Advanced Mathematical Economics

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Contents

Chapter 7

First order quasi-linear partial differential equations

7.1 Introduction

In this chapter we present introductory results on first-order partial differential equations (PDE) and some applications to demography and economics. Those equations can also be called hyperbolic PDEs, using a classification for second-order equations. They are another example of functional equations.

A first-order (or hyperbolic) PDEs is a known function of one or more unknown functions of more than one independent variable, together with its first-order derivatives. If there is only one unknown function we call the PDE scalar, and if there are two unknown functions we call it planar PDE.

In physics these equations model advection, travelling or transportation behaviors. They are used in mathematical demography for modelling age-dependent dynamics of population.

In economics, usually one of the independent variables is time and the other independent variable is the support of some distribution. They can be applied in continuous time overlapping generations models, vintage capital models, interest rate term-structure models in continuous time. They also provide an elegant and effective way of modelling the dynamics of distribution in heterogeneous agent economies.

The Hamilton-Jacobi equation for deterministic optimal control problems with a finite horizon and a constraint given by a ordinary differential equation is also usually a non-linear first-order PDE.

The field is very large in terms of equations studied and methods involved, and is not generally in the toolbox of economists. We will only present a very brief introduction allowing to study very simple linear models.

There are two benefits from studying these equations: first, they provide a convenient modelling framework for setting up and characterizing the solution for models with heterogeneity, which is becoming topical in economics, second, they provide a better understanding of the implicit assumptions which are introduced when using ODE (or difference equations) models for studying dynamics of heterogeneity.

We assume throughout that there are only two independent variables $\mathbf{x} = (x_1, x_2) \in \mathrm{X} \subseteq \mathbb{R}^2$ and deal mainly with equations of dimension one, that is mappings $u : \mathrm{X} \to \mathbb{R}$.

Definition A **first-order partial differential equation** in two independent variables $\mathbf{x} \in \mathrm{X} \subseteq \mathbb{R}^2$ is a known relation $F : D \to \mathbb{R}$ where $D \subset \mathbb{R}^5$ involving the unknown function $u : \mathrm{X} \to \mathbb{R}$ and its gradient

$$F(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x})) = 0 \tag{7.1}$$

where $\nabla u(\mathbf{x})$ is the gradient of $u(.)$, i.e.

$$\nabla u(\mathbf{x}) = \Big(\frac{\partial u(\mathbf{x})}{\partial x_1}, \frac{\partial u(\mathbf{x})}{\partial x_2}\Big)^\top \big(u_{x_1}(\mathbf{x}), u_{x_2}(\mathbf{x})\big)^\top$$

where we use the $u_{x_i}(.) = \frac{\partial u(.)}{\partial x_i} = \partial_{x_i} u(\mathbf{x})$.

Types of first-order PDE First-order PDE are classified into four categories:

- **linear PDE**: if $F(\cdot)$ is linear in the derivatives $\nabla u(\mathbf{x})$ and u,

$$a(\mathbf{x})\, u_{x_1} + b(\mathbf{x})\, u_{x_2} = c(\mathbf{x})\, u + d(\mathbf{x}) \tag{7.2}$$

  where $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ and $d(\cdot)$ are differentiable functions of $\mathbf{x}$;

- **semi-linear PDE**: if $F(\cdot)$ is linear in $\nabla u(\mathbf{x})$ and non-linear in u, which only enters into the right-hand side,

$$a(\mathbf{x})\, u_{x_1} + b(\mathbf{x})\, u_{x_2} = c(\mathbf{x}, u); \tag{7.3}$$

- **quasi-linear PDE**: if $F(\cdot)$ is linear in ∇u and non-linear in u as

$$a(\mathbf{x}, u)\, u_{x_1} + b(\mathbf{x}, u)\, u_{x_2} = c(\mathbf{x}, u) \tag{7.4}$$

- **non-linear PDE**: if $F(\cdot)$ is non-linear in $\nabla u(\mathbf{x})$ and u

$$F(\mathbf{x}, u, \nabla u) = 0$$

  where F is non-linear in u_{x_1} and/or u_{x_2}

Linear equations can be classified further as non-autonomous or autonomous, if a, b, c and d are constants, or non-homogeneous or homogeneous, if function $c(\mathbf{x}, u)$ is homogeneous in u.

Multi-dimensional first order PDE equations In addition we can consider systems of hyperbolic equations

$$\mathbf{F}(\mathbf{x}, \mathbf{u}(\mathbf{x}), D_{\mathbf{x}}\mathbf{u}(\mathbf{x}) = \mathbf{0}$$

where $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{u} : \mathbb{R}^m \to \mathbb{R}^n$, and $D_{\mathbf{x}}\mathbf{u}$ denotes the Jacobian.

For, instance a linear planar equation in two independent variables we have

$$\mathbf{A}\,\mathrm{D}_{\mathbf{x}}\mathbf{u}(\mathbf{x}) = \mathbf{B}\ \ \mathbf{u}(\mathbf{x})$$

where

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u_1(\mathbf{x}) \\ u_2(\mathbf{x}) \end{pmatrix} \text{ and } \mathrm{D}_{\mathbf{x}}\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \partial_{x_1} u_1(\mathbf{x}) & \partial_{x_2} u_1(\mathbf{x}) \\ \partial_{x_1} u_2(\mathbf{x}) & \partial_{x_2} u_2(\mathbf{x}) \end{pmatrix}$$

and

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Solutions A solution to a first-order PDE is a differentiable function $f(x, y)$ that satisfies the PDE. Existence and uniqueness of solutions for first-order PDE, and for problems involving them, are not guaranteed. Classic solutions are solutions such that $u \in C^1(\mathrm{X})$. Otherwise we call generalised or weak solutions (i.e, non-differentiable or discontinuous solutions).

There are several **methods for obtaining solutions** which can be applied to general or specific problems. The analytical methods for simpler equations are

- method of characteristics
- transformation methods (in particular application of Laplace transforms)

Those methods simplify the first-order PDE into a system of ODE's or a parameterised ODE.

Problems involving PDEs There are two main types of problems involving first-order PDE.

1. the Cauchy problem: there is a single constraint on $\mathbf{x}$ along a surface $\Gamma \in \mathrm{X}$:

$$\begin{cases} F(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x})) = 0, & \mathbf{x} \in \mathrm{X} \\ u \mid_{\Gamma} = \phi, & \mathbf{x} \in \Gamma \subset \mathrm{X} \end{cases}$$

   where ϕ is a constant;

2. problems may involve two constraints, associated with each independent variable, for instance

$$\begin{cases} F(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x})) = 0, & \mathbf{x} \in \mathrm{X} \\ u \mid_{x_1=0} = \psi(x_2), & (0, x_2) \in \mathrm{X} \\ u \mid_{x_2=0} = \phi(x_1), & (x_1, 0) \in \mathrm{X} \end{cases}$$

Well-posed problems Existence, uniqueness and properties of solutions vary widely. Again we have to distinguish existence properties of the PDE and of the problem involving the PDE (i.e., the PDE and the boundary conditions). A problem is **ill-posed** if, for instance, although the PDE has a solution the problem involving the PDE may not have a solution, in the same domain as the solution to the PDE. A problem is **well-posed** if the general solution to the PDE has a particular solution satisfying the constraints of the problem.

Linear PDEs, and well-posed problems involving liner PDEs, have explicit solutions.

Qualitative theory Let us consider the case in which time is one of the independent variables, i.e., $u = u(t, x)$ such that $u : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$. Therefore, the solution of a first-order PDE describes a solution of a distribution, or wave, travelling over X across time.

For linear and well-posed PDE's the distributional dynamics that characterizes their solution can be explicitly discovered.

For non-linear PDEs we are unaware of the existence of a qualitative theory as developed as the qualitative theory for ODE's. In particular, a Grobmann-Hartmann theorem for PDE does not seem to be available. There are phenomena that do not exist in ODE's: traveling waves, front waves, for instance.

An important case of first-order PDE's are equations of type

$$u_t + g(u)_x = 0$$

that satisfy a conversation law

$$\int_{\mathrm{X}} u(t,x)\,dx = \bar{u} \text{ for every } t \in \mathbb{R}_+.$$

where $\bar{u}$ is a constant. Given an initial function $u(0, x) = \phi(x)$ such that

$$\int_{\mathrm{X}} \phi(x)\,dx = \bar{u} \text{ for } t = 0,$$

the solution $u(t, x)$ conserves the same mass throughout time. Assuming that the solution exists, we may be interested in characterizing the asymptotic behavior of the distribution $\lim_{t\to\infty} u(t, x)$.This is the analog of studying the long-run behavior for ODE's. However, while the solution of an ODE can converge asymptotically to a steady state (a point) the solution of a PDE can converge asymptotically to a function (or a steady state distribution).

In the rest of the chapter, in section 7.2 we solve scalar linear equations with an infinite domain through the method of characteristics. In section 7.3 we solve some linear equations in the semi-infinite domain by using Laplace transform methods. In section 7.4 we discsuss the qualitatitive analysis of first-order PDE's when time is one independent variable. In Section 7.6 has several applications to economics and demography are presented.

7.2 Scalar equations in the infinite domain and the method of characteristics

7.2.1 Introduction: the method of characteristics

In this section we solve hyperbolic PDE in the infinite domain. We denote the independent variables by $\mathbf{x}$ and assume that the domain of $\mathbf{x}$ is the whole set $\mathrm{X} = \mathbb{R}^2$. We consider scalar function $u : \mathbb{R}^2 \to \mathbb{R}$ as our dependent variable and consider quasi-linear equations of type [1]

$$a(\mathbf{x}, u)\, u_{x_1}(\mathbf{x}) + b(\mathbf{x}, u)\, u_{x_2}(\mathbf{x}) = c(\mathbf{x}, u(\mathbf{x})),\ \mathbf{x} \in \mathrm{X}$$

and $a(.)$, $b(.)$, and $c(.)$ are known functions.

One useful method to solve the hyperbolic PDE in the infinite domain is the **method of characteristics**.

The following definition is useful

Definition 1. ***Directional derivative*** *Consider a function $f(\mathbf{x})$, the derivative of f in the direction given by vector $\mathbf{v} = (v_{x_1}, v_{x_2})^\top$ is*

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1 + h\, v_{x_1}, x_2 + h\, v_{x_2}) - f(\mathbf{x})}{h}$$

if the limit exists.

