma}+\frac{Y}{r}\right)=0
\end{aligned}$$

if and only if $\bar{c}=c^*=(r-\gamma)\left(A_0+\dfrac{Y}{r}\right)$.

At last we get the solution

$$C^*(t)=c^*e^{\gamma t},\ A^*(t)=a^*(t)e^{\gamma t}$$

where

$$c^*=(r-\gamma)\left(a_0+\frac{Y}{r}\right),\ a^*(t)=A_0+\frac{Y}{r}\left(1-e^{-\gamma t}\right).$$

We see that

$$C^*(t)=(r-\gamma)\left(A^*(t)+\frac{Y}{r}\right),\text{for } t\in[0,\infty)$$

which means that the solution to the optimal control problem evolves along the eigenspace associated to the eigenvalue $\lambda\_$ (see figure 7.2).

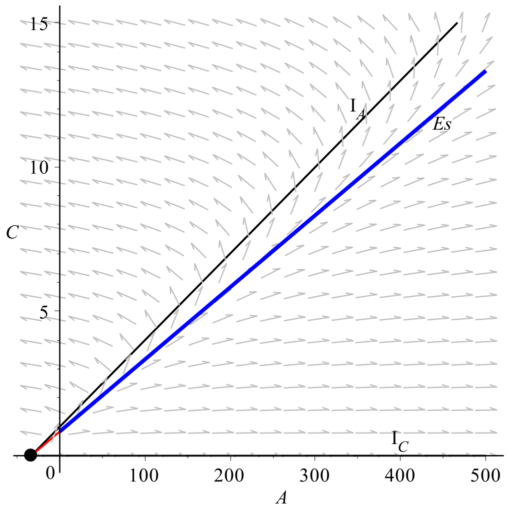

Figure 7.2: Phase diagram for the benchmark consumer problem for the case $r>\gamma$.

If $r<\rho$, and therefore $\gamma<0$, the solution evolves along the eigenspace associated to $\lambda\_$ butit converges to the steady state in which $A(\infty)=-Y/r$. In this case $\mathbb{E}^-=\mathbb{E}^s$ that is this is the stable eigenspace (which as the model is linear is the stable manifold).

From this we have a geometrical interpretation of the solution to the optimal control problem: if $r\neq\rho$ the solution will belong to the eigenspace $\mathbb{E}^-$, and it converges to the steady state if $r<\rho$ and diverges from it if $r>\rho$.

This illustrates, and reinforces, the fact that interpreting phase diagrams for MHDS of optimal control problems should be done with care: if the optimal control problem has a single solution, the geometrical analog of it is also unique.

7.6 Bibliography

- Introductory Kamien and Schwartz (1991), Chiang (1992);
- A little less introductory Weber (2011);
- Complete reference, particularly for optimal control Grass et al. (2008);
- The seminal contributions: optimal control Pontryagin et al. (1962);
- Other important contributions: Fleming and Rishel (1975);
- Historical contributions in economics: Arrow and Kurz (1970), Seierstad and Sydsæter (1987), Intriligator (2002),
- Applications: Kamien and Schwartz (1991) (management) , Grass et al. (2008) (economics, terror and drugs), economic theory Brock and Malliaris (1989); Growth applications: Acemoglu (2009).

Chapter 8

Dynamic programming

Consider again the free terminal state optimal control problem: among functions $y \in \mathrm{Y}$ and $u \in U$ satisfying

$$\dot{y} = G(y(t), u(t), t), \text{ for } t \in [0, \bar{t}] \tag{8.1}$$

and $y(0) = y_0$ find the pair (y^*, u^*) that maximize the functional

$$\mathsf{J}[y, u] \equiv \int_0^{\bar{t}} F(t, y(t), u(t))dt \tag{8.2}$$

where $\bar{t}$ is given and $y^*(\bar{t})$ is free.