If function $f(x, y)$ is differentiable, the directional derivative of f in the direction given by vector $\mathbf{v} = (v_x, v_y)^\top$ is equal to the dot product[2]

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v} = \left(f_{x_1}, f_{x_2}\right) \cdot (v_{x_1}, v_{x_2}) = f_{x_1}(\mathbf{x}) v_{x_1} + f_{x_2}(\mathbf{x}) v_{x_2}.$$

We start with simple linear PDE to illustrate their solution using the **method of characteristics**. A characteristic is a curve in the domain x, which we can write as $\mathbf{x} = \mathbf{X}(\xi)$ such that such that $u(\mathbf{X}(\xi))$ behaves like an ODE. This means that for every $\xi \in \mathbb{R}$ the solution of the PDE behaves like and ODE.

It is very important to remember that we assume, in all this section, that there are no restrictions on the domain of the independent variables, x_1 and x_2 in this section, that is, we assume $\mathbf{x} \in \mathbb{R}^2$. In the next sections we introduce constraints on the domain X.

[1] The following notation sometimes is more convenient

$$a(\mathbf{x}, u)\partial_x u(\mathbf{x}) + b(\mathbf{x}, u)\partial_y u(\mathbf{x}) = c(\mathbf{x}, u(\mathbf{x})),\ \mathbf{x} \in \mathrm{X}.$$

[2] Observe there is a relationship with the total differential. Let $z = f(x, y)$, where $f(.)$ is differentiable. The total differential is $dz = f_x(x, y)dx + f_y(x, y)dy$. If we write $dx = v_x h$ and $dy = v_y h$ then $\nabla f(x, y) \cdot \mathbf{v} = \lim_{h \to 0} \frac{dz}{h}$.

7.2.2 The two simplest first order PDEs

We start with the two simplest first-order PDE: $u_{x_1}(\mathbf{x}) = 0$ and $u_{x_2}(\mathbf{x}) = 0$.

Proposition 1. *The equation*

$$u_{x_1}(\mathbf{x}) = 0, \ \mathbf{x} \in X = \mathbb{R}^2$$

has the general solution

$$u(\mathbf{x}) = f(x_2)$$

where $f \in C^1(\mathbb{R})$ is an arbitrary differentiable function.

Proof. First observe that the solution to equation $u_x = 0$ is any function that remains constant along direction $v = (1,0)^\top$. This can be proved by observing that the directional derivative along that direction is zero,

$$\nabla u(x,y) \cdot (1,0) = u_x \times 1 + u_y \times 0 = u_x = 0.$$

This is equivalent to any function function, $f(.)$, that remains unchanged along any variation parallel to the x-axis, that is $f(y)$. □

In order to have a better intuition on this result, consider an ODE $u_x(x) = 0$, where $u(x)$ is an unknown function of single independent variable $u : \mathbb{R} \to \mathrm{X} \subseteq \mathbb{R}$. This equation has the solution $u(x) = k$ where k is an arbitrary **point** in the domain of $u(.)$, $\mathrm{X} \subseteq \mathbb{R}$. In the case of the PDE $u_{x_1}(\mathbf{x}) = 0$ the solution is $u(\mathbf{x}) = f(x_2)$ where $f(x_2)$ is an arbitrary differentiable **function** over $\mathbb{R}$.

Proposition 2. *The equation*

$$u_{x_2}(\mathbf{x}) = 0, \ \mathbf{x} \in X = \mathbb{R}^2$$

has the general solution

$$u(\mathbf{x}) = f(x_1)$$

where $f \in C^1(\mathbb{R})$ is an arbitrary differentiable function.

Proof. Not the PDE solution is constant along the direction $v = (0,1)^\top$, because it is equivalent to the directional derivative along that direction being equal to zero,

$$\nabla u(x_1, x_2) \cdot (0,1) = u_{x_2} = 0.$$

In this case, the solution is any function function, $f(.)$, that remains unchanged along any variation parallel to the y-axis, that is $f(x_1)$. □

From those two previous results we can understand more general linear first order scalar PDE's as being constant along particular directions, which are called **characteristics**.

7.2.3 Linear equation with constant coefficients

Next we consider linear equations without side constrains and Cauchy problems for linear hyperbolic equations defined in the infinite domain.

A. Free boundary problems

Consider the first order linear autonomous PDE

$$u_{x_1}(\mathbf{x}) + a\, u_{x_2}(\mathbf{x}) = 0,\ \mathbf{x} \in \mathrm{X} = \mathbb{R}^2 \tag{7.5}$$

where $a \neq 0$ is an arbitrary constant.

Proposition 3. *The general solution of PDE* (7.5) *is*

$$u(\mathbf{x}) = f(x_2 - a\, x_1),$$

where $f \in C^1(\mathbb{R})$ is an arbitrary differentiable function.

Proof. First, observe that the PDE (7.5) determines a function $u(x,y)$ which is constant along the direction $v = (1,a)^\top$, because

$$\nabla u \cdot (1,a) = u_x + a u_y = 0.$$

To interpret this geometrically consider the two-dimensional surface

$$S \equiv \{\ (x, y, u(x,y))\} \subset \mathbb{R}^3.$$

A particular solution of the PDE $(x_0, y_0, u(x_0, y_0))$ belongs to the surface S. But, next we show that a solution to the PDE traces out a curve C over the surface S in which u remains constant. We call this curve a **characteristic curve**.

In order to determine curve C we parametrize the two independent variables as $x = X(s)$, $y = Y(s)$, where $s \in \mathbb{R}$. Then, we get a parameterized value for u, as $u = U(s) = u(X(s), Y(s))$. A characteristic curve is defined as

$$C = \{\ (X_1(s), X_2(s), U(s)) : U(s) = \text{constant}\}.$$

Taking derivatives to $u = U(s) = u(X_1(s), X_2(s))$ we find

$$\frac{dU}{ds} = \frac{du(X_1(s), X_2(s))}{ds} = u_{x_1}\frac{dX_1}{ds} + u_{x_2}\frac{dX_2}{ds}$$

The PDE will hold if and only if the following conditions hold:

- the characteristic system

$$\begin{aligned}\frac{dX_1}{ds} &= 1\\ \frac{dX_2}{ds} &= a\end{aligned}$$

- the compatibility condition

$$\frac{dU}{ds} = 0$$

Solving the characteristic system and the compatibility equation we get

$$\begin{aligned} x_1 &= X_1(s) = s + c_1 \\ x_2 &= X_2(s) = as + c_2 \\ u &= U(s) = U(0) \end{aligned}$$

where c_1 and c_2 are arbitrary constants, and $U(0)$ is an arbitrary function, say $f(k)$. We can set $c_1 = 0$ and $c_2 = k$. Eliminating s in the first two equations we find

$$s = x_1 = \frac{x_2 - k}{a}.$$

This implies that there, a characteristic curve is a straight line in (x_1, x_2) with a constant value $x_2 - a\,x_1 = k$ is constant. Then we find the general solution for (7.5) to be constant along the direction $(1, a)^\top$,

$$u(x, y) = U(s) = f(k) = f(x_2 - a\,x_1),$$

where f is an arbitrary C^1 function. □

Verification: In order to check that this is a solution, assume that $u(x_1, x_2) = f(x_2 - a\,x_1)$ for $f(\cdot) \in C^1(\mathbf{R})$ Then

$$u_{x_1}(x_1, x_2) + a\,u_{x_2}\,(x_1, x_2) = -af^{'}(x_2 - a\,x_1) + af^{'}(x_2 - a\,x_1) = 0$$

which is equation (7.5).

We call **projected characteristic** to the line $x_2 = k + a\,x_1$, where $k \in \mathbb{R}$ is arbitrary, and we call $f(x_2 - a\,x_1)$ the **first integral** of the PDE.

Figure 7.1 depicts projected characteristic lines for cases $a > 0$ and $a < 0$. These curves correspond to the projection in the space (x_1, x_2) of the solution curves of the PDE (7.5) over which $u(x_1, x_2)$ is constant.

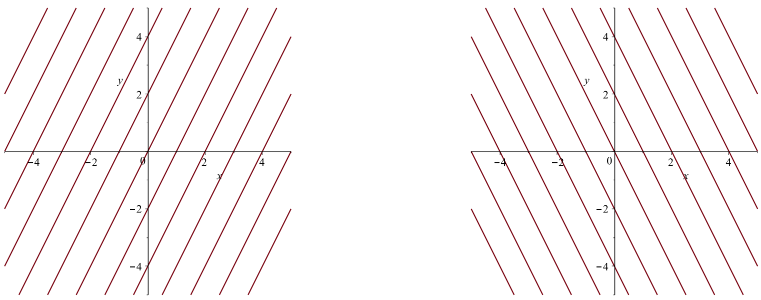


Figure 7.1: Characteristic lines for equation (7.5) for $a > 0$ (left figure) and $a < 0$ (right figure)

Linear right hand side Next we introduce the linear first-order PDE with an homogeneous right-hand side

$$u_{x_1} + a\,u_{x_2} = b\,u, \mathbf{x} \in \mathbb{R}^2 \tag{7.6}$$

where $a \neq 0$ and $b \neq 0$ are constants.

Proposition 4. *The general solution of PDE* (7.6) *is*

$$u(\mathbf{x}) = f(x_2 - a\,x_1)\,e^{b\,x_1}$$

where $f(.)$ is an arbitrary C^1 function.

Proof. To solve it by using the method of characteristics we parameterize again both the independent variables, $x_1 = X_1(s)$ and $x_2 = X_2(s)$, and the unknown function $u = u(X_1(s), X_2(s)) = U(s)$ and solve the system

$$\frac{dX_1}{ds} = 1,\ \frac{dX_2}{ds} = a,\ \frac{dU}{ds} = bU$$

which have solutions

$$x_1 = X_1(s) = s,\ x_2 = X_2(s) = a\,s + k,\ u = U(s) = g(k)\,e^{bs}.$$

where k is an arbitrary constant and $g(k)$ is an arbitrary function. Then $s = x_1$ and the projected characteristic if again $x_2 - a\,x_1 = k$ and $u = g(k)e^{bc_1}e^{b\,x_1} = f(k)e^{b\,x_1}$. □

This equation has the same projected characteristics as shown in figure 7.1 but now, the value of $u(.)$ will not remain constant along the characteristics, as in the case of equation (7.5): it will grow or decay along the characteristic at the rate b, respectively, if $b > 0$ or if $b < 0$.

Cauchy problems

Consider again equation (7.5) and assume that we know the distribution for x_2 for a particular value of x_1, say $x_1 = 0$. If x_1 is interpreted as time, and x_2 as another independent variable, we call the problem an **initial-value problem** (which is a particular case of the Cauchy problem)

$$\begin{cases} u_{x_1}(\mathbf{x}) + a\,u_{x_2}(\mathbf{x}) = 0, & \mathbf{x} \in \mathbb{R}_+ \times \mathbb{R} \\ u = \phi(x_2), & \mathbf{x} \in \{x_1 = 0\} \times \mathbb{R} \end{cases} \tag{7.7}$$

where ϕ is a **known** C^1 function. We can write the initial condition as $u(0, x_2) = \phi(x_2)$ where $\phi(.)$ is known.

Proposition 5. *The general solution to the Cauchy problem* (7.7) *is*

$$u(x_1, x_2) = \phi(x_2 - a\,x_1),\ (x_1, x_2) \in X = \mathbb{R}^2$$

Proof. In the three-dimensional surface S, previously presented, the **constraint defines a curve** $(0, x_2, \phi(x_2))$ that has a **projection** in the (x_1, x_2) space characterized by a curve passing through point $\{\ (0, x_2)\}$. Using the same method that we used to determine the characteristic curve C, we parameterize the constraint Γ by a new variable r, such that it defines a direction $\Gamma = \{\ (0, r)\}$.

Introducing the two parameterizations (associated to the characteristic curve and the initial condition) we define

$$x_1 = X_1(s, r),\ x_2 = X_2(s, r)$$

implying

$$u = u(X_1(s,r), X_2(s,r)) = U(s,r).$$

The characteristic system and the compatibility condition become the system of parameterized (by r) intitial value problems where the ODE's have the independent variable s

$$\begin{aligned}
\frac{\partial X_1(s,r)}{\partial s} &= 1, \quad \text{s.t} \quad X_1(0,r) = 0,\\
\frac{\partial X_2(s,r)}{\partial s} &= a, \quad \text{s.t} \quad X_2(0,r) = r\\
\frac{\partial U(s,r)}{\partial s} &= 0, \quad \text{s.t} \quad U(0,r) = \phi(r).
\end{aligned}$$

The solution to the three ODE initial value problems allows us to obtain a relationship between the initial independent variables and the parameters related to the characteristic and the initial condition

$$x_1 = X_1(s,r) = s \tag{7.8}$$

$$x_2 = X_2(s,r) = as + r \tag{7.9}$$

and

$$u = U(s,r) = U(0,r) = \phi(r), \text{ for any } \ s \in R$$

To get the solution in the original independent variables, we have to obtain the reversed relationships, say $s = S(x,y)$ and $r = R(x,y)$. In order to get it, observe that the solution for the characteristic system can be written as $(x_1, x_2) = G(s,r)$. If this system is invertible then $(s,r) = G^{-1}(x_1,x_2)$. The system (7.8)-(7.9) can provide this solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} s \\ r \end{pmatrix} \Leftrightarrow \begin{pmatrix} s \\ r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - a\,x_1 \end{pmatrix}$$

Therefore, $s = x_1$ and $r = x_2 - a\,x_1$. Then $u(x_1,x_2) = U(s,r) = \phi(r) = \phi(x_2 - a\,x_1)$ $\square$

Example If the initial distribution is $u(0,x_2) = \phi(x_2) = e^{-x_2^2}$, then the solution to the Cauchy problem (7.7) is

$$u(x_1,x_2) = e^{-(x_1-x_2)^2}.$$

Figure 7.2 illustrates this case. The projected characteristics are again as those depicted in figure 7.1.

Next, consider an equation (7.6) and the associated Cauchy problem

$$\begin{cases} u_{x_1}(\mathbf{x}) + u_{x_2}(\mathbf{x}) = b\,u(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2 \\ u = \phi(x_2), & \mathbf{x} \in \{\ x_1 = 0\} \ \times \mathbb{R} \end{cases} \tag{7.10}$$

where $c \neq 0$ is a constant.

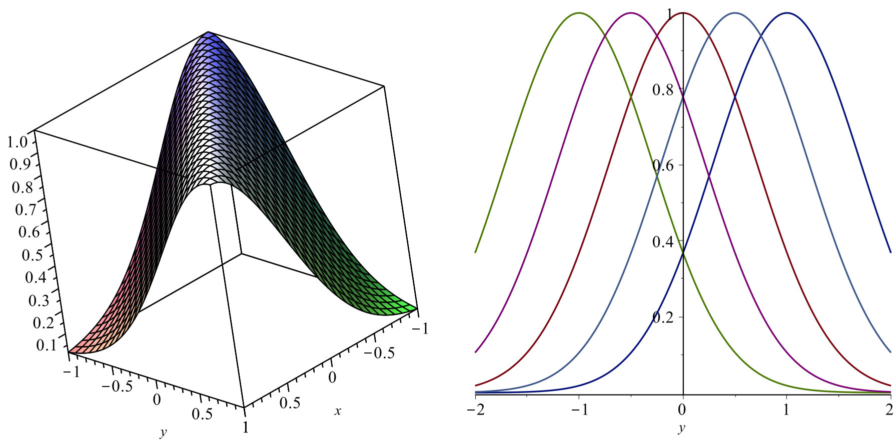


Figure 7.2: Solution for the problem $u_x + u_y = 0$ and $u(0, x_2) = e^{-x_2^2}$, 3d plot and 2d plot for $x_1 \in \{-1, -0.5, 0, 0.5, 1\}$

Proposition 6. *The solution to problem* (7.10) *is*

$$u(\mathbf{x}) = \phi(x_2 - x_1)e^{b\,x_1}$$

Exercise: prove this.

In Figure 7.3 we present an illustration. Observe that for $b > 0$ the solution has both a advection (i.e., transport) and a growing behavior.

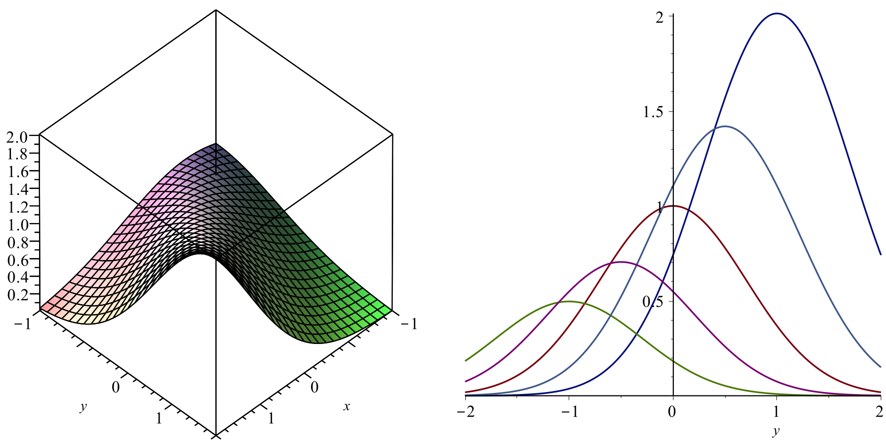


Figure 7.3: Solution for the problem $u_x + u_y = 0.7u$ and $u(0, y) = e^{-y^2}$, 3d plot and 2d plot for $x \in \{-1, -0.5, 0, 0.5, 1\}$

Next, we will see what we can learn from the application of the method of characteristics to solving the semi-linear and the quasi-linear equations.

7.2.4 Semi-linear equation

We consider first one simple semi-linear equation that can be solved by transformation to a linear equation. Next we present conditions for the existence of solutions to more general semi-linear equations, with or without a zero right-hand side, i.e, with $c(\mathbf{x}, u) = 0$ or $c(\mathbf{x}, u) \neq 0$.

Semi-linear equation with zero right-hand-side

We consider the problem for a more general case, in which the coefficient functions are not specified

$$\begin{cases} a(\mathbf{x})u_{x_1} + b(\mathbf{x})u_{x_2} = 0, & \mathbf{x} \in \mathbb{R}^2 \\ u \mid_{\Gamma} = \phi, & \mathbf{x} \in \Gamma \subset \mathbb{R}^2 \end{cases} \tag{7.11}$$

where $a(.)$, $b(.)$ and $c(.)$ are C^1 functions in $\mathbb{R}^2$, and there is a constraint given by curve Γ. The solution to this problem depends on the form of the constraint surface Γ.

Let us consider the points in the constrained set parameterised by r, write $\Gamma = \{\ (x_1, x_2) = (\gamma_1(r), \gamma_2(r))\}$ and define

$$A(r) \equiv a(\gamma_1(r), \gamma_2(r))$$
$$B(r) \equiv b(\gamma_1(r), \gamma_2(r))$$

We say that the constraint Γ is **characteristic** if it is tangent to the projected characteristic and Γ is **non-characteristic** if it is not tangent to the projected characteristic.

We will see next that, Γ is characteristic if

$$\frac{A(r)}{B(r)} = \frac{\gamma_1'(r)}{\gamma_2'(r)}$$

and Γ is non-characteristic if

$$\frac{A(r)}{B(r)} \neq \frac{\gamma_1'(r)}{\gamma_2'(r)}. \tag{7.12}$$

Proposition 7. *Consider the Cauchy problem* (7.11)*. A unique solution exists if* Γ *is non-characteristic in all its domain. The local solution to the problem exist and is unique, and can be written as*

$$u(x_1, x_2) = \phi(G|_{\Gamma}^{-1}(x_1, x_2)).$$

where $\det G|_{\Gamma} \neq 0$*.*

Proof. In order to see this we proceed in two phases.