8.1 The finite horizon case

Proposition 1 (Necessary conditions according to the principle of dynamic programming). *Consider the optimal state and control functions $y^* \in \mathrm{Y}$ and $u^* \in U$ for the optimal control problem with free terminal state. Then the* ***Hamilton-Jacobi-Bellman*** *equation must hold*

$$-V_t(t, y) = \max_{u \in \mathcal{U}} \left\{ F(t, y, u) + V_y(t, y)G(t, y, u) \right\} \tag{8.3}$$

for all $t \in [0, \bar{t})$ and all $y \in Y \subseteq \mathbb{R}$.

Proof. (heuristic) We define the functional over the state and control functions continuing from an arbitrary time $t \geq 0$: $(y, u) : [t, \bar{t}] \to Y \times U \subseteq \mathbb{R}^2$

$$\mathsf{J}[y, u](t) = \int_t^{\bar{t}} F(s, y(s), u(s))ds.$$

and call **value function** to

$$V(t, y(t)) \equiv \max_{(u(s)|s \in [t, \bar{t}])} \mathsf{J}[y, u; t]$$

for $y(t) \in Y$.

The **Principle of dynamic programming optimality** states the following: for every $(t, y) \in [0, \bar{t}] \times Y$ and every $\Delta t \in (0, \bar{t} - t]$ the value function satisfies

$$V(t, y(t)) = \max_{(u(s)|s\in[t,t+\Delta t])} \left\{ \int_t^{t+\Delta t} F(s, y(s), u(s))ds + V(t + \Delta t, y(t + \Delta t)) \right\}$$

where

$$y(t + \Delta t) = y(t) + G(t, y(t), u(t))\Delta t + o(\Delta t).$$

Performing a first-order Taylor expansion we get

$$V(t + \Delta t, y(t + \Delta t)) = V(t, y(t)) + V_t(t, y(t))\Delta t + V_y(t, y(t))G(t, y(t), u(t))\Delta t + o(\Delta t)$$

(this requires that V is C^1). If the interval Δt is sufficiently small we can use the mean-value theorem

$$\int_t^{t+\Delta t} F(s, y(s), u(s))ds = F(t, y(t), u(t))\Delta t$$

Then

$$V(t, y(t)) = \max_{(u(s)|s\in[t,t+\Delta t])} \left\{ F(t, y(t), u(t))\Delta t + V(t, y(t)) + V_t(t, y(t))\Delta t + V_y(t, y)g(t, y(t), u(t))\Delta t + o(\Delta t) \right\}.$$

Cancelling out $V(t, y(t))$, dividing by Δt, taking $\Delta t \to 0$ and observing that the pair $(t, y(t))$ is an arbitrary element of $T \times Y$ we get the HJB equation (8.3). □

For solving the optimal control problem, while the Pontriagyin's principle provides necessary conditions in a form of a initial-terminal value problem for a planar ODE, the principle of the dynamic programming provides a formula for evaluating the value of our resource in a recursive way and independent of time.

The HJB equation (8.3) is a PDE (partial differential equation).

8.2 Infinite horizon discounted optimal control problem

The infinite horizon discounted optimal control problem is, again, to find functions $u^* \in \mathrm{U}$ and $u^* \in \mathrm{Y}$ satisfying

$$\begin{cases} \dot{y} = g(y(t), u(t)), & t \in [0, \infty) \\ y(0) = y_0, \\ \lim_{t\to\infty} h(t)y(t) \geq 0 \end{cases}$$

that maximize the objective functional

$$\mathsf{J}[y, u] \equiv \int_0^\infty e^{-\rho t} f(y(t), u(t))dt$$

Proposition 2 (Necessary conditions according to the principle of dynamic programming for the infinite horizon problem). *Let (y^*, u^*) be the solution to the discounted infinite horizon problem. Then it satisfies the HJB equation*

$$\rho v(y) = \max_u \left\{ f(y,u) + v^{'}(y) g(y,u) \right\} \tag{8.4}$$

Proof. For $y(t) = y$ the value function is

$$V(t,y) \equiv \int_t^{\infty} e^{-\rho s} f(y^*(s), u^*(s)) ds$$

Multiplying by a inverse of the discount factor, the value function becomes independent of the initial time,

$$e^{\rho t} V(t,y) = \int_t^{\infty} e^{-\rho(s-t)} f(y^*(s), u^*(s)) ds = v(y).$$