- First: applying the same method as before, we introduce the change in coordinates $x_1 = X_1(s,r)$, $x_2 = X_2(s,r)$, implying $u = u(x_1,x_2) = u(X_1(s,r), X_2(s,r)) = U(s,r)$. The characteristic system and the compatibility condition become

$$\begin{aligned}\frac{\partial X_1(s,r)}{\partial s} &= a(X_1(s,r), X_2(s,r))\\ \frac{\partial X_2(s,r)}{\partial s} &= b(X_1(s,r), X_2(s,r))\\ \frac{\partial U(s,r)}{\partial s} &= 0\end{aligned}$$

  and the constraints on their values introduced by Γ that we associate with $s=0$ are

$$\begin{aligned}X_1(0,r) &= \gamma_1(r)\\ X_2(0,r) &= \gamma_2(r)\\ U(0,r) &= \phi(r).\end{aligned}$$

  If we solve the ODE characteristic system together with the initial conditions we obtain the transformation $(x_1,x_2) = G(s,r)$, where

$$x_1 = X_1(s,r) \tag{7.13}$$

$$x_2 = X_2(s,r). \tag{7.14}$$

  In order to obtain the solution satisfying $u = U(0,r) = \phi(r)$ we need to solve system (7.13)-(7.14), that is, we need to find $(s,r) = G^{-1}(x_1,x_2)$.

- Second: The system is locally invertible to $s = S(x,y)$ and $r = R(x,y)$ if we can apply the inverse function theorem $(s,r) = G^{-1}(x_1,x_2)$. This is possible if the Jacobian of G has a non-zero determinant evaluated at points $(0,r)$.

  The Jacobian of system (7.13)-(7.14) evaluated at point $(s,r) = (0,r)$ is

$$D(G)|_\Gamma = \begin{pmatrix} \frac{\partial X_1}{\partial s}(0,r) & \frac{\partial X_1}{\partial r}(0,r) \\ \\ \frac{\partial X_2}{\partial s}(0,r) & \frac{\partial X_2}{\partial r}(0,r)\end{pmatrix} = \begin{pmatrix} a(\gamma_1(r),\gamma_2(r)) & \gamma_1'(r) \\ b(\gamma_1(r),\gamma_2(r)) & \gamma_2'(r)\end{pmatrix} = \begin{pmatrix} A(r) & \gamma_1'(r) \\ B(r) & \gamma_2'(r)\end{pmatrix}$$

Then $\det(D(G|_\Gamma)) \neq 0$ if condition (7.12) holds, and, using the inverse function theorem, we can (at least locally) determine $(s,r)|_{s=0} = G^{-1}(x_1,x_2)$, and the solution will have the generic form $u(x,y) = \phi(G^{-1}(x_1,x_2))$ □

This means that, geometrically, the solution will propagate not along parallel characteristic lines but along lines which can change slope depending on the values of x_1 and x_2.

Exercise Consider next the special case of (7.11)

$$\begin{cases} u_{x_1} + b\,x_2\,u_{x_2} = 0,\ \mathbf{x} \in \mathbb{R}^2 \\ u(0,x_2) = \phi(x_2)\,\mathbf{x} \in \{\ x_1 = 0\}\ \times \mathbb{R}\end{cases}$$

where $\phi(\cdot)$ is an arbitrary $C^1(\mathbb{R})$ function.

Show that the solution is

$$u(\mathbf{x}) = \phi(x_2\, e^{-b\, x_1}) \text{ for any } \mathbf{x} \in \mathbb{R}^2.$$

In this case we can observe that the characteristics are

$$x_2 = k\, e^{b\, x_1}$$

they are still parallel as in the linear case, but behave differently depending on the sign of b:

1. if $b > 0$ then x_2 diverges exponentially for increasing values of x_1
2. if $b = 0$ then x_2 is constant for any values of x_1
3. if $b < 0$ then x_2 converges exponentially to zero for increasing values of x_1.

General semi-linear equation

The Cauchy problem for a semi-linear equation and an associated boundary in a surface Γ is

$$\begin{cases} a(\mathbf{x})u_{x_1}(\mathbf{x}) + b(\mathbf{x})u_{x_2}(\mathbf{x}) = c\big(\mathbf{x}, u(\mathbf{x})\big), & \mathbf{x} \in \mathbb{R}^2 \\ u\,|_{\Gamma} = \phi, & \mathbf{x} \in \Gamma \subset \mathbb{R}^2 \end{cases} \tag{7.15}$$

where $a(.)$ and $b(.)$ are C^1 functions in $\mathbb{R}^2$ and $c(.)$ is a C^1 function in $\mathbb{R}^3$. Observe that the function u enters, possibly in a non-linear from, in the right-hand side.

Again we introduce a parameterisation associated with the characteristic surface and the boundary surface by a pair (s, r) and set $x_1 = X_1(s, r)$ and $x_2 = X_2(s, r)$ and $u = U(s, r) = u(X_1(s, r), X_2(s, r))$

In this case the characterstic equation system and the compatibility condition become

$$\begin{aligned} \frac{\partial X_1(s,r)}{\partial s} &= a\left(X_1(s,r), X_2(s,r)\right) \\ \frac{\partial X_2(s,r)}{\partial s} &= b\left(X_1(s,r), X_2(s,r)\right) \\ \frac{\partial U(s,r)}{\partial s} &= c\left(X_1(s,r), X_2(s,r), U(s,r)\right) \end{aligned}$$

and the constrains on their values introduced by Γ that we associate with $s = 0$

$$\begin{aligned} X_1(0,r) &= \gamma_1(r) \\ X_2(0,r) &= \gamma_2(r) \\ U(0,r) &= \phi(r) \end{aligned}$$

We observe again that from the solution of the two first ODE's we get a relationship $(x_1, x_2) = G(s, r)$ and if Γ is non-characteristic we get, at least locally $(s, r) = G^{-1}(x_1, x_2)$, which allows for uniqueness and existence of solutions for the PDE problem. The only difference is related to the fact that now the right hand side of the compatibility condition for U depends on U.

7.2.5 Quasi-linear equations

Let us consider the semi-linear equation and an associated boundary in a surface Γ

$$\begin{cases} a(\mathbf{x}, u)u_{x_1} + b(\mathbf{x}, u)u_{x_2} = c(\mathbf{x}, u), & \mathbf{x} \in \mathbb{R}^2 \\ u \mid_\Gamma = \phi, & \mathbf{x} \in \Gamma \subset \mathbb{R}^2 \end{cases}$$

where $a(.)$, $b(.)$ and $c(.)$ is a C^1 functions in $\mathbb{R}^3$.

Again we introduce a parameterisation associated with the characteristic surface and the boundary surface by a pair (s, r) and set $x_1 = X_1(s, r)$ and $x_2 = X_2(s, r)$ and $u = U(s, r) = u(X_1(s, r), X_2(s, r))$

In this case the characterstic equation system and the compatibility condition become

$$\begin{aligned} \frac{\partial X_1(s,r)}{\partial s} &= a\left(X_1(s,r), X_2(s,r), U(s,r)\right) \\ \frac{\partial X_2(s,r)}{\partial s} &= b\left(X_1(s,r), X_2(s,r), U(s,r)\right) \\ \frac{\partial U(s,r)}{\partial s} &= c\left(X_1(s,r), X_2(s,r), U(s,r)\right). \end{aligned}$$

This system, differently from the previous cases, lost their recursive structure, in the sense that we cannot separate the determination of the solutions for $X_1(.)$ and $X_2(.)$ from $U(.)$: the two independent variables, X_1 and X_2, and the dependent variable, U, are jointly determined. In order to solve the system, Γ provides the boundary conditions for $s = 0$:

$$\begin{aligned} X_1(0,r) &= \gamma_1(r) \\ X_2(0,r) &= \gamma_2(r) \\ U(0,r) &= \phi(r) \end{aligned}$$

Now the non-characteristic conditions for (Γ, ϕ) are more involved because all three differential equations depend on (X_1, X_2, U) and the conditions for the application of the non-characteristic condition may not hold.

The geometric meaning is the following: while for linear and semi-linear PDE the characteristic lines are parallel and do not cross, for the quasi-linear case this may not be the case. At singularity points the uniqueness and even the existence of solutions may break down.

A well known quasi-linear first-order PDE is the inviscid Burger's equation (see `https://en.wikipedia.org/wiki/Burgers%27_equation`)

$$\begin{cases} u_t + uu_x = 0, & (t, x) \in \mathbb{R}^2 \\ u(0, x) = \phi(x) & (t, y) \in \{\ t = 0\} \ \times \mathbb{R} \end{cases}$$

It can be proved that the characteristic equations can intersect which implies that the solutions cannot be unique at those singular points. Introducing some solvability conditions, gives birth to shock waves, which is a type of behavior not presented in linear hyperbolic PDE's.

7.3 The linear equation in the semi-infinite domain and Laplace transform methods

In the previous cases we assumed that the independent variables were defined in the space $\mathrm{X} = \mathbb{R}^2$. The solution of the first-order PDE and/or of the associated problems varies both in terms of the existence and of the methods of determination if the domain is different, that is $\mathrm{X} \subset \mathbb{R}^2$. In this case we may have as solutions not functions (single-valued continuous mappings) but generalized functions (also called weak solutions).

7.3.1 Linear equation with zero right-hand side

To illustrate this, assume that $\mathrm{X} = \mathbb{R}^2_{++}$, that is $x > 0$ and $y > 0$ and consider the problem

$$\begin{cases} u_x + a u_y = 0, & (x,y) \in \mathbb{R}^2_{++} \\ u(x,0) = \psi(x), & (x,y) \in \mathbb{R}_{++} \times \{y = 0\} \\ u(0,y) = \phi(y), & (x,y) \in \{x = 0\} \times \mathbb{R}_{++} \times \end{cases} \tag{7.16}$$

A convenient way to solve this equation is to use **Laplace transforms** instead of the method of characteristics (see the Appendix). In order to do this we pick one of the independent variables as a parameter (for instance x) and keep one variable as an independent variable (for instance y)[3]. Laplace transforms are convenient because the domain of transformation is the semi-infinite interval $[0, \infty)$.

The method of solution follows the steps:

1. First, we apply Laplace transforms to go from the PDE into a parameterized ODE
2. Second, we solve the ODE and apply the transforms of the boundary conditions
3. Finally, we apply inverse Laplace transforms to obtain the solution

Proposition 8. *The solution to Cauchy problem* (7.16) *is*

$$u(x,y) = \phi(y - ax)H(y - ax) + a\, \mathcal{L}^{-1}\left[\int_0^x \psi(s) e^{-a\xi(x-s)} ds\right](y) \tag{7.17}$$

where

$$H(z) = \begin{cases} 0, & \text{if } z \le 0 \\ 1, & \text{if } z > 0 \end{cases}$$

is the Heaviside generalized function and $\mathcal{L}^{-1}[f(x)]\,(y)$ *is the inverse Laplace transform. Therefore*

$$\phi(y - ax)H(y - ax) = \begin{cases} 0 & \text{if } y - ax \le 0 \\ \phi(y - ax) & \text{if } y - ax > 0. \end{cases}$$

[3]The choice can be done in a way to simplify the solution of the problem, given the constraints.

Proof. Let $U(x,\xi)$ be the Laplace transform of $u(x,y)$ taking variable x as a parameter, that is

$$U(x,\xi) \equiv \mathcal{L}[u(x,y)](\xi) = \int_0^\infty e^{-\xi y} u(x,y)dy$$

where $\xi > 0$. The Laplace transforms of $u_x(\cdot)$ and $u_y(\cdot)$ are

$$\mathcal{L}[u_x(x,y)](\xi) = \int_0^\infty e^{-\xi y} u_x(x,y)dy = U_x(x,\xi)$$

and

$$\mathcal{L}[u_y(x,y)](\xi) = \int_0^\infty e^{-\xi y} u_y(x,y)dy = \xi U(x,\xi) - u(x,0) = \xi U_y(x,\xi) - \psi(x).$$

From the continuity and differentiability properties of $u(\cdot)$, $u_x(x,y) + au_y(x,y) = 0$ holds if and only if

$$\int_0^\infty e^{-\xi y}\left(u_x(x,y) + au_y(x,y)\right)dy = 0.$$

Then the PDE, in Laplace transforms, is equivalent to the linear ODE in the variable x, parameterized by the transformed variable ξ [4]

$$U_x(x,\xi) + a(\xi U(x,\xi) - \psi(x)) = 0.$$

In order to solve the Cauchy problem, we also need to introduce the Laplace transform of $\phi(y)$, that is

$$\mathcal{L}[u(0,y)](\xi) = \int_0^\infty e^{-\xi y}\phi(y)dy = \Phi(\xi).$$

Then we get an initial-value problem for the parameterized (by ξ) ODE

$$\begin{cases} U_x(x,\xi) = -a\xi U(x,\xi) + a\psi(x), & x > 0 \\ U(0,\xi) = \Phi(\xi), & x = 0. \end{cases}$$

The solution is

$$U(x,\xi) = \Phi(\xi)e^{-a\xi x} + a\int_0^x \psi(s)e^{-a\xi(x-s)}ds.$$

The inverse Laplace transform is

$$u(x,y) = \mathcal{L}^{-1}\left[\, U(x,\xi)\right]\ (y) = \frac{1}{2\pi i}\lim_{Y\to\infty}\int_{\gamma-iY}^{\gamma+iY} e^{\xi y}\ F(z)dz.$$

[4]We can obtain the same functional equation if we evaluate directly the integral, because

$$\begin{aligned}\int_0^\infty e^{-\xi y}\left(u_x(x,y) + au_y(x,y)\right)dy &= \int_0^\infty e^{-\xi y}\frac{\partial u}{\partial x}(x,y)dy + a\int_0^\infty e^{-\xi y}\frac{\partial u}{\partial y}\ (x,y)dy = \\ &= U_x(x,\xi) + a\left(\int_0^\infty e^{-\xi y}u(x,y) - \int_0^\infty u(x,y)\frac{d}{dy}\ \left(e^{-\xi y}\right)dy\right) = \\ &= U_x(x,\xi) - au(x,0) + a\xi U(x,\xi) = 0\end{aligned}$$

applying integration by parts.

Therefore

$$
\begin{aligned}
u(x,y) = \mathcal{L}^{-1}\big[U(x,\xi)\big] &= \\
&= \mathcal{L}^{-1}\big[\Phi(\xi)e^{-a\xi x}\big] + a\,\mathcal{L}^{-1}\big[\int_0^x \psi(s)e^{-a\xi(x-s)}ds\big].
\end{aligned}
$$

The solution is (7.16). This can be proved by the fact that

$$
\begin{aligned}
e^{-ax\xi}\Phi(\xi) = e^{-ax\xi}\int_0^\infty e^{-\xi s}\phi(s)ds &= \int_0^\infty e^{-\xi(s+ax)}\phi(s)ds \\
&= \int_{ax}^\infty e^{-\xi\, y}\phi(y-a\,x)dy,\ (s+ax=y) \\
&= \int_0^\infty e^{-\xi\, y}\phi(y-a\,x)\,H(y-ax)dy = \\
&= \mathcal{L}\big[\ \phi(y-a\,x)\,H(y-ax)\big]\ (\xi)
\end{aligned}
$$

and $\mathcal{L}^{-1}\Big[\ \mathcal{L}\big[\ \phi(y-a\,x)\,H(y-ax)\big]\ (\xi)\Big]\ = \phi(y-a\,x)\,H(y-ax).$ □

Example 1. In order to have an intuition on the solution consider the case: $a>0$, $\phi(y)=0$ and $\psi(x)=\psi$, a constant. In this case

$$
U(x,\xi) = a\psi\int_0^x e^{-a\xi(x-s)}ds = \psi\ \left(\frac{1-e^{-a\xi x}}{\xi}\right).
$$

Then

$$
u(x,y) = \psi\mathcal{L}^{-1}\left[\ \frac{1-e^{-a\xi x}}{\xi}\right](y) = \psi H(ax-y)
$$

That is the solution is

$$
u(x,y) = \begin{cases} \psi & \text{for } 0<y<a\,x \\ 0 & \text{for } y\geq a\,x \end{cases}
$$

In this case the solution takes a constant value for $\{(x,y): 0<y<ax\}$ where $a>0$ and $x>0$, and it is equal to zero elsewhere.

Example 2. If, instead, we had the case $\phi(y)=e^{by}$ and $\psi(x)=0$ we would have

$$
u(x,y) = \phi(y-ax)H(y-ax) = e^{b(y-ax)}\ H(y-ax)
$$

$$
u(x,y) = \begin{cases} 0 & \text{for } 0<y\leq ax \\ e^{b(y-ax)} & \text{for } y>ax. \end{cases}
$$

In this case the projected characteristics are as in figure :

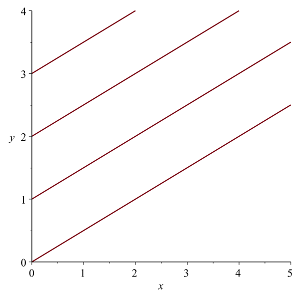


Figure 7.4: Characteristic lines for (7.16) for $a > 0$ $\psi(x) = 0$ and $\phi(y) = e^{by}$

7.3.2 Linear equation with homogeneous right-hand side

Now consider the problem

$$\begin{cases} u_x + u_y = au, & x > 0, y > 0 \\ u(x,0) = e^{bx}, & x > 0 \\ u(0,y) = 0, & y > 0 \end{cases}$$

Using the same method we find the solution

$$u(x,y) = e^{\frac{b}{a}\,(ax-y)}(1 - H(y - a\,x)).$$

or, equivalently,

$$u(x,y) = \begin{cases} e^{\frac{b}{a}\,(ax-y)}, & \text{if } 0 < y < a\,x \\ 0, & \text{if } y \geq a\,x \end{cases}$$

To prove this we can go back to the Laplace transform of equation (7.17),

$$U(x,\xi) = a\int_0^x e^{b\,s}\;e^{-a\xi(x-s)}ds = \;a\,\frac{e^{b\,x}\quad - e^{-a\,\xi\,x}}{b + a\xi},$$

applying the inverse Laplace transform, we find

$$u(x,y) = \mathcal{L}^{-1}[\;U(x,\xi)](y) = e^{\frac{b}{a}\,(ax-y)}(1 - H(y - a\,x)).$$

A graphical depiction of the solution for $a = 1$ and $b = 2$ presented in Figure 7.5. The projected characteristics are as in Figure 7.4 but, differently from that case where u is constant, now the solution growth at the rate a along the characteristic lines.

7.4 Evolution equations

In this section we assume that the independent variables are time and one set of characteristics x, and that the PDE takes the form

$$u_t + \partial_x q(u) = 0\,(t,x) \in \mathbb{R}_+ \;\times \mathbb{R}.$$

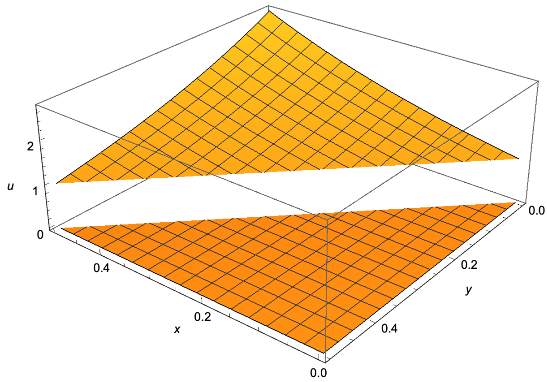


Figure 7.5: Solution for the problem $u_x + u_y = u$, $u(0, y) = 0$, $u(x, 0) = e^{2x}$ and defined for $x > 0$ and $y > 0$

The solution for generic non-linear first order PDE's can display several types of behavior (see (Dafermos, 2000, p.13) :

(1) blow-up if the the solution becomes infinite when a certain level for x is reached in finite time,

(2) globally bounded solutions while x goes to infinite in infinite time;

(3) progressive concentration along time tending asymptotically to a degenerate distribution concentrated at a finite value for x, x^*, in infinite time;

(4) shock-waves such that, after a surface $\Gamma(t, x)$ is reached, the solution becomes non-smooth and multivalued; or

(5) rarefaction waves such that the distribution becomes increasingly dispersed.

From the above results we can conclude the following as regards the linear case

$$u_t + b(x)\, u_x = cu\, (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

(see (Olver, 2014, p.29)). The characteristic curves have the following generic properties:

1 for each point $(t, x) \in \mathbb{R}^2$, there is a unique characteristic passing through that point

2 characteristic curves cannot cross each other

3 if $x = h(t)$ is a characteristic curve, then $x = h(t) + k$ is also a characteristic curve for $k \in \mathbb{R}$, if $c = 0$ or $x = k\, e^{-ct}$

4 the path traced out by a characteristic curve $x = h(t)$ for increasing values of t always moves in the same direction and cannot change the direction of propagation

5 as $t \to \infty$ the characteristic curve either converges to a fixed point, $x(t) \to x^*$ where $c(x^*) = 0$ or go to $\pm\infty$ either in finite or infinite time.