Then we can write

$$V(t,y) = e^{-\rho t} v(y)$$

and upon substituting in equation (8.3) we get equation (8.4). □

In the case of the discounted infinite horizon the HJB equation is not a PDE but an ODE in implicit form. In order to see this we need to determine another important element of the DP approach: the policy function.

If we define the function $h(u,y) \equiv f(y,u) + v^{'}(y) g(y,u)$ the HJB equation (8.4) can be written as

$$\rho v(y) = \max_u h(u,v).$$

We can obtain the optimal control from the first-order condition

$$\frac{\partial h(u,y)}{\partial u} = 0.$$

If function $h(u,v)$ is monotonic as regards u, by appealing to the implicit function theorem, we can obtain the optimal control as a function of the state variable, $u^* = \pi(y)$. Function $\pi(.)$ in the DP literature is called **policy function**. It gives the optimal control as a function of the state variable. This is why it is called a **feedback control** problem.

The reason for this is the following. If we substitute the policy function in equation (8.4) we finally obtain the HJB equation as an ODE in implicit form

$$\rho v(y) = f(\pi(y), y) + v^{'}(y) g(\pi(y), y)$$

where the state variable y is the independent variable and the value function, $v(y)$, is the unknown function.

If we are able to determine a solution to this equation, we can usually specify the utility function, which means that we are able to obtain the optimal control as a function of the state variable. We can obtain the solution to the optimal control problem by substituting in the ODE constraint to get

$$\dot{y} \ = g(y, \pi(y)),\ t \in [0, \infty)$$

which, together with the initial condition $y(0) = y_0$, would, hopefully, allow for the determination of the solution for the state variable.

If we can find the policy function, then obtaining the optimal dynamics for y reduces to solving an initial-value problem instead of a mixed initial-terminal value problem (or two-point boundary value problem) as is the case when we use the calculus of variations of the Pontriyagin's principle approaches.

However, only in a very small number of cases we can obtain closed form solutions to the HJB equation. Next we show some cases in which this is possible.

8.3 Applications

8.3.1 Example 1: The resource depletion problem

We solve again resource-depletion problem for an infinite horizon

$$\max_{C} \int_0^{\infty} e^{-\rho t} \ln{(C(t))} dt,\ \text{s.t}\ \dot{W} = -C,\ W(0) = W_0$$

by using the DP principle.

The HJB equation is

$$\rho v(W) = \max_{C} \Big[\ \ln{(C)} + v^{'}(W)(-C)\Big]$$

Policy function

$$\frac{1}{C^*} - v^{'}(W) = 0 \Leftrightarrow C^* = \left(v^{'}(W)\right)^{-1}$$

Then the HJB becomes

$$\rho v(W) = -\ \ln{(v^{'}(W))} - 1$$

The textbook method for solving the HJB equation through is by using the **method of undetermined coefficients** after we make a conjecture over the form of the value function (no constructive way here).

Assume the trial function

$$v(W) = a + b\ln{(W)}$$

As $v^{'}(W) = b/W$ and substituting and collecting terms we get

$$\rho a + 1 + \ln{(b)} = \ln{(W)}\,(1 - \rho b)$$

then $b = 1/\rho$ and $a = (\ln \rho - 1)/\rho$.

Then:

$$v(W) = \frac{\ln \rho - 1 + \ln (W)}{\rho},\ C^* = (v'(W))^{-1} = \rho W$$

A second method: the HJB equation is an ODE, where W is the independent variable, so we can try to solve it (this is a constructive method).