To illustrate the last point consider the problem

$$\begin{cases} u_t + \beta(x - x^*) = 0, \text{ for } (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(0, x) = \phi(x), \text{ for } (t, x) \in \{\ t = 0\}\ \times \mathbb{R}. \end{cases}$$

The solution to this Cauchy problem is

$$u(t, x) = \phi\Big(x^* + (x - x^*)\, e^{-\beta\, t},\ \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

The asymptotic dynamics depend on the sign of β:

1. if $\beta > 0$ we have a rarefaction wage, i, e. $\lim_{t\to\infty} u(t, x) = \phi(x^*)$ which is a constant for all $x \in X$
2. if $\beta < 0$ we have a rarefaction wage, i, e. $\lim_{t\to\infty} u(t, x) = \delta_u(x^*)$ is a degenerate distribution whose mass is concentrated at $x = x^*$.

7.5 Applications

7.5.1 The transport equation

We consider a simple example called the **transport equation**. To simplify assume that the independent variables are (t, x) and that their domain is unbounded, i.e., $(t, x) \in \mathbb{R}^2$ [5] :

$$\begin{cases} \partial_t u(t, x) + \partial_x\left(\mu\, x\, u(t, x)\right) = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(0, x) = \phi(x), & (t, x) \in \{t = 0\} \times \mathbb{R} \end{cases} \tag{7.18}$$

The transport equation describes a conservation phenomenon. If, for instance,

$$\int_{-\infty}^{\infty} \phi(x) dx = 1$$

Proposition 9. *The solution to the transport equation Cauchy problem* (7.18) *is*

$$u(t, x) = e^{-\mu t} \phi\left(x e^{-\mu t}\right).$$

[5] We use the notation $\partial_{x_i} f(\mathbf{x}) = \dfrac{\partial f(\mathbf{x})}{\partial x_i}$, where $\mathbf{x} = (x, \dots, x_i, \dots x_n) \in \mathbb{R}^n$.

Proof. The PDE can be equivalently written as

$$u_t(t,x) + \mu\, x\, u_x(t,x) + \mu\, u(t,x) = 0.$$

Let us consider a change in variables: $x = X(y) = e^{\mu y}$ and

$$v(t,y) = e^{\mu t}\, u(t, X(y)).$$

Taking derivatives for t and y and applying the relationship in equation (7.18) we find that

$$v_t(t,y) + v_y(t,y) = 0$$

if and only if $u_t(t,x) + \mu x u_x(t,x) + \mu u(t,x) = 0$. This equation has the form of (7.5), with $a = 1$. As $v(0,y) = u(0, X(y)) = \phi(X(y))$ we can use the solution to Cauchy problem (7.7) to obtain

$$v(t,y) = \phi(X(y-t)) = \phi\left(e^{\mu(y-t)}\right) = \phi\left(X(y)e^{-\mu t}\right).$$

To obtain the solution we just need to transform back to the original function $u(.)$ and substitute $X(y) = x$. $\square$

The projected characteristics are as in Figure 7.6 for $\mu > 0$. Differently from the previous cases we see that the characteristics are non-linear, and, in this case exponentially growing.

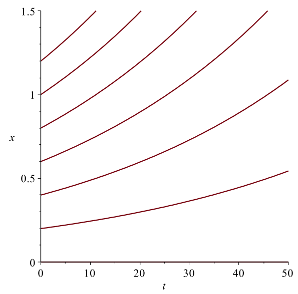


Figure 7.6: Characteristic lines for (7.18) for $a > 0$

7.5.2 Age-structured population dynamics

The exponential model for population dynamics, $\dot{n} = \mu n$, where μ is the difference between the fertility rate and the mortality rate, has the solution $n(t) = n(0)e^{\mu t}$,

Although this model may be a good approximation asymptotically, in the shorter run there is a large deviation. One of the reasons for the deviation is related to the fact that both fertility and mortality rates are age-dependent. If we introduce age-dependent mortality and fertility rates the dynamics of the population is governed by a first-order PDE.

Let $N(a,t)$ be the number of females of age a at time t in a population. The total population at time t is

$$N(t) = \int_0^{a_{max}} N(a,t)\, da$$

Defining the proportion of population with age a by $n(a,t) = N(a,t)/N(t)$ we have

$$\int_0^{a_max} n(a,t)\,da \quad = 1.$$

The proportion of population, measured by the proportion of females, between ages a_1 and $a_2 > a_1$ at time t is

$$n([a_1,a_2],t) = \int_{a_1}^{a_2} n(a,t)da.$$

If there is no mortality, the instantaneous change in $n([a_1,a_2],t)$ is

$$\frac{d}{dt}\int_{a_1}^{a_2} n(a,t)da = n(a_1,t) - n(a_2,t)$$

where $n(a_1,t)$ and $n(a_2,t)$ is the flow of females which is just entering and leaving the interval $[a_1,a_2]$. As $\frac{d}{dt}\int_{a_1}^{a_2} n(a,t)da = \int_{a_1}^{a_2} u_t(a,t)da$ and, using the fundamental theorem of calculus $n(a_1,t) - n(a_2,t) = -\int_{a_1}^{a_2} u_a(a,t)da$ then

$$\frac{d}{dt}\int_{a_1}^{a_2} n(a,t)da - (n(a_1,t) - n(a_2,t)) = \int_{a_1}^{a_2} (n_t(a,t) + n_a(a,t))\,da = 0$$

and there is a conservation law. However, introducing mortality, the population in the interval $[a_1,a_2]$ will not remain constant

$$\int_{a_1}^{a_2} (n_t(a,t) + n_a(a,t))\,da - \int_{a_1}^{a_2} \mu(a,t)n(a,t)da = 0$$

Therefore we have the equation for an age-dependent population

$$n_t(a,t) + n_a(a,t) = \mu(a,t)n., \text{ for } (a,t) \in [0,a_{max}] \ \times \mathbb{R}_+.$$

The McKendry model further assumes an initial population distribution and an age-dependent fertility

$$\begin{cases} n_t + n_a = -\mu(a,t)n, & (a,t) \in (0,\mathrm{X}) \times (0,\infty) \\ n(a,0) = n_0(a), & (a,t) \in (0,\mathrm{X}) \times \{t=0\} \\ n(0,t) = b(t), & (a,t) \in \{a=0\} \ \times (0,\infty) \end{cases} \tag{7.19}$$

$\mathrm{X} = \max\{a\}$, the maximum age of the population, $n_0(a)$ is the initial age-distribution of the population, and the number of newborns is

$$b(t) = \int_0^{a_{max}} \beta(a,t)n(a,t)da$$

where $\beta(a,t)$ is the age-distribution of fertility at time t. If we compared to the PDE already presented, the McKendrick model has two new features:

1. first, it has two boundary conditions: an initial distribution for the population (at $t=0$) and for the population at age $a=0$;

2. second, the boundary condition referring to the newborns is non-local, that is, it depends on the distribution of the total population. This last feature implies that it is hard to solve, requiring the solution of an integral equation.

Assuming away that global nature of fertility, the Mc-Kendrick equation features a different type of dynamics depending in the difference between a and t: for $a < t$ the dynamics depends on the newborns, i.e., population with age $a = 0$, while for $a > t$ the dynamics is governed by the initial age-distribution of the population. Of course, asymptotically the first type of behavior prevails.

Consider the case

$$\begin{cases} n_t + n_a = -\mu n, & (a,t) \in (0, \mathrm{X}) \times (0, \infty) \\ n(a,0) = n_0(a), & (a,t) \in (0, \mathrm{X}) \times \{t = 0\} \\ n(0,t) = b(t), & (a,t) \in \{a = 0\} \ \times (0, \infty) \end{cases}$$

where $n_0(a)$ is the initial distribution of population, and $\phi(t)$ is the number of offspring here assumed as exogenous, i.e., independent of the distribution of population.

Prove that the solution is

$$n(a,t) = \begin{cases} b(t-a)\pi(a), & \text{if } a \leq t \\ n_0(a-t)\frac{\pi(a)}{\pi(a-t)}, & \text{if } a \geq t \end{cases}$$

where

$$\pi(a) = e^{-\mu a}$$

is the probability of survival until age a. Therefore, $\frac{\pi(a)}{\pi(a-t)} = e^{-\mu t}$.

A simpler version of this model, assumes that there is only one fertile age $0 < \alpha < \mathrm{X}$, leading to

$$b(t) = \int_0^{a_{max}} \beta\, \delta(a-\alpha)\, n(a,t) da = \beta n(\alpha, t).$$

The solution displays an "echo effect" with period equal to α.

Reference: McKendrick (1926) and for a recent textbook presentation Kot (2001)

7.5.3 Cohort's budget constraint

Let $w(a,t)$ be the financial wealth of an agent with age a at time t. The budget constraint is

$$w_t + w_a = s(a,t) + rw(a,t) \tag{7.20}$$

where $s(a,t)$ is the savings at age a at time t and r is the interest rate. If we assume that the initial stock of wealth is unbounded then $w : (0, A) \times (0, T) \to \mathbb{R}$ and the initial wealth distribution is $w(0,t) = 0$.

The general solution of equation (7.20) is

$$w(a,t) = \left(\int_0^a s(z, z-a+t) e^{-rz} dz + f(t-a) \right) e^{ra}$$

for an arbitrary $f(.)$.

If we assume that there are no bequests, that is no wealth at birth, $w(a,t) = 0$ and $s(a,t) = e^{ba(K-a)+gt} - c$ the solution becomes

$$w(a,t) = \frac{\sqrt{\pi}}{2\sqrt{b}} \left(\Phi\left(\frac{Kb+g-r}{2\sqrt{b}} \right) - \Phi\left(\frac{(K-2a)b+g-r}{2\sqrt{b}} \right) \right) e^{\frac{K^2b^2((2K+4(t-a))g-2(K-2a)r)b+(g-r)^2}{4b}} - \\ - \frac{c}{r}\left(1-e^{ra}\right) \quad (7.21)$$

where $\Phi(x) = \text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-z^2} dz$. Figure **??** illustrates equation (7.21). It displays a life-cycle behavior of savings: the agent tends to be a net borrower at young age and lender at older ages, although it dissaves later in life.