The HJB is equivalent to

$$v'(W) = e^{-(1+\rho v(W))}$$

ODE $y'(x) = e^{(a+by(x))}$ has the closed form solution

$$y(x) = \frac{1}{b}\left(-a + \ln\left(-\frac{1}{b(k+x)}\right)\right)$$

where k is an arbitrary constant. Then we determine

$$v(W) = -\frac{1}{\rho}\left(1 + \ln\left(\frac{1}{\rho(W+k)}\right)\right)$$

and

$$C^* = \left(V'(W)\right)^{-1} = \rho(W+k)$$

Substituting in the constraint $\dot{W} = -C = -\rho(W+k)$, we get the solution

$$W(t) = -k + (W(0)+k)e^{-\rho t}.$$

The problem is somewhat incompletely specified, which reveals a potential problem when using the DP approach.

In our case, as it is natural to assume that $\lim_{t\to\infty} W(t) = 0$ we would obtain $k = 0$ and therefore we would get the same solution as from using the CV and Pontriyagin's approaches:

$$C^*(t) = \rho W_0 e^{-\rho t},\ W^*(t) = W_0 e^{-\rho t},\ \text{for } t \in [0, \infty).$$

8.3.2 Example 2: The benchmark consumption-savings problem

Applying the HJB equation (8.4) to our problem we have

$$\rho v(A) = \max_C \left\{ \frac{C^{1-\theta} - 1}{1-\theta} + v'(A)(Y - C + rA) \right\}. \tag{8.5}$$

Define a indirect utility function by

$$\tilde{u}(v'(A)) = \max_C \left\{ \frac{C^{1-\theta} - 1}{1-\theta} - v'(A)\, C \right\}$$

and total wealth, summing up human and financial wealth, by $W(A) \equiv \frac{Y}{r} + A$, then the HJB equation (8.5) at the optimum is a implicit ODE

$$\rho v(A) = \tilde{u}(v'(A)) + r\, v'(A)\, W(A). \tag{8.6}$$

Solving the static utility problem we get the optimum policy for consumption

$$C^* = \pi(A) \equiv \left(v^{'}(A)\right)^{-\frac{1}{\theta}}.$$

as a function of the (unknown) marginal value function, and upon substitution yields

$$\tilde{u}(v'(A)) = \frac{1}{1-\theta}\Big(\left(v^{'}(A)\right)^{\frac{\theta-1}{\theta}} - 1\Big).$$

Therefore, equation (8.6) becomes

$$\rho\, v(A) = \frac{\theta}{1-\theta}\left(v^{'}(A)\right)^{\frac{\theta-1}{\theta}} - \frac{1}{1-\theta} + r\, v^{'}(A)\, W(A) \tag{8.7}$$

To solve this (implicit ODE) equation, we use again the method of undetermined coefficients. Conjecturing the trial function

$$v(A) = a + b\, W(A)^{1-\theta},$$

with arbitrary parameters a and b. Then

$$v^{'}(A) = b\,(1-\theta)\, W(A)^{-\theta}$$

and after substitution in equation (8.7) we get

$$a\rho + \frac{1}{1-\theta} = W(A)^{1-\theta}\, b\,\theta\left[\,\Big(b\,(1-\theta)\Big)^{-1/\theta} - (r-\gamma)\right]$$

where we have again $\gamma \equiv (r-\rho)/\sigma$. Setting both sides to zero, yields

$$a = \frac{1}{\rho(\theta-1)} \text{ and } b = \frac{(r-\gamma)^{-\theta}}{1-\theta}$$

Then, the value function is

$$v(A) = \frac{1}{1-\theta}\Big[\,(r-\gamma)^{-\theta}\left(\frac{Y}{r} + A\right)^{1-\theta} - \frac{1}{\rho}\Big].$$

Taking the derivative as regards A and substituting in the policy function for C, we find the optimal consumption in feedback form

$$C^*(A) = (r-\gamma)\left(\frac{Y}{r} + A\right)$$

which only makes sense if $r > \gamma$.

We can get the optimal asset path by substituting optimal consumption in the budget constraint

$$\dot{A}^* = Y + rA - C^*(A) = \gamma\left(\frac{Y}{r} + A\right).$$

Solving this equation with $A(0) = A_0$ we get the optimal paths for asset holdings

$$A^*(t) = -\frac{Y}{r} + \left(\frac{Y}{r} + A_0\right)e^{\gamma t}, \text{ for } t \in [0, \infty),$$

and consumption

$$C^*(t) = (r-\gamma)\left(\frac{Y}{r} + A_0\right) e^{\gamma t}, \text{ for } t \in [0,\infty).$$

Exercise Prove, by setting $\theta = 1$, that the value function for $u(C) = \ln(C)$ is

$$V(A) = \frac{1}{\rho}\Big[\ \frac{r-\rho}{\rho} + \ln(\rho W(A))\ \Big].$$

Hint: use the property $f(x) = \exp(\ln f(x))$ and use the l'Hôpital theorem.

The utility function is a generalized logarithm $u(C) = \ln_\theta(C) = \dfrac{C^{1-\theta}-1}{1-\theta}$. Sometimes in the literature people write

$$u(C) = \begin{cases} \dfrac{C^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1 \\ \ln(C) & \textit{if}\, \theta = 1 \end{cases}$$

The problem with this formulation is that if we cannot obtain the value function for the logarithm utility by setting the limit of $\theta = 1$ for the general case $\theta = 1$, which is

$$v(A) = \frac{(r-\gamma)^{-\theta}}{1-\theta} W(A)^{1-\theta}.$$

8.3.3 Example 3: The Ramsey model

The HJB for the Ramsey model is

$$\rho v(k) = \max_c \Big\{\ u(c) + v'(k)\,(F(k) - c)\Big\}$$

The optimality condition is

$$u'(c) = v'(k)$$

if u is sufficiently smooth then we obtain the policy function $c = C(k) = (u')^{-1}(v'(k))$. Substituting back in the HJB equation yields the implicit ODE in $v(k)$

$$\rho v(k) = u(C(k)) + v'(k)(F(k) - C(k))$$

which does not have a closed form solution in general.

Exercise: for the case in which $u(c) = \dfrac{c^{1-\theta}-1}{1-\theta}$ and $F(k) = k^\alpha$, such that $\theta = \alpha$ prove that a closed form solution can be found.

8.3.4 Example 4: The AK model

The Rebelo (1991) AK model can be seen as a special case of the previous problem in which the HJB function is

$$\rho v(K) = \max_C \Big\{\ \frac{C^{1-\theta}}{1-\theta} + v'(K)\,(AK - C)\Big\}$$

Using the same steps as before, we get

$$\rho v(K) = \frac{\theta}{1-\theta} \left(v'(K)\right)^{\frac{\theta-1}{\theta}} + v'(K)AK \tag{8.8}$$

To solve the equation we use again the method of undetermined coefficients and find

$$v(K) = \frac{((A-\gamma)K)^{1-\theta}}{1-\theta}.$$

where

$$\gamma = \frac{A-\rho}{\theta}.$$

The consumption, in the feedback form is,

$$C^*(K) = (A-\gamma)K$$

and the budget constraint of the economy is

$$\dot{K}^* = AK^* - C^*(K) = \gamma K^*.$$

Considering the given initial level for capital $K(0) = K_0$ we get the optimal paths for capital and output

$$K^*(t) = K_0 e^{\gamma t},\ Y^*(t) = AK_0 e^{\gamma t},\ \text{for } t \in [0, \infty).$$

8.4 Bibliography

- The seminal contribution: Bellman (1957)

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