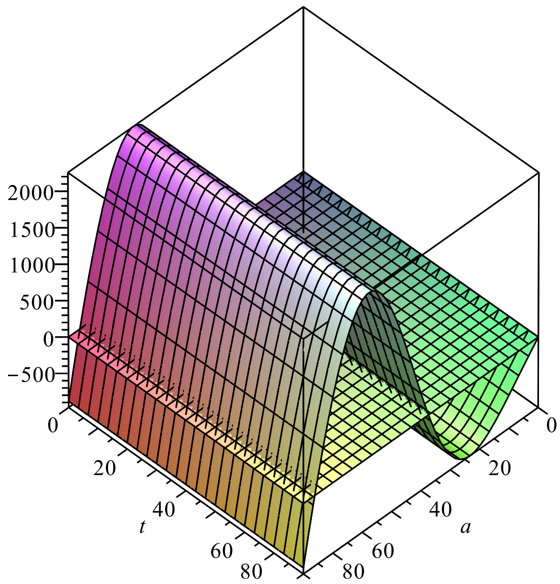


Figure 7.7: Illustration of equation (7.21) for $K = 88$, $b = 0.0029$, $r = 0.02$, $g = 0.02$ and $c = 50$.

7.5.4 Interest rate term-structure

We consider an economy in which there is perfect foresight (i.e, there is a deterministic setting) and in which there are two types of assets: a bank account and a continuum of bonds with maturity dates ranging from zero to infinity, $x \in [0, \infty)$. We show that, if there is absence of arbitrage opportunities, the forward rate at time t for a bound maturing at time $t + x$ perfectly anticipates the spot interest rate for time $t + x$.

First, the bank account's balance, for an initial deposit of $B(0)$, follows the process

$$B(t) = B(0)e^{\int_0^t r(\tau)d\tau}.$$

where $r(t)$ is the spot interest rate. Therefore, the rate of return for a bank account is

$$\frac{dB(t)}{dt}\frac{1}{B(t)} = r(t), \text{for any}\, t \geq 0.$$

Second, the price, at time t, for a bond[6] that matures at time $t+x$,

$$P(t,x) = e^{-\int_0^x f(t,y)dy} \tag{7.22}$$

where $f(t,x)$ is the forward interest rate for maturity $x \geq 0$. The spot interest rate and the forward rate with maturity zero are the same $f(t,0) = r(t)$.

The change in the price of the bond is

$$\begin{aligned} dP(t,x) &= \partial_t P(t,x)\,dt + \partial_x P(t,x)\,dx = \\ &= \Big(\partial_t P(t,x) - \partial_x P(t,x)\Big)\,dt \end{aligned}$$

because in we move dt on time the duration to maturity reduces such as $dx + dt = 0$. Taking derivatives in equation (7.22) we have

$$\begin{aligned} \partial_t P(t,x) &= -P(t,x) \int_0^x \partial_t f(t,y)dy \\ \partial_x P(t,x) &= -P(t,x)\, f(t,x). \end{aligned}$$

Therefore, the rate of return for a bond with maturity x

$$\frac{dP(t,x)}{dt}\frac{1}{P(t,x)} = f(t,x) - \int_0^x \partial_t f(t,y)\,dy.$$

Using the mean value theorem $f(t,x) - f(t,0) = \int_0^x \partial_x f(t,y)\,dy$, and noting that $f(t,0) = r(t)$ we find

$$\frac{dP(t,x)}{dt}\frac{1}{P(t,x)} = -\int_0^x \left(\partial_t f(t,y) - \partial_x f(t,y)\right) dy + r(t)$$

Third, If there are no arbitrage opportunities, the instantaneous rate of return for any two investments should be equal for every point in time t. In particular, the rate of return for the bank account should be equal to the rate of return for a bond of any maturity $x \geq 0$,

$$\frac{dB(t)/dt}{B(t)} = \frac{dP(t,x)/dt}{P(t,x)}.$$

This condition is equivalent to

$$\int_0^x \left(\partial_t f(t,y) - \partial_x f(t,y)\right) dy = 0, \text{ for every } x \geq 0.$$

Therefore the forward rate, in a deterministic setting, solves the following Cauchy problem

$$\begin{cases} \partial_t f(t,x) - \partial_x f(t,x) = 0, & (t,x) \in \mathbb{R}^2_+ \\ f(t,0) = r(t), & (t,x) \in \mathbb{R}_+ \times \{t=0\}\ . \end{cases}$$

Because the PDE has the general solution $f(t,x) = h(t+x)$, the solution to the Cauchy problem is

$$f(t,x) = r(t+x)$$

the forward rate, at time t, for a bond maturing at time $t+x$ is a perfect predictor to the spot rate at time $t+x$.

[6]It is implicitly assumed that the bond has face value equal to one and does not pay coupons.

7.5.5 Optimality condition for a consumer choice problem

Consider the following general consumer problem

$$\max_{c_1, c_2} \; u(c_1, c_2)$$

subject to the following constraints

$$\begin{cases} E(c_1, c_2) = p_1 c_1 + p_2 c_2 \leq W \\ 0 \leq c_1 \leq \bar{c}_1 \\ 0 \leq c_2 \leq \bar{c}_2 \end{cases}$$

Assume that $u(.)$ is continuous, differentiable, increasing and concave in both arguments. Forming The Lagrangean

$$\mathcal{L} = u(c_1, c_2) + \lambda(W - E(c_1, c_2)) - \eta_1 c_1 - \eta_2 c_2 + \zeta_1(\bar{c}_1 - c_1) + \zeta_2(\bar{c}_2 - c_2).$$

The solution (which always exists) (c_1^*, c_2^*) verifies the Karush-Kuhn-Tucker conditions

$$\begin{aligned} & u_{c_i}(c_1, c_2) - \lambda p_j - \eta_j - \zeta_j = 0, \; j = 1, 2 \\ & \eta_j c_j = 0, \; \eta_j \geq 0, \; c_j \geq 0, \; j = 1, 2 \\ & \zeta_j(\bar{c}_j - c_j) = 0, \; \zeta_j \geq 0, \; c_j \leq \bar{c}_j, \; j = 1, 2 \\ & \lambda(W - E(c_1, c_2)) = 0, \; \lambda \geq 0, \; E(c_1, c_2) \leq W \end{aligned}$$

We have three cases:

First, an interior solution, $c_1^* \in (0, \bar{c}_1)$ and $c_2^* \in (0, \bar{c}_2)$, with the optimality conditions

$$p_2 u_{c_1}(c_1^*, c_2^*) = p_1 u_{c_2}(c_1^*, c_2^*) \tag{7.23}$$

$$E(c_1^*, c_2^*) = W \tag{7.24}$$

Equation (7.23) is a first-order partial differential equation with solution

$$u(c_1^*, c_2^*) = v\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)$$

and, if we use equation (7.24) and define $w \equiv W/p_1$ in the optimum, we have

$$u(c_1^*, c_2^*) = v(w)$$

If the utility function is strictly concave then with very weak conditions (differentiability) we have an unique interior optimum. It is clear that the budget set, in real terms, is a projected characteristic.

Second, the first corner solution for c_1 is $c_1^* = 0$ and an interior solution for c_2 $c_2^* \in (0, \bar{c}_2)$ and the budget constraint be saturated, and satisfies the conditions

$$p_2 u_{c_1}(c_1^*, c_2^*) = p_1 u_{c_2}(c_1^*, c_2^*) - p_2 \eta_1 \tag{7.25}$$

$$E(c_1^*, c_2^*) = W \tag{7.26}$$

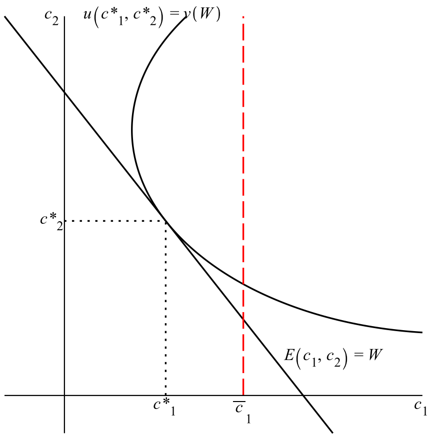


Figure 7.8: Interior optimum

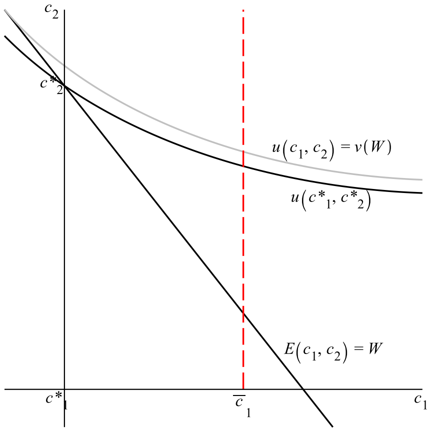


Figure 7.9: Corner solution: zero consumption

Equation (7.25) is a first-order partial differential equation with solution

$$u(c_1^*, c_2^*) = \frac{\eta_1 c_2^*}{p_1} + v\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)$$

If we use equation (7.28) in the optimum we have

$$u(c_1^*, c_2^*) = -\frac{\eta_1 p_2 c_2^*}{p_1} + v(w) < v(w)$$

then the indirect utility level is smaller than for the unconstrained case

Third, the second corner solution for c_1: lLet $c_1^* = \bar{c}_1$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated; It verifies the conditions

$$p_2 u_{c_1}(c_1^*, c_2^*) = p_1 u_{c_2}(c_1^*, c_2^*) + p_2 \zeta_1 \tag{7.27}$$

$$E(c_1^*, c_2^*) = W \tag{7.28}$$

Equation (7.27) is a first-order partial differential equation with solution

$$u(c_1^*, c_2^*) = -\frac{\zeta_1 c_2^*}{p_1} + v\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)$$

and if we use equation (7.28) in the optimum we have

$$u(c_1^*, c_2^*) = \frac{\zeta_1 p_2 c_2^*}{p_1} + v(w) < v(w)$$

then the indirect utility level is smaller than for the unconstrained case

We verify that the interior optimum has characteristics given by $c_1 + \frac{p_1}{p_2} c_2$, and in general they coincide with the budget constraint.

We can repeat the same exercise for the corner solutions for c_2.

7.5.6 Growth and inequality dynamics

Consider an economy in which the households are heterogenous as regards their capital endowments, let the endowments belong to the continuum $k \in [\underline{k}(t), \overline{k}(t)] \subset \mathbb{R}_+$, and let $N(t, k)$ be the number of people having an asset endowment k, at time t.

If we denote the total population by $N(t)$ then

$$N(t) = \int_{\underline{k}(t)}^{\overline{k}(t)} N(t, k) dk.$$

We can denote the **population density** by $n(t, k) = N(t, k)/N(t)$. In this case

$$\int_{\underline{k}(t)}^{\overline{k}(t)} n(t, k) dk = 1.$$

Assume that the capital accumulates in linearly as

$$\frac{dk}{dt} = \gamma k(t).$$

Then, using Leibniz rule

$$\begin{aligned}
\frac{d}{dt}\int_{\underline{k}(t)}^{\overline{k}(t)} n(t, k) dk &= \int_{\underline{k}(t)}^{\overline{k}(t)} n_t(t, k) dk + n(t, \overline{k}(t)) \frac{d\overline{k}(t)}{dt} - n(t, \underline{k}(t)) \frac{d\underline{k}(t)}{dt} \\
&= \int_{\underline{k}(t)}^{\overline{k}(t)} n_t(t, k) dk + n(t, \overline{k}(t)) \gamma \overline{k}(t) - n(t, \underline{k}(t)) \gamma \underline{k}(t) = \\
&= \int_{\underline{k}(t)}^{\overline{k}(t)} n_t(t, k) + \frac{\partial \left(\gamma k n(t, k)\right)}{\partial k} dk
\end{aligned}$$

by the mean-value theorem. Then the density satisfies the PDE

$$n_t(t, k) + \gamma k n_k(t, k) + \gamma n(t, k) = 0$$

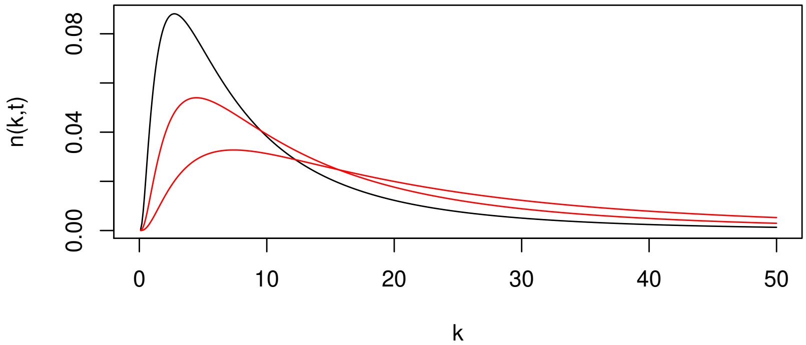


Figure 7.10: Density for several increasing dates

which has the form of the transport equation (7.18).

Given an initial density $n_0(k)$ the solution is then

$$n(t,k) = e^{-\gamma t} n_0\left(ke^{-\gamma t}\right), \ (t,k) \in \mathbb{R}_+.$$

Assuming a log-normal distribution

$$\phi(k) = \left(2\pi k^2 \sigma^2\right)^{-\frac{1}{2}} \exp\left(-\frac{(\log(k) - \mu)^2}{2\sigma^2}\right)$$

then

$$n(t,k) = \left(2\pi k^2 \sigma^2\right)^{-\frac{1}{2}} \exp\left(-\frac{(\log(k) - \gamma t - \mu)^2}{2\sigma^2}\right)$$

Figure **??** presents the dynamics of capital distribution, that is, the dynamics of the density of population for several levels of capital

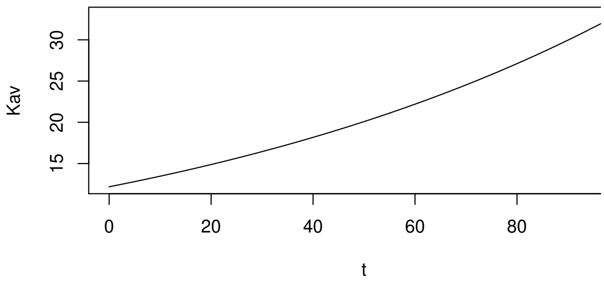
Growth can be seen as travelling of the density of population for higher levels of capital wealth. We can compute several statistics to characterize the growth and distributional facts from this simple model (see Figure 7.11:

1. There is unbounded growth, with a constant growth rate $\gamma(t) = \gamma$
2. The distribution is non-ergodic (the average and variance tends to infinity)

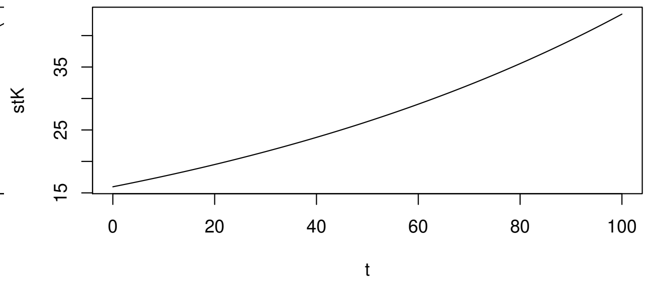
$$\bar{k}(t) = e^{\gamma t + \mu + \frac{\sigma^2}{2}}, \ \sigma_k(t) = (\bar{k}(t))^2 \left(e^{\sigma^2} - 1\right)$$

3. But the inequality measures are constant: Gini and Theil indices

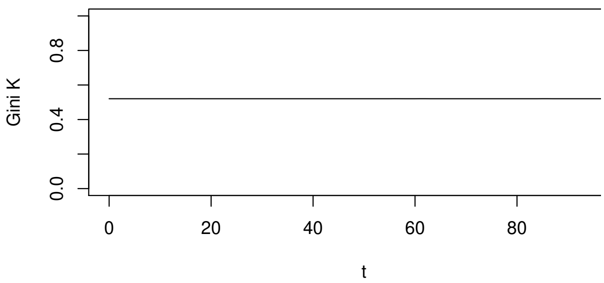
$$G(t) = \text{erf}\left(\frac{\sigma}{2}\right), \ \text{Th}(t) = \frac{\sigma^2}{2}$$



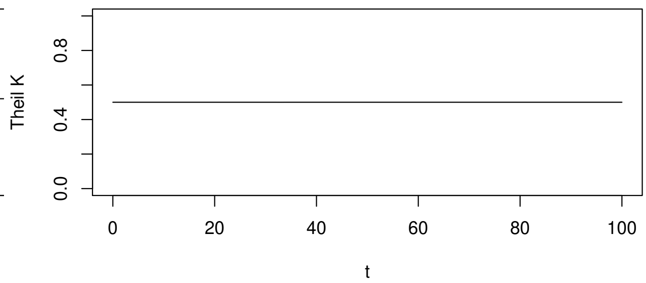
(a) Increasing average



(b) Increasing st. dev.



(c) Constant Gini coefficient



(d) Constant Theil index

Figure 7.11: Linear accumulation function for $\gamma > 0$ and an initial log-normal distribution

4. Ratio of the quantiles is also constant

$$\frac{k_{90}}{k_{10}} = -\sigma\sqrt{2}\left[\mathrm{erf}^{-1}\left(1-2\frac{9}{10}\right) - \mathrm{erf}^{-1}\left(1-2\frac{1}{10}\right)\right]$$

7.6 References

- Mathematics of PDE: introductory Olver (2014) more advanced (Evans, 1998, ch 3) or Chechkin and Goritsky (2009)
- Applications to economics: Hritonenko and Yatsenko (2013)
- Application to mathematical demography: (Kot, 2001, ch. 23)
- A useful site: `http://eqworld.ipmnet.ru/en/solutions/fpde/fpdetoc1.htm`

7.A Laplace transforms and inverse Laplace transforms

Consider function $f(x)$ where $x > 0$. The Laplace transform of $f(x)$ is

$$\mathcal{L}[f(x)](s) = \int_0^\infty e^{-sx} f(x)dx = F(s).$$

The application of Laplace transforms to the solution of differential equations is convenient because it allows for the transformation of a ODE into an non-differential equation and the transformation of a PDE into an ODE.

The Laplace transform of $f^{'}(x) = df(x)/dx$ is

$$\mathcal{L}[f^{'}(x)](s) = \frac{d}{dx}\left(\int_0^\infty e^{-sx} f^{'}(x)dx\right) = s\int_0^\infty e^{-sx} f(x)dx + e^{-sx}f(x)\mid_{x=0}^\infty = sF(s) - f(0)$$

if the function $f(.)$ is bounded.

Example: consider the differential equation

$$f^{'}(x) = af(x).$$

Applying the Laplace transform to both sides, yields

$$\mathcal{L}[f^{'}(x)](s) = a\mathcal{L}[f(x)](s),$$

which is equivalent to the the algebraic equation in $F(s)$

$$sF(s) - f(0) = aF(s).$$

Therefore

$$F(s) = \frac{f(0)}{s-a}.$$

To go back to the solution as a function of the independent variable x, we apply the inverse Laplace transform

$$\mathcal{L}^{-1}[F(s)](x) = f(0)\mathcal{L}^{-1}[\frac{1}{s-a}](x).$$

But

$$f(x) = \mathcal{L}^{-1}[F(s)](x) = \int_0^\infty e^{-xs}F(s)ds$$

and

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)(x) = e^{ax}$$

Therefore,

$$f(x) = f(0)\, e^{ax}.$$

Some Laplace transforms used in the main text are presented in Table 7.1

The Laplace transform and the inverse Laplace transforms are tabulated in many textbooks on calculus or in the web, see `http://tutorial.math.lamar.edu/pdf/Laplace_Table.pdf`. We can compute them using Mathematica `http://reference.wolfram.com/language/ref/LaplaceTransform.html` and `http://reference.wolfram.com/language/ref/InverseLaplaceTransform.html`.

Table 7.1: Laplace transforms and inverse Laplace transforms

$f(x)$	$F(s)$	
a	$\frac{1}{s}$	
x	$\frac{1}{s^2}$	
e^{ax}	$\frac{1}{s-a}$	
$H(a-x)$	$\frac{e^{-as}}{s}$	$a>0$
$f(x-a)\,H(x-a)$	$F(s)\,e^{-as}$	$a>0$

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